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# ON 3-DIMENSIONAL $\alpha$ -PARA KENMOTSU MANIFOLDS

#### KRISHANU MANDAL AND U.C. DE

ABSTRACT. The aim of the present paper is to study 3-dimensional  $\alpha$ -para Kenmotsu manifolds. First we consider 3-dimensional Ricci semisymmetric  $\alpha$ -para Kenmotsu manifolds and obtain some equivalent conditions. Next we study cyclic parallel Ricci tensor in 3-dimensional  $\alpha$ -para Kenmotsu manifolds. Moreover, we investigate  $\eta$ -parallel Ricci tensor in 3-dimensional  $\alpha$ -para Kenmotsu manifolds. Continuing our study, we consider locally  $\phi$ -symmetric 3-dimensional  $\alpha$ -para Kenmotsu manifolds. Next, we study gradient Ricci solitons in 3-dimensional  $\alpha$ -para Kenmotsu manifolds. Finally, we give an example of a 3-dimensional  $\alpha$ -para Kenmotsu manifold which verifies some results.

## 1. INTRODUCTION

In 1972, Chen and Yano [8] introduced the notion of quasi-constant curvature as a conformally flat manifold with the curvature tensor 'R of type (0,4) satisfies the condition

where R(X, Y, Z, W) = g(R(X, Y)Z, W), R is the curvature tensor of type (1,3), p, q are scalar functions and T is a non zero 1-form defined by

(1.2) 
$$g(X,\rho) = T(X).$$

It can be easily seen that if the curvature tensor 'R is of the form (1.1) then the manifold is of conformally flat. Vranceanu [28] defined the notion of almost constant curvature by the same expression (1.1). Later Mocanu [22] proved that the manifold introduced by Chen and Yano and the manifold introduced by Vranceanu are the same. Hence a Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor 'R satisfies the relation (1.1). In particular q = 0, the

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manifold is said to be a manifold of constant curvature. Noticing, a manifold is of quasi-constant curvature is a generalization of a manifold of constant curvature. A manifold of quasi-constant curvature have been studied by several authors such as Adati and Wang [3], Adati [1], Wang [29], De and Ghosh [11] and many others.

A Riemannian manifold M is called locally symmetric if its curvature tensor R is parallel, that is,  $\nabla R = 0$ , where  $\nabla$  is the Levi-Civita connection. The notion of semisymmetric manifold, a proper generalization of locally symmetric manifold, is defined by  $R(X, Y) \cdot R = 0$ , where R(X, Y) acts on R as a derivation. A complete intrinsic classification of these manifolds was given by Szabó in [26]. The classification results of Szabó were presented in the book [6]. Also in [19], Kowalski classified 3-dimensional Riemannian spaces satisfying  $R(X, Y) \cdot R = 0$ .

A Riemannian manifold is said to be Ricci semisymmetric if  $R(X, Y) \cdot S = 0$  where S denotes the Ricci tensor of type (0, 2). A general classification of these manifolds has been worked out recently by Mirzoyan [21].

A Ricci soliton [7] is a generalization of an Einstein metric. In a manifold M a Ricci soliton is a triplet  $(g, V, \lambda)$ , with g, a pseudo Riemannian metric, V a vector field (called the potential vector field) and  $\lambda$  a real scalar such that

(1.3) 
$$\pounds_V g + 2S + 2\lambda g = 0,$$

where  $\pounds_V$  is the Lie derivative with respect to V and S is the Ricci tensor of type (0, 2). The Ricci soliton is said to be shrinking, steady or expanding according as  $\lambda$  is negative, zero or positive, respectively. The compact Ricci solitons are the fixed points of the Ricci flow

$$\frac{\partial}{\partial t}g = -2S$$

projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings. If the complete vector field V is the gradient of a potential function -f, then g is said to be a gradient Ricci soliton and equation (1.3) takes the form

(1.4) 
$$\nabla \nabla f = S + \lambda g,$$

where  $\nabla$  denotes the Riemannian connection.

In a compact manifold a Ricci soliton has constant curvature in dimension 2 (Hamilton [15]) and also in dimension 3 (Ivey [16]). On the other hand a Ricci soliton on a compact manifold is a gradient Ricci soliton [24]. Recently Yildiz et al [32] have studied Ricci solitons in 3-dimensional f-Kenmotsu manifolds. For details, we refer to Chow and Knopf [9], Bejan and Crasmareanu [4] and Derdzinski [12]. Motivated by the above studies in the present paper we investigate certain curvature conditions on 3-dimensional  $\alpha$ -para Kenmotsu manifolds.

