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# On Subclasses Of Bi-Starlike Functions Defined By Tremblay Fractional Derivative Operator

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#### Abstract

In this paper, we introduce and investigate new subclasses of strongly bi-starlike and bi-starlike functions defined by Tremblay fractional derivative operator in the open unit disk. Also we obtain upper bounds for the coefficients  $|a_2|$  and  $|a_3|$  of functions belonging to these classes. Unlike recent studies, we use different technique for obtain the upper bounds on the coefficients  $|a_3|$ . Theorems proved in this paper generalizes the results given in [3].

*Keywords:* bi-starlike functions, Tremblay operator, Bi-univalent functions, Fractional derivative, strongly bi-starlike functions, Coefficient bounds 2010 Mathematics Subject Classification: 30C45, 30C50, 30C80

#### 1. Introduction

Let  $\mathscr{A}$  denote the class of functions f(z) which are analytic in the open unit disk  $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$  and normalized by the conditions f(0) = f'(0) - 1 = 0 and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$
 (1.1)

Also let we denote by  $\mathscr{S}$ , the subclass of  $\mathscr{A}$  which elements are univalent in  $\mathbb{U}([4])$ .

From The Koebe's One-Quarter Theorem ([4]) says that "the range of every  $f(z) \in \mathscr{S}$  contains the disk  $\{w \in \mathbb{C} : |w| < 1/4\}$ ". Therefore every  $f \in \mathscr{S}$  has an inverse and the inverse function  $f^{-1}$  satisfy the following:

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w$$
  $(|w| < r_0(f), r_0(f) \ge \frac{1}{4}).$ 

The function  $f^{-1}$  is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$$
(1.2)

$$= w + \sum_{n=2}^{\infty} b_n w^n$$

Let  $f \in \mathscr{A}$ . If both f(z) and  $f^{-1}(z)$  are univalent in  $\mathbb{U}$ , then we say that f is *bi-univalent* in  $\mathbb{U}$ . The class of all bi-univalent functions in  $\mathbb{U}$  given by (1.1) is denoted by  $\Sigma$ .

The reader can find a detailed information about the function class  $\Sigma$  in [16] (see also [3],[9],[22]).

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Coefficient estimates for various subclasses of bi-univalent functions have been previously studied by some authors including Ali *et al.* [2], Frasin [6], Kumar *et al.* [8], Sümer Eker [1],[21], Magesh and Yamini [10], Srivastava *et al.* [15],[19],[20]. We need to following definitions of fractional integral and fractional derivative for our results. (For details, see [11],[12], [17], [18]).

**Definition 1.** For a function f, the fractional integral of order  $\delta$  is defined, by

$$D_z^{-\delta}f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}} d\xi; \ (\delta > 0)$$

where f is an analytic function in a simply-connected region of complex z-plane containing the origin, and the multiplicity of  $(z - \xi)^{\delta - 1}$  is removed by requiring,  $log(z - \xi)$  to be real when  $z - \xi > 0$ .

**Definition 2.** The fractional derivative of order  $\delta$  is defined, for a function f, by

$$D_z^{\delta}f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi \ (0 \le \delta < 1),$$

where f is constrained, and the multiplicity of  $(z - \xi)^{-\delta}$  is removed, as in Definition 1. **Definition 3.** Under the hypotheses of Definition 2, the fractional derivative of order  $(n + \delta)$  is defined by

$$D_z^{n+\delta}f(z) = \frac{d^n}{dz^n} D_z^{\delta}f(z) \quad (0 \le \delta < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

By virtue of Definitions 1, 2 and 3, we have

$$D_z^{-\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)} z^{n+\delta} \qquad (n \in \mathbb{N}, \delta > 0)$$

and

$$D_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n-\delta+1)} z^{n-\delta} \qquad (n \in \mathbb{N}, 0 \le \delta < 1).$$

In his thesis, Tremblay [23] investigated a fractional calculus operator defined in terms of the Riemann-Liouville fractional differential operator. Recently, Ibrahim and Jahangiri [7] extended the Tremblay Operator in the complex plane.

