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RESULTS ON BETTI SERIES OF THE UNIVERSAL MODULES OF SECOND ORDER DERIVATIONS

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Abstract

Let R be the coordinate ring of an affine irreducible curve presented by $\frac{k[x,y]}{(f)}$ and m a maximal ideal of R. Assume that R_m, the localization of R at m, is not a regular ring. Let $\Omega_2(R_m)$ be the universal module of second order derivations of R_m. We show that, under certain conditions, $B(\Omega_2(R_m), t)$, the Betti series of $\Omega_2(R_m)$, is a rational function. To conclude, we give examples related to $B(\Omega_2(R_m), t)$ for various rings R.

Keywords: Universal module, Universal differential operators, Betti series, Minimal resolution.

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1. Introduction

Let R be a commutative $k-algebra$ over a field of characteristic zero. Consider the exact sequence

$$0 \to I \to R \otimes_k \xrightarrow{\varphi} R \to 0,$$

where $\varphi(a \otimes b) = ab$ for $a, b \in R$ and $I = \ker \varphi$.

For any $n \geq 1$, I^n is an ideal contained in I. Let us define a k-linear map $\Delta_n : R \to \frac{I}{I^{n+1}}$ by $\Delta_n(r) = 1 \otimes r - r \otimes 1 + I^{n+1}, \Delta_n(k) = 0$. The left R-module $\frac{I}{I^{n+1}}$ is called the *universal module of* n^{th} *order derivations*, and Δ_n is the *universal derivation of order* n. Denote $\frac{I}{I^{n+1}}$ by $\Omega_n(R)$. (A definition of $\Omega_n(R)$ may be found in [3]). We note that $\Omega_n(R) \otimes_R R_T \cong \Omega_n(R_T)$ and that $\Omega_n(R)$ is a finitely generated R-module when R is a finitely generated k-algebra, and that $\Omega_n(R)$ is a free R-module of rank $\binom{n+k}{k} - 1$ with basis

$$\left\{\Delta_n(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}) : 1 \leq \alpha_1 + \alpha_2 + \cdots + \alpha_k \leq n\right\}$$

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when $R = k[x_1, \ldots, x_k]$, (see [2]).

Assume that R is a local $k-$algebra with maximal ideal m. A resolution

$$\cdots F_2 \overset{\partial_2}{\to} F_1 \overset{\partial_1}{\to} F_0 \overset{\varepsilon}{\to} \Omega_n(R) \to 0$$

of $\Omega_n(R)$ by free modules of finite rank, such that $\partial_n(F_n) \subseteq mF_{n-1}$ for all $n \geq 1$, is called a *minimal resolution.* It is known that the minimal free resolution is unique up to isomorphism of complexes. The *Betti series* of $\Omega_n(R)$ is defined to be the series

$$B(\Omega_n(R), t) = \sum_{i \geq 0} \dim_{R/m} \operatorname{Ext}^i\Big(\Omega_n(R), \frac{R}{m}\Big) t^i$$

for all $n \geq 1$.

It is interesting to know if $B(\Omega_n(R), t)$ is a rational function. If R is a finitely generated regular algebra then $\Omega_n(R_m)$ is a free R_m-module, where m is a maximal ideal of R. Therefore $B(\Omega_n(R_m), t)$ is obviously a rational function for all $n \geq 1$.

2. Main results

We first state a known result:

2.1. Lemma. *Let* $R = \frac{k[x_1, x_2, \ldots, x_k]}{(f)}$. *Then we have an exact sequence*

$$0 \to \ker \alpha \to \frac{\Omega_n(k[x_1, x_2, \ldots, x_k])}{f\Omega_n(k[x_1, x_2, \ldots, x_k])} \to \Omega_n(R) \to 0$$

of $R-$*modules, (see* [1]*).* □

2.2. Lemma. *Let* $k[x, y]$ *be a polynomial algebra and* m *a maximal ideal containing* f. *Suppose that* $\Delta_2(yf), \Delta_2(xf)$ *and* $\Delta_2(f)$ *are elements of* $m\Omega_2(k[x, y])$. *Then the module generated by* $\{\Delta_2(g) : g \in fk[x, y]\}$ *is a submodule of* $m\Omega_2(k[x, y])$, *where*

$$\Delta_2 : k[x, y] \to \Omega_2(k[x, y])$$

is the second order derivation.

Proof. Since Δ_2 is a k-linear map, it suffices to show that $\Delta_2(x^i y^j f) \in m\Omega_2(k[x, y])$, By the definition of Δ_2 we have that

$$\Delta_2(x^i y^j f) = a(x, y)\Delta_2(xf) + b(x, y)\Delta_2(yf) + c(x, y)f\Delta_2(\alpha),$$

where $\alpha \in \Omega_2(k[x, y])$, $a(x, y), b(x, y), c(x, y) \in k[x, y]$. By assumption $\Delta_2(xf), \Delta_2(yf) \in m\Omega_2(k[x, y])$ and $f \in m$. Hence each summand belongs to $m\Omega_2(k[x, y])$. Therefore $\Delta_2(x^i y^j f) \in m\Omega_2(k[x, y])$, as required. □

We now give a well-known result.

