International Electronic Journal of Algebra
Volume 9 (2011) 114-123

FINITE GROUPS WHOSE IRREDUCIBLE CHARACTERS VANISH ONLY ON ELEMENTS OF PRIME POWER ORDER

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Received: 26 March 2010; Revised: 14 August 2010
Communicated by Abdullah Harmancı

Abstract. The aim of this note is to investigate the finite groups whose irreducible characters vanish only on elements of prime power order. Interesting, we give a new characterization of A_5, where A_5 is the alternating group of degree 5.

Mathematics Subject Classification (2000): 20C15
Keywords: finite groups, characters, zeros of characters

1. Introduction

It is well known that the set of values $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ has a strong influence on the group structure of G, where $\mathrm{Irr}(G)$ denotes the set of irreducible complex characters of G. The aim of this paper is to provide some evidence that also the zeros of irreducible characters encode non-trivial information of G.

Following [3], we say that an element x of G is a vanishing element if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) = 0$. Denote $\mathrm{Van}(G)$ the set $\{g \in G : \chi(g) = 0$ for some $\chi \in \mathrm{Irr}(G)\}$, $\mathrm{Vo}(G)$ the set $\{o(g) : g \in \mathrm{Van}(G)\}$ consisting of the orders of the elements in $\mathrm{Van}(G)$.

Recently, Malle, Navarro and Olsson [6] proved that every non-linear $\chi \in \mathrm{Irr}(G)$ vanishes on some element of prime power order. Naturally, we consider the following problem: if every element in $\mathrm{Vo}(G)$ is of prime power order, then what can be said about the structure of G?

Following [4], we call groups all of whose elements have prime power order $CP-$groups. Generally, we say that a group G is a $VCP-$group if every element in $\mathrm{Vo}(G)$ is of prime power order. Furthermore, a group G is called a VCP_1-group if every element in $\mathrm{Vo}(G)$ is of prime order.

This work was supported by the NSF of Sichuan University of Science and Engineering (Grant No. 2009XJKRL011), Sichuan Provincial Education Department Foundation of China (08ZA086), China Postdoctoral Science Foundation funded project (No: 20100480582).

It is known that the quotient group of the automorphism group of $\mathrm{L}_2(9)$ modulo the group of inner automorphisms is isomorphic to the elementary abelian group of order 4; in other words, there are 3 subgroups of index 2 in $\mathrm{Aut}(L_2(9))$, and we denote them by U, V, W. One of them, say U, is isomorphic to S_6 and is not a $VCP-$group; another one, V, is isomorphic to $\mathrm{PGL}_2(9)$ and possesses a vanishing element of order 10. Close inspection shows that the remaining subgroup W is in fact a $VCP-$group.

We first study the non-solvable $VCP-$groups, such groups are nearly $CP-$groups. We have the following easy result.

Theorem A. *Let G be a finite non-solvable $VCP-$group, and $Sol(G)$ denote the solvable radical of G. Then the following statements hold:*

(1) *If $Sol(G) = 1$, then G is isomorphic to one of the following groups:*

$$L_2(q) \text{ for } q = 5, 7, 8, 9, 17,\ L_3(4),\ Sz(8),\ Sz(32), \text{ or } W.$$

(2) *Assume that $Sol(G) > 1$. Let $N := O^p(\mathrm{Sol}(G))$ for some prime p such that $Sol(G)/N > 1$. Then $p = 2$ and G/N is a $CP-$group; furthermore, one of the following holds:*

(2.1) *$Sol(G)/N$ is elementary abelian and $G/\mathrm{Sol}(G)$ is isomorphic to $L_2(5)$.*

(2.2) *$Sol(G)/N$ is abelian and $G/\mathrm{Sol}(G)$ is isomorphic to $L_2(8)$.*

(2.3) *$Sol(G)/N$ is nilpotent of class at most 6 and $G/\mathrm{Sol}(G)$ is isomorphic to $Sz(8)$ or $Sz(32)$.*

If the zeros of the irreducible characters are elements of prime order, then we have the following result:

Theorem B. *Let G be a finite non-abelian and solvable group. If every irreducible character of G vanishes only on elements of prime order, then one of the following holds.*

(1) *G is a p-group of exponent p.*

(2) *$G = E \times F$, where E is an elementary abelian $p-$group (possibly $E = 1$) and F is a Frobenius group with complement of order p.*

Applying Theorem A and Theorem B, we easily get the following result, which is a new characterization of A_5, where A_5 is the alternating group of degree 5.