**Definition 4.** Let  $f \in \mathscr{A}$ . The Tremblay fractional derivative operator  $T_z^{\mu,\gamma}$  of a function f is defined, for all  $z \in \mathbb{U}$ , by

$$\mathbf{T}_{z}^{\mu,\gamma}f(z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)} z^{1-\gamma} D_{z}^{\mu-\gamma} z^{\mu-1} f(z)$$

 $(0<\gamma\leq 1;\; 0<\mu\leq 1,\; 0\leq \mu-\gamma< 1,\; \mu\geq \gamma).$ 

Obviously, if we choose  $\mu = \gamma = 1$ , we obtain

$$\mathbf{T}_{z}^{1,1}f(z) = f(z).$$

In [5], Esa et al. defined modified of Tremblay operator of analytic functions in complex domain as follows:

**Definition 5.** Let  $f(z) \in \mathscr{A}$ . The modified Tremblay operator denoted by  $\mathfrak{T}^{\mu,\gamma} : \mathscr{A} \to \mathscr{A}$  and defined as:

$$\begin{aligned} \mathfrak{T}_{z}^{\mu,\gamma}f(z) &= \frac{\gamma}{\mu} \mathrm{T}_{z}^{\mu,\gamma}f(z) \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)} z^{1-\gamma} D_{z}^{\mu-\gamma} z^{\mu-1} f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)} a_{n} z \end{aligned}$$

 $(0 < \gamma \le 1; \ 0 < \mu \le 1, \ 0 \le \mu - \gamma < 1, \ \mu \ge \gamma).$ 

where  $T_z^{\mu,\gamma}$  denote the Tremblay fractional derivative operator. For more information about Tremblay Operator see [20]). The object of the present paper is to introduce new subclasses of strongly bi-starlike and bi-starlike functions defined by modified Tremblay operator and find estimates on the modulus of the coefficients  $|a_2|$  and  $|a_3|$  for functions in this class. In the sequel, it is assumed that  $0 < \gamma \leq 1$ ;  $0 < \mu \leq 1$ ,  $0 < \mu - \gamma < 1$ ,  $\mu \geq \gamma$ .

### 2. Main Results

**Definition 6.** A function f(z) given by (1.1) is said to be in the class  $\mathscr{N}_{\Sigma}(\alpha, \mu, \gamma)$  if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left| \arg \frac{z(\mathfrak{T}_{z}^{\mu,\gamma}f)'(z)}{\mathfrak{T}_{z}^{\mu,\gamma}f(z)} \right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \le 1, z \in \mathbb{U})$$

$$(2.1)$$

and

$$\left|\arg\frac{w(\mathfrak{T}_{w}^{\mu,\gamma}g)'(w)}{\mathfrak{T}_{w}^{\mu,\gamma}g(w)}\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \le 1, w \in \mathbb{U})$$

$$(2.2)$$

where the function g(w) is given by (1.2).

It is clear that for  $\gamma = \mu$ , this class is reduced to  $S_{\Sigma}^*(\alpha)$  of class of strongly bi-starlike of order  $\alpha$  ( $0 < \alpha \le 1$ ), which is introduced by Brannan and Taha [3].

**Theorem 1.** If f(z) given by (1.1) be in the class  $\mathscr{N}_{\Sigma}(\alpha, \mu, \gamma)$ , then

$$|a_2| \le 2\alpha(\gamma+1)\sqrt{\frac{(\gamma+2)}{(\mu+1)\left[4\alpha(\mu+2)(\gamma+1) + (1-3\alpha)(\mu+1)(\gamma+2)\right]}}$$
(2.3)

and

$$|a_3| \le \frac{2\alpha(\gamma+2)(\gamma+1)^2}{(\mu+1)[\gamma(\mu+3)+2]}.$$
(2.4)

*Proof.* For f given by (1.1), we can write from (2.1) and (2.2)

$$\frac{z(\mathfrak{T}_{z}^{\mu,\gamma}f)'(z)}{\mathfrak{T}_{z}^{\mu,\gamma}f(z)} = [p(z)]^{\alpha}$$
(2.5)

$$\frac{w(\mathfrak{T}_{w}^{\mu,\gamma}g)'(w)}{\mathfrak{T}_{w}^{\mu,\gamma}g(w)} = [q(w)]^{\alpha}$$
(2.6)

where p(z) and q(w) are in Caratheódory Class  $\mathscr{P}$ . So p(z) and q(w) are have the following series expansions:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$
(2.7)

and

 $q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots$ (2.8)