2.3. Lemma. *Let* R *be a local ring with maximal ideal* m. *Let* M *be a finitely generated* R*-module. Suppose that*

$$0 \to F_1 \overset{\partial}{\to} F_0 \to M \to 0$$

is a minimal resolution of M. *Then* $\operatorname{Ext}^1(M, R/m)$ *is not zero.*

Proof. Consider the given minimal resolution

$$0 \to F_1 \overset{\partial}{\to} F_0 \to M \to 0$$

of R-modules. Then we have the complex

$$0 \to \operatorname{Hom}(M, R/m) \overset{\varepsilon^*}{\to} \operatorname{Hom}(F_0, R/m) \overset{\partial^*}{\to} \operatorname{Hom}(F_1, R/m)$$

of R/m-modules. Since F_0 and F_1 are free modules it follows that ∂^* is a matrix whose entries are all in m. Hence $\operatorname{Im}\partial^* \subseteq m\mathrm{Hom}(F_1, R/m)$. By Nakayama's Lemma $m\mathrm{Hom}(F_1, R/m) \neq \mathrm{Hom}(F_1, R/m)$. Therefore $\mathrm{Ext}^1(M, R/m) = \dfrac{\mathrm{Hom}(F_1, R/m)}{\mathrm{Im}\partial^*}$ is nonzero as required. □

2.4. Proposition. *Let $k[x,y]$ be a polynomial ring and m a maximal ideal of $k[x,y]$ containing an irreducible element f. If $\Delta_2(yf), \Delta_2(xf)$ and $\Delta_2(f)$ are elements of $m\Omega_2(k[x,y])$, then $\Omega_2\Big(\big(\frac{k[x,y]}{(f)}\big)_{\bar{m}}\Big)$ admits a minimal resolution of $\Big(\frac{k[x,y]}{(f)}\Big)_{\bar{m}}$ modules.*

Proof. Set $S = \frac{k[x,y]}{(f)}$, $\bar{m}$ a maximal ideal of S. Consider the exact sequence

$$0 \to \ker \alpha_{\bar{m}} \to \left(\frac{\Omega_2(k[x,y])}{f\Omega_2(k[x,y])}\right)_{barm} \to \Omega_2\big(S_{\bar{m}}\big) \to 0$$

of $S_{\bar{m}}$-modules. By lemma 2.2 this exact sequence is minimal. To complete the proof we need to see that $\ker \alpha_{\bar{m}}$ is a free $S_{\bar{m}}$-module. Notice that the Krull dimension of $S_{\bar{m}}$ is one, and that $\left(\frac{\Omega_2(k[x,y])}{f\Omega_2(k[x,y])}\right)_{\bar{m}}$ is free of rank five. Let K be the field of fractions of $S_{\bar{m}}$. The transcendental degree of K is one. Hence $\dim_K \Omega_2\big(S_{\bar{m}}\big) \otimes_{S_{\bar{m}}} K = \dim_K \Omega_2(K) = 2$ as a K-vector space.

Therefore we have

$$\dim_K \ker \alpha_{\bar{m}} \otimes_{S_{\bar{m}}} K = \dim_K \left(\frac{\Omega_2(k[x,y])}{f\Omega_2(k[x,y])}\right)_{\bar{m}} \otimes_{S_{\bar{m}}} K - \dim_K \Omega_2(K) = 5-2 = 3.$$

On the other hand, $\ker \alpha_{\bar{m}}$ is generated by $\overline{\Delta_2(xf)}, \overline{\Delta_2(yf)}$ and $\overline{\Delta_2(f)}$ as an S-module. Therefore $\ker \alpha_{\bar{m}}$ must be a free $S_{\bar{m}}$-module, as required. □

2.5. Theorem. *Let $k[x,y]$ be a polynomial ring and m a maximal ideal containing an irreducible polynomial f. Suppose that $R = \frac{k[x,y]}{(f)}$ is not a regular ring at $\bar{m} = \frac{m}{(f)}$, and that $\overline{\Delta_2(xf)}, \overline{\Delta_2(yf)}$ and $\overline{\Delta_2(f)}$ are elements of $m\Omega_2(k[x,y]$. Then $B(\Omega_2(R_{\bar{m}}), t)$ is a rational function.*

Proof. By Proposition 2.4 we have the minimal resolution

$$0 \to F_1 \to F_0 \to \Omega_2(R_{\bar{m}}) \to 0$$

of $\Omega_2(R_{\bar{m}})$. By Lemma 2.3, $\mathrm{Ext}^1\Big(\Omega_2\big(R_{\bar{m}}\big), \dfrac{R_{\bar{m}}}{_{\bar{m}}R_{\bar{m}}}\Big) \neq 0$. Hence the result follows. □

We now give some examples.

2.6. Example. Let $R = k[x,y,z]$ with $y^2 = xz$, $z^2 = x^3$ over a field k of characteristic zero and let $m = (x,y,z)$ be the maximal ideal corresponding to the origin. It is known that R is not a regular ring at $\bar{m}$, that is the origin is a singular point of the variety. It was seen in [1] that

$$0 \to m \to R^2 \to R^6 \to R^8 \to J_2(R) \to 0$$

is an exact sequence of R-modules. Therefore the projective dimension of $J_2(R)$ is not finite. Now we may conjecture that $B(\Omega_2(R_m), t)$ is a rational function. Here $J_2(R) = \Omega_2(R) \oplus R$.

2.7. Example. Let $R = k[x,y,z]$ with $y^2 = x^3$. R is not a regular ring at $m = (x,y)$, the maximal ideal. It is known that the projective dimension of $J_1(R)$ and $J_2(R)$ is one. Hence $B(J_1(R_m), t)$ and $B(J_2(R_m), t)$ are rational functions. Here $J_1(R) = \Omega_1(R) \oplus R$ and $J_2(R) = \Omega_2(R) \oplus R$.

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