The paper is organized as follows: In section 2, we give a brief account on almost paracontact metric manifolds, normal almost paracontact metric manifolds and curvature properties of  $\alpha$ -para Kenmotsu manifolds. Next in section 3 we consider 3-dimensional Ricci semisymmetric  $\alpha$ -para Kenmotsu manifolds and obtain some equivalent conditions. Section 4 deals with cyclic parallel Ricci tensor in 3-dimensional  $\alpha$ -para Kenmotsu manifolds and prove that a 3-dimensional  $\alpha$ para Kenmotsu manifold M has cyclic parallel Ricci tensor if and only if the scalar curvature  $r = -6\alpha^2$ . Section 5 is devoted to study  $\eta$ -parallel Ricci tensor in 3dimensional  $\alpha$ -para Kenmotsu manifolds and prove that a 3-dimensional  $\alpha$ -para Kenmotsu manifold M has  $\eta$ -parallel Ricci tensor if and only if the scalar curvature  $r = -2\alpha^2$ . We discuss locally  $\phi$ -symmetric 3-dimensional  $\alpha$ -para Kenmotsu manifolds in section 6. Next, we study gradient Ricci solitons in 3-dimensional  $\alpha$ -para Kenmotsu manifolds. Finally, we give an example of a 3-dimensional  $\alpha$ -para Kenmotsu manifold which verifies some results.

# 2. Preliminaries

2.1. Almost paracontact metric manifolds. A differential manifold M of dimension 2n + 1 is called an almost paracontact manifold ([20],[23]) equipped with the structure  $(\phi, \xi, \eta)$  where  $\phi$  is a tensor field of type (1, 1), a vector field  $\xi$  and a 1-form  $\eta$  satisfying

(2.1) 
$$\phi^2 = I - \eta \otimes \xi \text{ and } \eta(\xi) = 1.$$

From equation (2.1) it can easily deduced that

(2.2) 
$$\phi \xi = 0, \ \eta(\phi X) = 0 \text{ and } \operatorname{rank}(\phi) = 2n.$$

If an almost paracontact manifold admits a pseudo-Riemannian metric g satisfying

(2.3) 
$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

where signature of g is (n + 1, n) for any vector fields  $X, Y \in \chi(M)$ , (where  $\chi(M)$  is the set of all differential vector fields on M) then the manifold is called almost paracontact metric manifold.

An almost paracontact structure is said to be a paracontact structure if  $g(X, \phi Y) = d\eta(X, Y)$  with the associated metric g [33]. For an almost paracontact metric manifold, there always exists a special kind of local pseudo orthonormal  $\phi$  basis  $\{X_i, \phi X_i, \xi\}, X_i$ 's and  $\xi$  are space-like vector fields and  $\phi X_i$ 's are time-like. Thus, an almost paracontact metric manifold is an odd dimensional manifold.

2.2. Normal almost paracontact metric manifolds. An almost paracontact metric manifold is said to be normal if the induced almost paracomplex structure J on the product manifold  $M^{2n+1} \times \mathbb{R}$  defined by

(2.4) 
$$J(X, f\frac{d}{dt}) = (\phi X + f\xi, \eta(X)\frac{d}{dt})$$

is integrable where X is tangent to M, t is the coordinate of  $\mathbb{R}$  and f is a smooth function on  $M^{2n+1} \times \mathbb{R}$ . The condition for being normal is equivalent to vanishing of the (1, 2)-type torsion tensor  $N_{\phi}$  defined by  $N_{\phi} = [\phi, \phi] - 2d\eta \otimes \xi$ , where  $[\phi, \phi]$ is the Nijenhuis torsion of  $\phi$ .