(see for details [4]). Now, equating the coefficients (2.5) and (2.6), we find that

$$\frac{\mu+1}{\gamma+1}a_2 = \alpha p_1,\tag{2.9}$$

$$\frac{2(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}a_3 - \left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2,$$
(2.10)

$$-\frac{\mu+1}{\gamma+1}a_2 = \alpha q_1 \tag{2.11}$$

and

$$\left[\frac{4(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)} - \left(\frac{\mu+1}{\gamma+1}\right)^2\right]a_2^2 - \frac{2(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}a_3 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2.$$
(2.12)

From (2.9) and (2.11), we get

 $p_1 = -q_1$ 

and

$$2\left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2).$$
(2.14)

Also from (2.10), (2.12) and (2.14), we get

$$a_2^2 = \frac{\alpha^2 (p_2 + q_2)(\gamma + 2)(\gamma + 1)^2}{(\mu + 1)[4\alpha(\mu + 2)(\gamma + 1) + (1 - 3\alpha)(\mu + 1)(\gamma + 2)]}.$$
(2.15)

It is well known that from the Caratheódory Lemma, the coefficients of  $|p_n| \le 2$  and  $|q_n| \le 2$  for  $n \in \mathbb{N}$  (see [4]). If we take absolute value of both side of  $a_2^2$  and if we apply the Carathéodory Lemma to coefficients  $p_2$  and  $q_2$  we obtain

$$|a_2| \le \sqrt{\frac{4\alpha^2(\gamma+2)(\gamma+1)^2}{(\mu+1)\left[4\alpha(\mu+2)(\gamma+1)+(1-3\alpha)(\mu+1)(\gamma+2)\right]}}$$

This gives desired bound for  $|a_2|$  as asserted in (2.3).

Now, in order to find the bound on  $|a_3|$ , from (2.10), (2.12) and (2.13), we can write

$$4a_{3} = \alpha\lambda \left\{ \frac{4(\mu+2)(\gamma+1) - (\mu+1)(\gamma+2)}{(\mu+2)(\mu+1)}p_{2} + \frac{(\gamma+2)}{\mu+2}q_{2} + \frac{2(\alpha-1)(\gamma+1)}{\mu+1}p_{1}^{2} \right\}$$
(2.16)

where

$$\lambda = \frac{(\gamma+1)(\gamma+2)}{\gamma(\mu+3)+2}.$$

If  $\alpha = 1$ , then

$$|a_3| \le \frac{2\lambda(\gamma+1)}{\mu+1}$$

Now, we consider the case  $0 < \alpha < 1$ . From (2.16), we can write

$$4Re(a_3) = \alpha\lambda Re\left\{\frac{4(\mu+2)(\gamma+1) - (\mu+1)(\gamma+2)}{(\mu+2)(\mu+1)}p_2 + \frac{(\gamma+2)}{\mu+2}q_2 + \frac{2(\alpha-1)(\gamma+1)}{\mu+1}p_1^2\right\}$$
(2.17)

From Herglotz's Representation formula (see [13]) for the functions p(z) and q(w), we have

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu_1(t),$$

and

$$q(w) = \int_0^{2\pi} \frac{1 + we^{-it}}{1 - we^{-it}} d\mu_2(t),$$

where  $\mu_i(t)$  are increasing on  $[0, 2\pi]$  and  $\mu_i(2\pi) - \mu_i(0) = 1$ , i = 1, 2. We also have

$$p_n = 2 \int_0^{2\pi} e^{-int} d\mu_1(t), \qquad n = 1, 2, \dots,$$

and

$$q_n = 2 \int_0^{2\pi} e^{-int} d\mu_2(t), \qquad n = 1, 2, \dots.$$

Now (2.17) can be written as follows :

$$4Re(a_{3}) = 2\lambda\alpha \left\{ \left( \frac{4(\gamma+1)}{\mu+1} - \frac{\gamma+2}{\mu+2} \right) \int_{0}^{2\pi} \cos 2t d\mu_{1}(t) + \frac{\gamma+2}{\mu+2} \int_{0}^{2\pi} \cos 2t d\mu_{2}(t) \right\} - \frac{8\lambda\alpha(1-\alpha)(\gamma+1)}{\mu+1} \left[ \left( \int_{0}^{2\pi} \cos t d\mu_{1}(t) \right)^{2} - \left( \int_{0}^{2\pi} \sin t d\mu_{1}(t) \right)^{2} \right]$$