Theorem C. *Suppose G is a finite group. If $Vo(G) = \{2, 3, 5\}$, then $G \cong \mathrm{A}_5$.*

In this paper, G always denotes a finite group. Notation is standard and taken from [5]. In particular, denote $\mathrm{Irr}_1(G)$ the set of non-linear irreducible complex characters of G, $\mathrm{Sol}(G)$ the solvable radical of G.

2. On non-solvable $VCP-$group

The following Proposition comes from [4, Theorems 6 and 8].

Proposition 2.1. *Let G be a non-solvable $CP-$group, and let $O_2(G)$ be its largest normal $2-$subgroup. Then one of following holds:*

(1) If $O_2(G) = 1$, then G is isomorphic to one of the following groups: $L_2(q)$ for $q = 5, 7, 8, 9, 17$, $L_3(4)$, $Sz(8)$, $Sz(32)$ or W.

(2) Suppose that $O_2(G) > 1$. Then G satisfies one of the following statements:

(2.1) $O_2(G)$ is elementary abelian and $G/O_2(G)$ is isomorphic to $L_2(5)$.

(2.2) $O_2(G)$ is abelian and $G/O_2(G)$ is isomorphic to $L_2(8)$.

(2.3) $O_2(G)$ is nilpotent of class at most 6 and $G/O_2(G)$ is isomorphic to $Sz(8)$ or $Sz(32)$.

Let p be a prime number. Recall that a character $\chi \in \mathrm{Irr}(G)$ is said to be of p-defect zero if p does not divide $|G|/\chi(1)$. By a fundamental result of R. Brauer (see [5, Theorem 8.17]), if $\chi \in \mathrm{Irr}(G)$ is of p-defect zero then, for every element $g \in G$ such that p divides $o(g)$, we have $\chi(g) = 0$.

Lemma 2.2. [3, Proposition 2.1] *Let G be a non-abelian simple group and p a prime number. If G is of Lie type, or if $p \geq 5$, then there exists $\chi \in Irr(G)$ of p-defect zero.*

Remark 2.3. Let G be a non-abelian simple group. By Burnside $p^a q^b$-theorem, we conclude that $|G|$ has a prime divisor p such that $p \geq 5$. Then by Lemma 2.2, there exists $\chi \in \mathrm{Irr}(G)$ such that χ is of p-defect zero.

Lemma 2.4. *Let G be a non-abelian simple group. If G is a $VCP-$group, then G is isomorphic to one of the following groups: $L_2(q)$ for $q = 5, 7, 8, 9, 17$, $L_3(4)$, $Sz(8)$, or $Sz(32)$.*

Proof. Let $G \cong \mathrm{A}_n$ for some $n \geq 14$.

For odd n, set

$a = (1, ..., n-9)(n-8, n-7, n-6, n-5)(n-4, n-3, n-2, n-1, n).$

For even n, set

$a = (1, ..., n-8)(n-7, n-6)(n-5, n-4, n-3, n-2, n-1).$

By Lemma 2.2, we may assume that there exists $\chi \in \mathrm{Irr}(G)$ such that χ is of 5-defect zero. Clearly, χ vanishes on a. Hence the hypothesis yields that $n < 14$. Then by [2], G is isomorphic to $\mathrm{L}_2(5)$ or $\mathrm{L}_2(9)$ (note that $\mathrm{L}_2(5) \cong \mathrm{A}_5$ and $\mathrm{L}_2(9) \cong \mathrm{A}_6$).

If G is a sporadic simple groups, then G is not a $VCP-$group, as one can check in [2]. By the classification theorem of the finite simple groups we can now suppose that G is a simple group of Lie type. Let p be a prime divisor of $|G|$. It is known that G has an irreducible character of $p-$defect zero (see Lemma 2.2). Since characters of $p-$defect zero vanish on elements of order divisible by p, it follows that no simple group of Lie type can have a nonidentity non-vanishing element. Hence the hypothesis implies that every element of G has prime power order, then by Proposition 2.1(1), we complete the proof. □

Proposition 2.5. *Let G be a non-solvable group. If G is a $VCP-$group, then G has an unique non-cyclic composition factor.*