**Proposition 2.1.** [30] For a 3-dimensional almost paracontact metric manifold M, the following conditions are mutually equivalent

- (a) M is normal,
- (b) there exist differential functions  $\alpha, \beta$  on M such that

$$(\nabla_X \phi)Y = \beta \{g(X, Y)\xi - \eta(Y)X\} + \alpha \{g(\phi X, Y)\xi - \eta(Y)\phi X\},\$$

(c) there exist differential functions  $\alpha, \beta$  on M such that

$$\nabla_X \xi = \alpha \{ X - \eta(X) \xi \} + \beta \phi X,$$

where  $\nabla$  is the Levi-Civita connection of the pseudo-Riemannian metric gand  $\alpha, \beta$  are given by

$$2\alpha = Trace\{X \to \nabla_X \xi\}, 2\beta = Trace\{X \to \phi \nabla_X \xi\}.$$

**Definition 2.1.** A 3-dimensional normal almost paracontact metric manifold M is said to be

- (1) paracosymplectic if  $\alpha = \beta = 0$  [10],
- (2)  $\alpha$ -para Kenmotsu if  $\alpha$  is a non-zero constant and  $\beta = 0$  [31], in particular para Kenmotsu if  $\alpha = 1$  [5],
- (3) quasi-para Sasakian if and only if  $\alpha = 0$  and  $\beta \neq 0$  [13],
- (4)  $\beta$ -para Sasakian if and only if  $\alpha = 0$  and  $\beta$  is a non-zero constant, in particular para Sasakian if  $\beta = -1$  [33].

2.3. Curvature properties of 3-dimensional  $\alpha$ -para Kenmotsu manifolds. In a 3-dimensional  $\alpha$ -para Kenmotsu manifold, it follows that [25]

(2.5)  

$$R(X,Y)Z = (\frac{r}{2} + 2\alpha^{2})[g(Y,Z)X - g(X,Z)Y] - (\frac{r}{2} + 3\alpha^{2})[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi + (\frac{r}{2} + 3\alpha^{2})[\eta(X)Y - \eta(Y)X]\eta(Z),$$

where r is the scalar curvature of the manifold and g, pseudo-metric. Clearly, (2.5) is of the form (1.1) and hence the manifold is of quasi-constant curvature. In addition, if  $r = -6\alpha^2$  then the manifold is of constant curvature. Contracting the equation (2.5) with respect to Z, we have

(2.6) 
$$S(X,Y) = (\frac{r}{2} + \alpha^2)g(X,Y) - (\frac{r}{2} + 3\alpha^2)\eta(X)\eta(Y).$$

**Proposition 2.2.** A 3-dimensional  $\alpha$ -para Kenmotsu manifold is of quasi-constant curvature.

In a 3-dimensional  $\alpha$ -para Kenmotsu manifold the following relations hold [25]:

(2.7) 
$$g(\phi X, Y) = -g(X, \phi Y),$$

(2.8) 
$$g(X,\xi) = \eta(X),$$

(2.9) 
$$S(X,\xi) = -2\alpha^2 \eta(X),$$

(2.10) 
$$R(X,Y)\xi = -\alpha^2 \{\eta(Y)X - \eta(X)Y\},\$$

(2.11) 
$$(\nabla_X \eta) Y = \alpha \{ g(X, Y) - \eta(X) \eta(Y) \},$$

(2.12) 
$$(\nabla_X \phi) Y = \alpha \{ g(\phi X, Y) \xi - \eta(Y) \phi X \},$$

(2.13) 
$$\nabla_X \xi = \alpha \{ X - \eta(X) \xi \},$$

for all vector fields X, Y, Z and  $W \in \chi(M)$ .

## 3. Ricci semisymmetric $\alpha$ -para Kenmotsu manifolds

In this section we consider a 3-dimensional Ricci semisymmetric  $\alpha$ -para Kenmotsu manifold. A Riemannian or pseudo-Riemannian manifold is said to be Ricci semisymmetric [2] if

$$(3.1) R(X,Y) \cdot S = 0,$$

where R(X, Y) denotes the derivation in the tensor algebra at each point of the manifold. It follows from equation (3.1)

(3.2) 
$$(R(X,Y) \cdot S)(U,V) = 0.$$

This implies

(3.3) 
$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0$$

Using (2.5) and (2.9) we get

$$S(R(X,Y)U,V) = (\frac{r}{2} + 2\alpha^2)[g(Y,U)S(X,V) - g(X,U)S(Y,V)] + 2\alpha^2(\frac{r}{2} + 3\alpha^2)[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\eta(V) + (\frac{r}{2} + 3\alpha^2)[S(Y,V)\eta(X) - S(X,V)\eta(Y)]\eta(U).$$
(3.4)