$$\leq 2\lambda\alpha \left\{ \left( \frac{4(\gamma+1)}{\mu+1} - \frac{\gamma+2}{\mu+2} \right) \int_0^{2\pi} \cos 2t d\mu_1(t) + \frac{\gamma+2}{\mu+2} \int_0^{2\pi} \cos 2t d\mu_2(t) + \frac{4(1-\alpha)(\gamma+1)}{\mu+1} \left( \int_0^{2\pi} \sin t d\mu_1(t) \right)^2 \right\}$$

$$= 2\lambda\alpha \left\{ \left( \frac{4(\gamma+1)}{\mu+1} - \frac{\gamma+2}{\mu+2} \right) \left( \int_0^{2\pi} (1-2\sin^2 t) d\mu_1(t) \right) + \frac{\gamma+2}{\mu+2} \left( \int_0^{2\pi} (1-2\sin^2 t) d\mu_2(t) \right) + \frac{4(1-\alpha)(\gamma+1)}{\mu+1} \left( \int_0^{2\pi} \sin t d\mu_1(t) \right)^2 \right\}.$$

By Jensen's inequality (see [14]), we have

$$\left(\int_0^{2\pi} |sint| d\mu(t)\right)^2 \le \int_0^{2\pi} sin^2 t d\mu(t)$$

Hence

$$4Re(a_3) \le 2\lambda\alpha \left\{ \frac{4(\gamma+1)}{\mu+1} - \left(\frac{4(1+\alpha)(\gamma+1)}{\mu+1} - \frac{2(\gamma+2)}{\mu+2}\right) \int_0^{2\pi} \sin^2 t d\mu_1(t) - \frac{2(\gamma+2)}{\mu+2} \int_0^{2\pi} \sin^2 t d\mu_2(t) \right\}$$

and thus

$$Re(a_3) \leq \frac{2\lambda \, \alpha(\gamma+1)}{\mu+1}$$

which implies

$$|a_3| \leq \frac{2\lambda\alpha(\gamma+1)}{\mu+1}.$$

This completes the proof of theorem.

If we choose  $\gamma = \mu$ , in the Theorem 1, we have the following corollary.

**Corollary 1.** [3] Let f(z) given by (1.1) belong to  $S^*_{\Sigma}(\alpha)$  ( $0 < \alpha \le 1$ ). Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}}$$
 and  $|a_3| \leq 2\alpha$ .

## 3. The Class $\mathscr{N}_{\Sigma}(\beta,\mu,\gamma)$ and Coefficient Estimates For The Functions In This Class

**Definition 7.** A function f(z) given by (1.1) is said to be in the class  $\mathcal{N}_{\Sigma}(\beta, \mu, \gamma)$  if the following conditions are satisfied:

$$f \in \Sigma \quad and \quad Re\left\{\frac{z(\mathfrak{T}_{z}^{\mu,\gamma}f)'(z)}{\mathfrak{T}_{z}^{\mu,\gamma}f(z)}\right\} > \beta \qquad (0 \le \beta < 1, z \in \mathbb{U})$$

$$(3.1)$$

and

$$Re\left\{\frac{w(\mathfrak{T}_{w}^{\mu,\gamma}g)'(w)}{\mathfrak{T}_{w}^{\mu,\gamma}g(w)}\right\} > \beta \qquad (0 \le \beta < 1, w \in \mathbb{U})$$

$$(3.2)$$

where the function *g* is inverse of the function *f* given by (1.2), For  $\gamma = \mu$ , the class of  $\mathcal{N}_{\Sigma}(\beta, \mu, \gamma)$  is reduced to  $S_{\Sigma}^*(\beta)$  of bi-starlike of order  $\beta$  ( $0 \le \beta < 1$ ), which is introduced by Brannan and Taha [3].