Proof. By induction, we may assume that $\mathrm{Sol}(G)$ is trivial. Let N be a (non-solvable) minimal normal subgroup of G. If N is not a non-abelian simple group, then $N = N_1 \times ... \times N_s$ is a direct product of isomorphic simple groups N_i, where $s \geq 2$. Let $\theta_i \in \mathrm{Irr}(N_i)$ be of p-defect zero, where $p \geq 5$ is a prime divisor of N_i (see Remark 2.3), and set

$$\theta = \theta_1 \times \ldots \times \theta_s.$$

Let χ_0 be an irreducible constituent of θ^G, let $x_1 \in N_1$ be of a prime order p and let $x_2 \in N_2$ be of a prime order $q\,(q \neq p)$. Clearly, θ^g is of p-defect zero for any $g \in G$, then we have

$$\theta^g(x_1) = \theta^g(x_1x_2) = 0.$$

This implies that

$$\chi_0(x_1) = \chi_0(x_1x_2) = 0.$$

Then we obtain a contradiction, hence N is a simple group.

Suppose that G/N is non-solvable. Note that $\mathrm{Out}(N)$ is solvable by the the classification of the finite simple groups, it follows that $C_G(N)$ is non-solvable and hence contains a non-solvable minimal normal subgroup M of G as $\mathrm{Sol}(C_G(N)) = 1$. Arguing as the above step, we conclude that M is s simple group.

Set $T = M \times N$. By Remark 2.3, there exist $\psi \in \mathrm{Irr}(M)$ and $\theta \in \mathrm{Irr}(N)$ such that ψ is of q-defect zero and that θ is of p-defect zero, where q, $p \geq 5$ are prime divisors of $|M|$ and $|N|$, respectively. Let $x \in M$, $z \in N$ be of order q, r, respectively, where $r \neq p$ and $r \neq q$. Then for any irreducible constituent χ of $(\psi \times \theta)^G$, we see that

$$\chi(x) = \chi(xz) = 0.$$

The contradiction completes the proof. □

Let G be finite group, $\pi(G)$ be the set of all prime divisors of its order, and $\omega(G)$ be the spectrum of G, that is, the set of all of its element orders. A graph $GK(G) = V(GK(G)), E(GK(G))$, where $V(GK(G))$ is a vertex set and $E(GK(G))$ is an edge set, is called the *Gruenberg* $-$ *Kegel graph* (or *prime graph*) of G if $V(GK(G)) = \pi(G)$ and the edge (r, s) is in $E(GK(G))$ iff $rs \in \omega(G)$. Denote by $\pi_i(G)$, $i = 1, ..., s(G)$, the ith connected component of $GK(G)$. If G has even order then we put $2 \in \pi_1(G)$.

Recall that a vertex set of a graph is called a clique if all vertices in that set are pairwise adjacent. The following result is part of [8, Corollary 7.6].

Lemma 2.6. *Let G be a finite non-abelian simple group, and let all connected components of its prime graph $GK(G)$ be cliques. If G is not of Lie type, then G is one of the groups in the following list:*

(1) sporadic groups M_{11}, M_{22}, J_1, J_2, J_3, and HS.

(2) alternating groups Alt_n, where $n = 5, 6, 7, 9, 12, 13$.

Lemma 2.7. *Let G be a non-solvable $VCP-$group. If every non-trivial quotient group of G is solvable, then G is isomorphic to one of the following: $L_2(q)$ for $q = 5, 7, 8, 9, 17$, $L_3(4)$, $Sz(8)$, $Sz(32)$, or W.*

Proof. Let N be the unique minimal normal subgroup of G. Then by Proposition 2.5, N is a non-abelian simple. In particular, $G \leq \mathrm{Aut}(N)$ and $G/N \leq \mathrm{Out}(N)$.

Assume that $\chi_p \in \mathrm{Irr}(N)$ such that χ_p be of p-defect zero where p is an prime of N, and let ψ be an irreducible constituent χ_p^G. Observe that $\chi_p^g(x) = 0$ for any $g \in G$ and any $x \in N$ of order divisible by p. It follows that $\psi(x) = 0$ whenever $x \in N$ is of order divisible by p.