Interchanging U and V in (3.4) yields

$$S(R(X,Y)V,U) = \left(\frac{r}{2} + 2\alpha^{2}\right)[g(Y,V)S(X,U) - g(X,V)S(Y,U)] + 2\alpha^{2}\left(\frac{r}{2} + 3\alpha^{2}\right)[g(Y,V)\eta(X) - g(X,V)\eta(Y)]\eta(U) + \left(\frac{r}{2} + 3\alpha^{2}\right)[S(Y,U)\eta(X) - S(X,U)\eta(Y)]\eta(V).$$
(3.5)

With the help of the equations (3.4) and (3.5), we obtain from (3.3)

$$(\frac{r}{2} + 2\alpha^{2})\{g(Y,U)S(X,V) - g(X,U)S(Y,V) + g(Y,V)S(X,U) - g(X,V)S(Y,U)\} + 2\alpha^{2}(\frac{r}{2} + 3\alpha^{2})\{g(Y,U)\eta(X)\eta(V) - g(X,U)\eta(Y)\eta(V) + g(Y,V)\eta(X)\eta(U) - g(X,V)\eta(Y)\eta(U)\} + (\frac{r}{2} + 3\alpha^{2})\{S(Y,U)\eta(X)\eta(V) - S(X,U)\eta(Y)\eta(V) + S(Y,V)\eta(X)\eta(U) - S(X,V)\eta(Y)\eta(U)\} = 0.$$
(3.6)

Using (2.6) it follows from (3.6)

(3.7) 
$$(\frac{r\alpha^2}{2} + 3\alpha^4) \{ g(Y,U)\eta(X)\eta(V) - g(X,U)\eta(Y)\eta(V) + g(Y,V)\eta(X)\eta(U) - g(X,V)\eta(Y)\eta(U) \} = 0.$$

Putting  $X = U = \xi$  in (3.7) we have

(3.8) 
$$(\frac{r\alpha^2}{2} + 3\alpha^4) \{g(Y, V) - \eta(Y)\eta(V)\} = 0$$

By making contraction (3.8) and using the fact  $\alpha \neq 0$  yields (3.

$$(9) r = -6\alpha^2.$$

Hence the manifold M is of constant curvature. Consequently the manifold M is an Einstein manifold. Indeed we have

(3.10) 
$$S(X,Y) = -2\alpha^2 g(X,Y).$$

Conversely, if the manifold is an Einstein manifold then

$$R(X,Y) \cdot S = 0.$$

By the above discussions we have the following:

**Theorem 3.1.** In a 3-dimensional  $\alpha$ -para Kenmotsu manifold M, the following conditions are equivalent:

- (a) M is Ricci semisymmetric, that is,  $R(X,Y) \cdot S = 0$  for any X,Y,
- (b) the scalar curvature  $r = -6\alpha^2$ ,
- (c) the manifold M is of constant curvature,
- (d) M is an Einstein manifold.

4.  $\alpha$ -para Kenmotsu manifolds with cyclic parallel Ricci tensor

This section deals with  $\alpha$ -para Kenmotsu manifolds with cyclic parallel Ricci tensor. We prove the following:

**Theorem 4.1.** A 3-dimensional  $\alpha$ -para Kenmotsu manifold M satisfies cyclic parallel Ricci tensor if and only if the scalar curvature  $r = -6\alpha^2$ .

*Proof.* Suppose the manifold under consideration satisfies cyclic parallel Ricci tensor [14], then the Ricci tensor S satisfies

(4.1) 
$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0,$$

for all  $X, Y, Z \in \chi(M)$ . It is known [17] that Cartan hypersurfaces are manifolds with non-parallel Ricci tensor, satisfying (4.1). From (4.1), it follows that r =constant. Then

(4.2)  

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y)$$

$$= -(\frac{r}{2} + 3\alpha^2)[(\nabla_X \eta)Y\eta(Z) + (\nabla_X \eta)Z\eta(Y) + (\nabla_Y \eta)X\eta(Z)$$

$$+ (\nabla_Y \eta)Z\eta(X) + (\nabla_Z \eta)X\eta(Y) + (\nabla_Z \eta)Y\eta(X)].$$

From (4.1) and (4.2) we get

(4.3) 
$$(\frac{r}{2} + 3\alpha^2)[(\nabla_X \eta)Y\eta(Z) + (\nabla_X \eta)Z\eta(Y) + (\nabla_Y \eta)X\eta(Z) + (\nabla_Y \eta)Z\eta(X) + (\nabla_Z \eta)X\eta(Y) + (\nabla_Z \eta)Y\eta(X)] = 0.$$