**Theorem 2.** If f(z) given by (1.1) in the class  $\mathscr{N}_{\Sigma}(\beta, \mu, \gamma)$ , then

$$|a_2| \le \sqrt{\frac{2\lambda(1-\beta)(\gamma+1)}{\mu+1}} \tag{3.3}$$

and

$$\left|a_{3}\right| \leq \frac{2(1-\beta)\lambda(\gamma+1)}{\mu+1} \tag{3.4}$$

where

 $\lambda = \frac{(\gamma+1)(\gamma+2)}{\gamma(\mu+3)+2}$ 

*Proof.* We can write the inequalities in (3.1) and (3.2) as in the following:

$$\frac{z(\mathfrak{T}_{z}^{\mu,\gamma}f)'(z)}{\mathfrak{T}_{z}^{\mu,\gamma}f(z)} = \beta + (1-\beta)p(z)$$
(3.5)

and

$$\frac{w(\mathfrak{T}^{\mu,\gamma}_{w}g)'(w)}{\mathfrak{T}^{\mu,\gamma}_{w}g(w)} = \beta + (1-\beta)q(w)$$
(3.6)

where p(z) and q(w) are given by (2.7) and (2.8), respectively. Like the proof of Theorem 1, by equating coefficients of (3.5) and (3.6) yields,

$$\frac{\mu+1}{\gamma+1}a_2 = (1-\beta)p_1, \tag{3.7}$$

$$\frac{2(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}a_3 - \left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2 = (1-\beta)p_2,$$
(3.8)

$$-\frac{\mu+1}{\gamma+1}a_2 = (1-\beta)q_1 \tag{3.9}$$

and

$$\left[\frac{4(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)} - \left(\frac{\mu+1}{\gamma+1}\right)^2\right]a_2^2 - \frac{2(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}a_3 = (1-\beta)q_2.$$
(3.10)

From (3.7) and (3.9) we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$2\left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2 = (1-\beta)^2 (p_1^2+q_1^2). \tag{3.12}$$

Also from (3.8) and (3.10) we obtain

$$\frac{2(\mu+1)}{\lambda(\gamma+1)}a_2^2 = (1-\beta)(p_2+q_2),\tag{3.13}$$

where  $\lambda = rac{(\gamma+1)(\gamma+2)}{\gamma(\mu+3)+2}.$  Thus, clearly we have

$$|a_2|^2 \le \frac{\lambda(\gamma+1)}{2(\mu+1)} (1-\beta) \left( |p_2| + |q_2| \right).$$
(3.14)

If we apply the Carathéodory Lemma to coefficients of  $p_2$ ,  $q_2$  we find the upper bound on  $|a_2|$  as given in (3.3). For the purpose of to find the bound on  $|a_3|$ , we multiply  $\frac{4(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)} - \frac{(\mu+1)^2}{(\gamma+1)^2}$  and  $\frac{(\mu+1)^2}{(\gamma+1)^2}$  to the equations (3.8) and (3.10) respectively and on adding them we obtain:

$$\frac{4(\mu+1)(\mu+2)}{\lambda}a_3 = (1-\beta)\left\{(3\mu\gamma+2\mu+7\gamma+6)p_2 + (\mu+1)(\gamma+2)q_2\right\}.$$
(3.15)

Now, let's take absolute value of the both side of (3.15). After then, if we apply the Carathéodory Lemma to coefficients of  $p_2$ ,  $q_2$  we find

$$|a_3|\leq rac{2(1-eta)\lambda(\gamma+1)}{\mu+1},$$

which is asserted in (3.4).

In the Theorem 2, if we choose  $\gamma = \mu$ , we obtain:

**Corollary 2.** [3] If f(z) given by (1.1) belongs to  $S^*_{\Sigma}(\beta)$   $(0 \le \beta < 1)$ , then

$$|a_2| \le \sqrt{2(1-\beta)}$$
 and  $|a_3| \le 2(1-\beta)$ .

#### 4. Conclusion

In our present paper, we introduce new subclasses  $\mathcal{N}_{\Sigma}(\alpha,\mu,\gamma)$  and  $\mathcal{N}_{\Sigma}(\beta,\mu,\gamma)$  of strongly bi-starlike and bi-starlike functions using Tremblay fractional derivative operator. Furthermore we obtained upper bounds for  $|a_2|$  and  $|a_3|$  for the functions in these classes. Unlike recent studies about bi-univalent functions, we have used Brannan and Taha's technique for obtain the upper bounds on the coefficients  $|a_3|$ . For  $\gamma = \mu$ , we can concluded the results which were given by Brannan and Taha [3]. In fact, our Theorems generalizes the results given in [3].

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