We, first, suppose that N is not a $VCP-$group. Hence we may assume that $g \in \mathrm{Van}(N)$ such that the number of prime divisors of $o(g)$ is greater than 1. Let p be a prime divisor of $o(g)$. If N is of Lie type, then by lemma 2.2, there exists $\chi_p \in \mathrm{Irr}(N)$ such that χ_p is of $p-$defect zero. So arguing as the above paragraph, we obtain a contradiction. Hence we may assume that N is not of Lie type. If $p \geq 5$, then by Lemma 2.2, there exists $\chi_p \in \mathrm{Irr}(N)$ such that χ_p is of $p-$defect zero. Then arguing as the above paragraph, we also obtain a contradiction. Therefore, we may assume that $\pi_1(N) = \{2, 3\}$ and that the other connected components contain only one prime divisor. Then applying Lemma 2.6, N is isomorphic to M_{11}, or M_{22} (note that $\mathrm{L}_2(5) \cong \mathrm{A}_5$ and $\mathrm{L}_2(9) \cong \mathrm{A}_6$). So G is not a $VCP-$group, as one can check in [2].

We, now, suppose that N is a $VCP-$group; then by Proposition 2.1(1), we conclude that N is isomorphic to one of the following: $\mathrm{L}_2(q)$ for $q = 5, 7, 8, 9, 17$, $\mathrm{L}_3(4)$, $\mathrm{Sz}(8)$, or $\mathrm{Sz}(32)$. Recall that $G \leq \mathrm{Aut}(N)$, then we conclude from [2] that the result is true. $\square$

The proof of the Theorem A.

Proof. We identify the irreducible characters of $G/\mathrm{Sol}(G)$ with the irreducible characters of G that contain $\mathrm{Sol}(G)$ in the kernel. So, if G is a $VCP-$group, then $G/\mathrm{Sol}(G)$ is also a $VCP-$group. By Proposition 2.5, we have that $G/\mathrm{Sol}(G)$ satisfies the hypothesis of Lemma 2.7. Then by lemma 2.7, $G/\mathrm{Sol}(G)$ is isomorphic to one of the following: $\mathrm{L}_2(q)$ for $q = 5, 7, 8, 9, 17$, $\mathrm{L}_3(4)$, $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$, or W. So $G/\mathrm{Sol}(G) - \{1\} = \mathrm{Van}(G/\mathrm{Sol}(G))$, as one can check in [2].

If $\mathrm{Sol}(G) = 1$, then G satisfies (1) of the theorem. Hence we may assume that $\mathrm{Sol}(G) > 1$. Note that $\mathrm{Sol}(G)$ is solvable, so we may choose a prime divisor p of $|\mathrm{Sol}(G)|$ such that $\mathrm{Sol}(G)/O^p(\mathrm{Sol}(G)) > 1$. Set $N := O^p(\mathrm{Sol}(G))$. Note that $G - \mathrm{Sol}(G) \subseteq \mathrm{Van}(G)$, then every element in $G - \mathrm{Sol}(G)$ has prime power order. Since $\mathrm{Sol}(G)/N$ is a p-group, all elements in G/N have prime power order, hence G/N is a non-solvable CP-group.

Let $O_2(G/N)$ be the largest normal 2-subgroup of G/N. If $O_2(G/N) = 1$, then by Proposition 2.1, G/N is isomorphic to one of the following: $\mathrm{L}_2(q)$ for $q = 5, 7, 8, 9, 17$, $\mathrm{L}_3(4)$, $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$, or W. This is impossible since $\mathrm{Sol}(G)/N > 1$, and thus $O_2(G/N) > 1$. Hence $p = 2$ and $\mathrm{Sol}(G)/N = O_2(G/N)$. Then by Proposition 2.1, we complete the proof. $\square$

3. On solvable VCP_1-groups

In this section, to prove Theorem B, we will use the following easy result.

Lemma 3.1. *Let G be a $VCP-$group. Let M be a normal subgroup of G and $\chi \in$ Irr(G/M). If χ vanishes on $x \in G$ and gcd$(|M|, o(x)) = 1$, then $C_M(x) = 1$.*

Proof. By the hypothesis, for every $y \in C_M(x)$ we have $\chi(xy) = 0$, and thus xy is an element of prime power. Since $o(xy) = o(x)o(y)$ and $(|M|, o(x)) = 1$, we have $C_M(x) = 1$, and we are done. $\square$

The group $G = [N]H$ means a semidirect product of a normal subgroup N and a complement subgroup H.

Proposition 3.2. [7, Theorem 1] *Let G be a group, and Let N be a proper normal subgroup of G. If every element in $G - N$ has prime order, then G is one of the following groups:*

(1) $G \cong \mathrm{A}_5$ and $N = 1$.

(2) G is a Frobenius group with a complement A of prime order and the kernel F of prime power order, where $N < F$ (N is a proper normal subgroup of F).

(3) G is a p-group.