By (2.11) it follows from (4.3)

$$(\frac{r}{2} + 3\alpha^2)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + g(Y,X)\eta(Z) + g(Y,Z)\eta(X)]$$

(4.4) 
$$+g(Z,X)\eta(Y) + g(Z,Y)\eta(X) - 6\eta(X)\eta(Y)\eta(Z)] = 0.$$

Let  $\{e_i\}(1 \le i \le 3)$  be an orthonormal basis of the tangent space at any point of the manifold. Then putting  $X = Y = e_i$  in (4.4) and taking summation over *i*, we get

(4.5) 
$$(\frac{r}{2} + 3\alpha^2)\eta(Z) = 0.$$

Since  $\eta(Z) \neq 0$ , thus we have from (4.5)

$$(4.6) r = -6\alpha^2.$$

On the other hand if  $r = -6\alpha^2$ , then from the equation (4.2) it follows that (4.1) holds. Thus our theorem is proved.

Again if  $r = -6\alpha^2$ , then the manifold M is of constant curvature and consequently Einstein. Hence from Theorem 3.1 we arrive to the following assertion:

**Corollary 4.1.** If a 3-dimensional  $\alpha$ -para Kenmotsu manifold M satisfies cyclic parallel Ricci tensor, then the manifold is Ricci semisymmetric.

# 5. $\alpha$ -para Kenmotsu manifolds with $\eta$ -parallel Ricci tensor

This section is devoted to study  $\alpha\mbox{-}\mathrm{para}$  Kenmotsu manifolds with  $\eta\mbox{-}\mathrm{parallel}$  Ricci tensor.

**Definition 5.1.** The Ricci tensor S of an  $\alpha$ -para Kenmotsu manifold M is called  $\eta$ -parallel if it satisfies the condition

(5.1) 
$$(\nabla_X S)(\phi Y, \phi Z) = 0, \text{ for all } X, Y, Z \in \chi(M).$$

The notion of  $\eta$ -parallel Ricci tensor for Sasakian manifolds was introduced by Kon [18].

We have from (2.6) and (2.3)

(5.2) 
$$S(\phi Y, \phi Z) = -(\frac{r}{2} + \alpha^2) \{ g(Y, Z) - \eta(Y) \eta(Z) \}.$$

Now differentiating (5.2) covariently along the vector field X we obtain

(5.3) 
$$(\nabla_X S)(\phi Y, \phi Z) = -\frac{dr(X)}{2} \{g(Y, Z) - \eta(Y)\eta(Z)\} + (\frac{r}{2} + \alpha^2)[(\nabla_X \eta)Y\eta(Z) + (\nabla_X \eta)Z\eta(Y)].$$

Using (2.11), the above equation implies

(5.4) 
$$(\nabla_X S)(\phi Y, \phi Z) = -\frac{dr(X)}{2} \{g(Y, Z) - \eta(Y)\eta(Z)\}$$
  
 
$$+\alpha(\frac{r}{2} + \alpha^2)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y)$$
  
 
$$-2\eta(X)\eta(Y)\eta(Z)].$$

Suppose the manifold under consideration satisfies the condition (5.1). Therefore

(5.5)  

$$-\frac{dr(X)}{2}\{g(Y,Z) - \eta(Y)\eta(Z)\} + \alpha(\frac{r}{2} + \alpha^{2})[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0.$$

Let  $\{e_i\}(1 \le i \le 3)$  be an orthonormal basis of the tangent space at any point of the manifold. Then putting  $Y = Z = e_i$  in (5.5) and taking summation over *i*, we get

dr(X) = 0.

Consequently

(5.7) 
$$r = \text{constant.}$$

From (5.5) and (5.6) yields

(5.8) 
$$\alpha(\frac{1}{2} + \alpha^2)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0.$$

By making suitable contraction of (5.8), we obtain

$$(5.9) r = -2\alpha^2.$$

Conversely, if we take  $r = -2\alpha^2$ , then the equation (5.1) holds. This leads to the following:

**Theorem 5.1.** A 3-dimensional  $\alpha$ -para Kenmotsu manifold M has  $\eta$ -parallel Ricci tensor if and only if the scalar curvature  $r = -2\alpha^2$ .

## 6. Locally $\phi$ -symmetric $\alpha$ -para Kenmotsu manifolds

In this section we discuss locally  $\phi$ -symmetric  $\alpha$ -para Kenmotsu manifolds.

**Definition 6.1.** A 3-dimensional  $\alpha$ -para Kenmotsu manifold is said to be locally  $\phi$ -symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z = 0,$$

for all vector fields X, Y, Z, W orthogonal to  $\xi$ .

This notation was introduced for Sasakian manifolds by Takahashi [27]. Differentiating (2.5) covariently along the vector field W we obtain

$$(\nabla_W R)(X,Y)Z = \frac{dr(W)}{2} \{g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi \\ +g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\ -(\frac{r}{2} + 3\alpha^2)\{g(Y,Z)\eta(X)\nabla_W\xi - g(X,Z)\eta(Y)\nabla_W\xi \\ +g(Y,Z)(\nabla_W\eta)(X)\xi - g(X,Z)(\nabla_W\eta)(Y)\xi \\ -(\nabla_W\eta)X\eta(Z)Y - (\nabla_W\eta)Z\eta(X)Y \\ +(\nabla_W\eta)Y\eta(Z)X + (\nabla_W\eta)Z\eta(Y)X\}.$$
(6.1)

Let us suppose that the manifold M under consideration is locally  $\phi$ -symmetric. With the help of (2.11), (2.2) and (6.1) we obtain

(6.2) 
$$\phi^2(\nabla_W R)(X,Y)Z = \frac{dr(W)}{2} \{g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y\}.$$

Using (2.1) it follows that

(6.3) 
$$\frac{dr(W)}{2} \{g(Y,Z)X - g(X,Z)Y\} = 0.$$

Taking inner product of (6.3) with  $\xi$ , we have

(6.4) 
$$\frac{dr(W)}{2} \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} = 0.$$

Contracting Y, Z in (6.4) yields

$$\frac{dr(W)}{2}=0,$$

that is,

$$r = \text{constant}.$$

Conversely, if r = constant, then from (6.2) it follows that the manifold is locally  $\phi$ -symmetric. Hence we can state the following:

**Theorem 6.1.** A 3-dimensional  $\alpha$ -para Kenmotsu manifold is locally  $\phi$ -symmetric if and only if the scalar curvature of the manifold is constant.

#### 7. Gradient Ricci solitons

In [4], Bejan and Crasmareanu have been studied Ricci solitons in 3-dimensional normal paracontact geometry and obtain the necessary conditions for their existence, moreover in the case of  $\alpha$ -para Kenmotsu manifolds the Ricci tensor S satisfies the condition  $S = -(2\alpha^2 + \alpha)g + \alpha\eta \otimes \eta$ . Let us suppose that a 3-dimensional  $\alpha$ -para Kenmotsu manifold admits a Ricci soliton defined by (1.3). It is well known that  $\nabla g = 0$ . Since  $\lambda$  is constant, thus  $\nabla \lambda g = 0$ . Therefore  $\pounds_V g + 2S$  is parallel. Here we recall a lemma proved by Srivastava and Srivastava in [25]:

**Lemma 7.1.** A parallel symmetric (0,2) tensor field in a 3-dimensional non-paracosymplectic  $\alpha$ -para Kenmotsu manifold is a constant multiple of the associated metric tensor.

By Lemma 7.1 we can say that  $\pounds_V g + 2S$  is a constant multiple of pseudo-metric tensor g. Thus  $\pounds_V g + 2S = kg$ , where k is constant. Hence  $\pounds_V g + 2S + 2\lambda g$  reduces to  $(k + 2\lambda)g$ . By (1.3) we have  $\lambda = -\frac{k}{2}$ . This leads to the following:

**Proposition 7.1.** In a 3-dimensional non-para-cosymplectic  $\alpha$ -para Kenmotsu manifold, the Ricci soliton  $(g, V, \lambda)$  is shrinking or expanding according as k is positive or negative.

Let M be a 3-dimensional non-para-cosymplectic  $\alpha$ -para Kenmotsu manifold with constant structure function  $\alpha$  and g a gradient Ricci soliton. Then the equation (1.4) can be put in of the form

(7.1) 
$$\nabla_Y Df = QY + \lambda Y$$

for all vector fields  $Y \in \chi(M)$ , where D denotes the gradient operator of pseudometric g.