(4) $G = [O_{p'}(G) \times O_p(G)]A$, where A is of prime order p, $N = O_{p'}(G) \times O_p(G)$, $[O_p(G)]A \in Syl_p(G)$, $O_{p'}(G) > 1$ and A acts fixed point freely on $O_{p'}(G)$.

Now we are ready to prove Theorem B.

Proof. For a non-linear irreducible complex character χ of G, write $\upsilon(\chi) = \{g \in G \mid \chi(g) = 0\}$. Let $\psi \in \mathrm{Irr}_1(G/G'')$ (since G is non-abelian and solvable, such ψ exists). Then $\psi_{G'/G''}$ is not irreducible, and so $\psi_{G'}$ is not irreducible (note that we identify the character χ of G/G'' with a suitable character of G). It follows by [5, Theorem 6.22] that G has a proper subgroup N such that $G' \leq N < G$ and $G - N \subseteq \upsilon(\psi)$. The hypothesis yields that all elements outside N are of prime order. Then by Proposition 3.2, we have that G satisfies (2), (3), or (4) of Proposition 3.2.

If G is the group in Proposition 3.2(2) then G satisfies (2) of Theorem B. If G is the group in Proposition 3.2(3), then by [1, Corollary 2.10], we see that G satisfies (1) of Theorem B.

Suppose that G has the structure described in Proposition 3.2(4). We set $E := O_p(G) < G$. Let $P := [O_p(G)]A$, and let $K := O_{p'}(G)$.

We, now, prove that $Z(G) = O_p(G)$. Note that $G/O_p(G)$ is a Frobenius group with complement $P/O_p(G)$; thus $Z(G/O_p(G)) = 1$, which implies $Z(G) \leq O_p(G)$.

We next claim:

$$C_K(x) = 1 \text{ for every } x \in P - Z(P). \tag{+}$$

Namely, if $x \in P - Z(P)$ (if any), then by [1, Lemma 2.9] there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) = 0$, since $G/K \cong P$. Thus, by Lemma 3.1, we see that $C_K(x) = 1$.

Since $O_p(G)$ centralizes $O_{p'}(G)$, from (+) it follows $O_p(G) \leq Z(P)$ and hence $O_p(G) \leq Z(G)$. Therefore, $Z(G) = O_p(G)$.

As $O_p(G) \leq Z(P)$ and A is a group of order p, we have that P is abelian. Let $\chi \in \mathrm{Irr}_1(G/Z(G))$, and let x be a nonidentity element of A. Clearly χ vanishes on x. For every $y \in Z(G)$, we have $\chi(xy) = 0$ and thus

$$x^p = 1 = (xy)^p = x^p y^p = y^p.$$

Then the group P is an elementary abelian p-group. We hence obtain $G = E \times F$, where F is a Frobenius group with complement of order p. Hence the proof is completed. $\square$

4. A new characterization of $\mathbf{A}_5$

We start by stating a consequence of Theorem A.

Corollary 4.1. *Suppose that G is a finite non-solvable group. If $Vo(G) = \{p, q, r\}$, where p, q and r are prime, then $G \cong \mathrm{A}_5$*

Proof. Clearly, G is a non-solvable $VCP-$group. Then applying Theorem A, it follows from [2] that $G/\mathrm{Sol}(G)$ is isomorphic to $\mathrm{L}_2(5)$. Obviously, $G - \mathrm{Sol}(G) \subseteq \mathrm{Van}(G)$. Hence $G-\mathrm{Sol}(G)$ consists of elements of prime order. Then by [7, Theorem 1], we obtain that $G \cong \mathrm{A}_5$. $\square$

Following [1], We will say that a group G belongs to the class υ_k, for a positive integer k, if every element in $\mathrm{Vo}(G)$ divides k. So, an abelian group belongs to υ_k for all k. The following result has already appeared in [1], here we give an easy and neat proof.

Lemma 4.2. *Let G be a Frobenius group and p a prime, $p \leq 5$. If $G \in \upsilon_p$, then the Frobenius kernel of G is abelian.*

Proof. Let G be a minimal counter example. Let C be a Frobenius complement of G and Q the kernel. Then by Theorem B, we have $|C| = p$. Let x be a generator of C. Since Q is nilpotent and the class υ_p is closed by images, the group Q is a q-group for some prime q, also Q' is minimal normal in G and $Q' \leq Z(Q)$. Observe that if $|C| \leq 3$ then we easily conclude that the result is true. Hence we may assume that $|C| = 5$.