From (7.1) it follows that

(7.2) 
$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X.$$

Using (2.6) in (7.2) we have

(7.3) 
$$R(X,Y)Df = \frac{dr(X)}{2}[Y - \eta(Y)\xi] - \frac{dr(Y)}{2}[X - \eta(X)\xi] - \alpha(\frac{r}{2} + 3\alpha^2)[\eta(Y)X - \eta(X)Y].$$

Substituting  $X = \xi$  in (7.3) yields

(7.4) 
$$R(\xi, Y)Df = \{\frac{dr(\xi)}{2} + \alpha(\frac{r}{2} + 3\alpha^2)\}\{Y - \eta(Y)\xi\}.$$

By (2.5) it follows that

(7.5)  $g(R(\xi, Y)Z, \xi) = -\alpha^2 \{ g(Y, Z) - \eta(Y)\eta(Z) \}.$ 

With the help of (7.4) and (7.5) we obtain

(7.6)  $g((\nabla_{\xi}Q)Y - (\nabla_{Y}Q)\xi,\xi) = 0.$ 

This implies

$$g(Y, Df) = \xi f \eta(Y),$$

that is,

 $Df = (\xi f)\xi.$ (7.7)Again using (2.13) and (7.7) we obtain (7.8) $g(\nabla_Y Df, X) = Y(\xi f)\eta(X) + \alpha(\xi f)\{g(X, Y) - \eta(X)\eta(Y)\}.$ From (7.1) and (7.8) we get (7.9) $S(X,Y) + \lambda g(X,Y) = Y(\xi f)\eta(X) + \alpha(\xi f)\{g(X,Y) - \eta(X)\eta(Y)\}.$ Putting  $X = \xi$  and using (2.6) we have  $Y(\xi f) = (\lambda - 2\alpha^2)\eta(Y).$ (7.10)Therefore we have from (7.10) and (7.8) $\nabla_Y Df = -2\alpha^2 \eta(Y)\xi + \lambda \eta(Y)\xi + \alpha(\xi f)Y - \alpha(\xi f)\eta(Y)\xi.$ (7.11)On the other hand (7.3) implies (7.12) $g(R(X,Y)Df,\xi) = 0.$ With the help of (7.11), (2.3) and (2.11) we obtain  $\nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df = \alpha(\xi f) [\nabla_X Y - \nabla_Y X]$  $-\alpha^2(\xi f)[X\eta(Y) - Y\eta(X)].$ (7.13)Again we have from (7.11) $\nabla_{[X,Y]} Df = (\lambda - 2\alpha^2 - \alpha(\xi f))\eta([X,Y])\xi + \alpha(\xi f)[\nabla_X Y - \nabla_Y X].$ (7.14)From (7.13) and (7.14) we get  $R(X,Y)Df = -\alpha^2(\xi f)[X\eta(Y) - Y\eta(X)]$  $-\{\lambda - 2\alpha^2 - \alpha(\xi f)\}\eta([X, Y])\xi.$ (7.15)Taking inner product of (7.15) with  $\xi$  yields  $g(R(X,Y)Df,\xi) = -(\lambda - 2\alpha^2 - \alpha(\xi f))\eta([X,Y]).$ (7.16)From (7.12) and (7.16) we get  $\lambda - 2\alpha^2 = \alpha(\xi f).$ (7.17)Again we have from (7.17) and (7.10)(7.18) $Y(\xi f) = \alpha(\xi f)\eta(Y).$ Using (7.17) we get from (7.11) $\nabla_Y Df = \alpha(\xi f)Y.$ (7.19)Making use of (7.18) and (7.17) we obtain from (7.9) $S(X,Y) = -2\alpha^2 q(X,Y).$ Hence M is an Einstein manifold. Thus we can state the following:

**Theorem 7.1.** If the pseudo-metric g of a 3-dimensional non-para-cosymplectic  $\alpha$ -para Kenmotsu manifold is a gradient Ricci soliton, then the manifold is an Einstein manifold.

Again it is well known that a 3-dimensional Einstein manifold is a manifold of constant curvature. Thus we have the following:

**Corollary 7.1.** If the pseudo-metric g of a 3-dimensional non-para-cosymplectic  $\alpha$ -para Kenmotsu manifold is a gradient Ricci soliton, then the manifold is a manifold of constant curvature.

8. Example of a 3-dimensional  $\alpha$ -para Kenmotsu manifold

**Example 8.1.** We consider the 3-dimensional manifold  $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$ , where (x, y, z) are standard coordinate of  $\mathbb{R}^3$ .