Let $\psi \in \mathrm{Irr}(Q)$ be of maximal degree. Recall that $Q' \leq Z(Q)$; then by [5, Corollary 2.30 and Theorem 2.31], there exist a subgroup Z of Q such that ψ vanishes on $Q - Z$ and that $Z \geq Q'$, $|Q/Z| = \psi(1)^2 = q^{2m}$.

Suppose that $q > 2$ or $m > 1$. Then there exists an irreducible character χ of G such that χ vanishes on $Q - \Delta$, where $\Delta := Z \cup Z^x \cup Z^{x^2} \cup Z^{x^3} \cup Z^{x^4}$. Therefore, the hypothesis yields that $Q = \Delta$. Thus $|Q| \leq 5|Z|$ and hence $|Q/Z| = q^{2m} \leq 5$, a contradiction. Hence we may assume that $q = 2$ and $m = 1$.

As ψ is of maximal degree, $\mathrm{cd}(Q) = \{1, 2\}$. It follows by [5, Theorem 12.11] that either $|Q : Z(Q)| = 8$ or Q has an abelian subgroup of index 2. Recall that $G/Z(Q)$ is a Frobenius group with the kernel $Q/Z(Q)$ and complement isomorphic to C, so $Q/Z(Q)$ is a C-module. We note that C has only two irreducible F_2-modules, namely the trivial module and a module of dimension 4. Hence it is impossible to $|Q : Z(Q)| = 8$.

Assume now that Q has an abelian subgroup E of index 2. Let ψ be any non-linear irreducible character of Q. Since E is abelian and $|Q : E| = 2$, it follows by [5, Theorem 6.2 and Corollary 6.19] that $\psi_E = \mu+\nu$, where μ and ν are irreducible and are conjugate to each other in Q. Hence $\psi = \mu^Q$, and so ψ vanishes on $Q - E$. Take $\chi \in \mathrm{Irr}(G)$ of degree 10. We see that χ vanishes on $Q - E$, hence we obtain a contradiction. □

The following result shows that characters of degree not divisible by some prime number p never vanish on $p-$elements.

Lemma 4.3. [1, Corollary 2.2] *If $\chi \in Irr(G)$ vanishes on a $p-$element, p prime, then p divides $\chi(1)$.*

We are now ready to prove Theorem C, which we state again:

Theorem C. *Suppose G is a finite group. If $Vo(G) = \{2, 3, 5\}$, then $G \cong \mathrm{A}_5$.*

Proof. Applying Corollary 4.1, we need prove that G is a non-solvable group. Assume that G is solvable. It follows from the hypothesis that G satisfies (2) of Theorem B. If $E > 1$, then we easily conclude from the hypothesis that G belongs to the class υ_p, a contradiction. So we may assume that $E = 1$ and thus G is a Frobenius group with kernel K and complement of order p. Let q be a prime divisor of $|K|$. Take $N = O_{q'}(K)$. Now consider the group G/N. Clearly, G/N is a Frobenius group with kernel K/N of prime power order and complement of order p. Suppose that G/N does not belong to the class υ_p. Then there exists $x \in K - N$ such that $xN \in \mathrm{Van}(G/N)$. Let $\chi \in \mathrm{Irr}_1(G/N)$ with $\chi(xN) = 0$. Let $x = x_q x_{q'}$, where x_q and $x_{q'}$ are the q-part and the q'-part of x, respectively. Since $\chi(xN) = \chi(x_q x_{q'} N) = \chi(x_q N) = 0$, we may suppose that such element x is a q-element. Assume that $N > 1$. Take $y \in N - 1$, we have

$$\chi(xy) = \chi(xyN) = \chi(xN) = \chi(x) = 0.$$

Since K is nilpotent and $\gcd(o(x), o(y)) = 1$, we have that $xy = yx$. Hence $o(xy)$ is not a prime number, and so we obtain a contradiction. Thus $N = 1$, and so $|G|$ have only two prime divisors, we also obtain a contradiction. Hence G/N belongs to the class υ_p. Recall that $p = 2$, 3 or 5; thus by Lemma 4.2, K/N is abelian. Since q is an arbitrary prime divisor of $|K|$, we get K is abelian. By Lemma 4.3, we easily see that G belongs to the class υ_p, a contradiction. The proof is complete. □

Acknowledgment. The authors would like to thank the referee for the valuable suggestions and comments. In addition, the authors would like to thanks to Professor Wujie Shi for his help.

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