The vector fields

$$e_1 = z \frac{\partial}{\partial x}, \ e_2 = z \frac{\partial}{\partial y}, \ e_3 = z \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let q be the pseudo-Riemannian metric defined by

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0,$$
  

$$g(e_1, e_1) = -1, \ g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let  $\eta$  be the 1-form defined by  $\eta(X) = g(X, e_3)$  for any  $X \in \chi(M)$ . Let  $\phi$  be the (1, 1) tensor field defined by  $\phi(e_1) = e_2$ ,  $\phi(e_2) = e_1$ ,  $\phi(e_3) = 0$ . Then we have

$$\eta(e_3) = g(e_3, e_3) = 1, \quad \phi^2 Z = Z - \eta(Z)e_3$$
$$g(\phi Z, \phi W) = -g(Z, W) + \eta(Z)\eta(W)$$

for any  $Z, W \in \chi(M)$ . Then for  $e_3 = \xi$ , the structure  $(\phi, \xi, \eta, g)$  defines an almost paracontact metric structure on M. Further one can easily verified  $[\phi, \phi](e_i, e_j) - 2d\eta(e_i, e_j) = 0$ ,  $1 \le i < j \le 3$ , which implies that the structure is normal. Let  $\nabla$ be the Levi-Civita connection with respect to the metric tensor g. Then we have

$$[e_1, e_3] = e_1 e_3 - e_3 e_1 \qquad = (z \frac{\partial}{\partial x})(z \frac{\partial}{\partial z}) - (z \frac{\partial}{\partial z})(z \frac{\partial}{\partial x})$$
$$= z^2 \frac{\partial^2}{\partial x \partial z} - z \frac{\partial}{\partial x} - z^2 \frac{\partial^2}{\partial z \partial x}$$
$$= -z \frac{\partial}{\partial x} = -e_1.$$

Similarly

$$[e_1, e_2] = 0, \ [e_2, e_3] = -e_2,$$
  
 $[e_1, e_1] = [e_2, e_2] = [e_3, e_3] = 0$ 

The Riemannian connection  $\nabla$  of the metric g is given by the Koszul's formula

(8.1) 
$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])$$

Using (8.1) we have,

(8.2) 
$$2g(\nabla_{e_1}e_3, e_3) = 0,$$

(8.3) 
$$2g(\nabla_{e_1}e_3, e_2) = 0,$$

(8.4) 
$$2g(\nabla_{e_1}e_3, e_1) = -2g(e_1, e_1)$$

From (8.2), (8.3) and (8.4) we get

(8.5) 
$$2g(\nabla_{e_1}e_3, X) = 2g(-e_1, X)$$

for any  $X \in \chi(M)$ . Therefore

$$\nabla_{e_1} e_3 = -e_1.$$

Further we get from (8.1)

$$\nabla_{e_1} e_1 = -e_3, \ \nabla_{e_1} e_2 = 0, \ \nabla_{e_1} e_3 = -e_1, \nabla_{e_2} e_1 = 0, \ \nabla_{e_2} e_2 = e_3, \ \nabla_{e_3} e_3 = -e_2,$$

$$\nabla_{e_3} e_1 = 0, \ \nabla_{e_3} e_2 = 0, \ \nabla_{e_3} e_3 = 0.$$

(8.6)

It is easily verify that the condition (2.12) holds. It is well known that

(8.7) 
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

With the previous results and using (8.7) we get the following:

$$\begin{aligned} R(e_1, e_2)e_1 &= -e_2, \ R(e_1, e_2)e_2 &= -e_1, \ R(e_1, e_2)e_3 &= 0, \\ R(e_2, e_3)e_1 &= 0, \ R(e_2, e_3)e_2 &= e_3, \ R(e_2, e_3)e_3 &= -e_2, \\ R(e_1, e_3)e_1 &= -e_3, \ R(e_1, e_3)e_2 &= 0, \ R(e_1, e_3)e_3 &= -e_1. \end{aligned}$$

Using the above expressions of the curvature tensor we get

$$S(e_1, e_1) = 2, \ S(e_2, e_2) = S(e_3, e_3) = 0.$$

In a pseudo-Riemannian manifold the scalar curvature

$$r = \sum_{i=1}^{3} g(e_i, e_i) S(e_i, e_i) = -2$$

Here  $\alpha = -1$ . Thus the manifold under consideration is an  $\alpha$ -para Kenmotsu manifold which verifies Theorem 5.1 and Theorem 6.1.

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