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# ON THE LIFTS OF $F_a(5,1)-$STRUCTURE ON TANGENT AND COTANGENT BUNDLE

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Abstract. This paper consist of three main sections. In the first part, we obtain the complete lifts of the $F_a(5,1)-$structure on tangent bundle. We have also obtained the integrability conditions by calculating the Nijenhuis tensors of the complete lifts of $F_a(5,1)-$structure. Later we get the conditions of to be the almost holomorfic vector field with respect to the complete lifts of $F_a(5,1)-$structure. Finally, we obtained the results of the Tachibana operator applied to the vector fields with respect to the complete lifts of $F_a(5,1)-$structure on tangent bundle. In the second part, all results obtained in the first section investigated according to the horizontal lifts of $F_a(5,1)-$structure in tangent bundle $T(M^n)$. In finally section, all results obtained in the first and second section were investigated according to the horizontal lifts of the $F_a(5,1)-$structure in cotangent bundle $T^*(M^n)$.

## 1. Introduction

The investigation for the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example [17]. Also, the idea of $F-$structure manifold on a differentiable manifold developed by Yano [14], Ishıhara and Yano [7], Goldberg [6] and among others. Moreover, Yano and Patterson [15, 16] studied on the horizontal and complete lifts from a differentiable manifold $M^n$ of class $C^\infty$ to its cotangent bundles. Andreu has studied the structure defined by a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying $F^5 + F = 0$ [1]. Later Ram Nivas and C.S. Prasad [11] studied on more form $F_a(5,1)-$structure. This paper consist of three main sections. In the first part,

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we obtain the complete lifts of the $F_a(5,1)-$structure on tangent bundle. We have also obtained the integrability conditions by calculating the Nijenhuis tensors of the complete lifts of $F_a(5,1)-$structure. Later we get the conditions of to be the almost holomorfic vector field with respect to the complete lifts of $F_a(5,1)-$structure. Finally, we obtained the results of the Tachibana operator applied to the vector fields with respect to the complete lifts of $F_a(5,1)-$structure on tangent bundle. In the second part, all results obtained in the first section investigated according to the horizontal lifts of $F_a(5,1)-$structure in tangent bundle $T(M^n)$. In finally section, all results obtained in the first and second section were investigated according to the horizontal lifts of the $F_a(5,1)-$structure in cotangent bundle $T^*(M^n)$.

Let $M^n$ be an $n-$dimensional differentiable manifold of class $C^\infty$. Suppose there exist on $M^n$, a $(1,1)$ tensor field $F(\neq 0)$ satisfying [11]

$$F^5 - a^2F = 0, \tag{1}$$

where $a$ is a complex number not equal to zero. If $a = i$ where $i = \sqrt{-1}$, our structure takes the form $F^5 + F = 0$ studied by Andreou [1].

Let us define on $M^n$, the operators $l$ and $m$ as follows :

$$l = (F^4/a^2) \text{ and } m = I - (F^4/a^2). \tag{2}$$

$I$ being unit tensor field.

In view of equations (1) and (2), we have

$$l^2 = l,\ m^2 = m \text{ and } l + m = I. \tag{3}$$

For a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying (1) the operators $l$ and $m$ defined by (2), when applied to the tangent space of $M^n$ at a point, are complementary projection operators.

Thus there exist complementary distributions $L$ and $M$ corresponding to the projection operators $l$ and $m$ respectively. If the rank of $F$ is constant every where or equal to $r$, the dimensions of $L$ and $M$ are $r$ and $n-r$ respectively [10]. Us call such a structure as $F_a(5,1)-$structure of rank $r$ [11].

For a tensor field $F(\neq 0)$ of type $(1,1)$ admitting $F_a(5,1)-$structure and for the projection operators $l$ and $m$ given by (2) we have

$$Fl = lF = F,\ Fm = mF = 0. \tag{4}$$

and

$$F^2l = lF^2 = F^2,\ F^2m = mF^2 = 0. \tag{5}$$

In the manifold $M^n$ endowed with $F_a(5,1)-$structure, the $(1,1)$ tensor field $\tilde{F}$ given by $\tilde{F} = l - m = (2F^4/a^2) - I$ gives an almost product structure [9].

1.1. **Complete Lift of $F_a(5,1)-$Structure on Tangent Bundle.** Let $M^n$ be an $n-$dimensional differentiable manifold of class $C^\infty$ and $T_P(M^n)$ the tangent space at a point $p$ of $M^n$ and

$$T(M^n) = \underset{p\in M^n}{U} T_P(M^n)$$

is the tangent bundle over the manifold $M^n$.

Let us denote by $T^r_s(M^n)$, the set of all tensor fields of class $C^\infty$ and of type $(r,s)$ in $M^n$ and $T(M^n)$ be the tangent bundle over $M^n$. The complete lift of $F^C$ of an element of $T^1_1(M^n)$ with local components $F^h_i$ has components of the form [16]

$$F^C = \begin{bmatrix} F^h_i & 0 \\ \delta^h_i & F^h_i \end{bmatrix}. \tag{6}$$

Now we obtain the following results on the complete lift of $F$ satisfying $F^5 - a^2F = 0$.

Let $F, G \in T^1_1(M^n)$. Then we have [16]

$$(FG)^C = F^C G^C. \tag{7}$$

Replacing $G$ by $F$ in (7) we obtain

$$(FF)^C = F^C F^C \text{ or } (F^2)^C = (F^C)^2. \tag{8}$$

Now putting $G = F^4$ in (7) since $G$ is $(1,1)$ tensor field therefore $F^4$ is also $(1,1)$ so we obtain $(FF^4)^C = F^C(F^4)^C$ which in view of (8) becomes

$$(F^5)^C = (F^C)^5. \tag{9}$$

Taking complete lift on both sides of equation $F^5 - a^2F = 0$ we get

$$(F^5)^C - (a^2F)^C = 0$$

which in consequence of equation (9) gives

$$(F^C)^5 - a^2F^C = 0. \tag{10}$$

Let $F$ satisfying $(1,1)$ be an $F-$structure of rank $r$ in $M^n$. Then the complete lifts $l^C = (F^4)^C$ of $l$ and $m^C = I - (F^4)^C$ of $m$ are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions $L^C$ and $M^C$ determined by $l^C$ and $m^C$, respectively.

1.2. **Horizontal Lift of $F_a(5,1)-$Structure on Tangent Bundle.** Let $F^h_i$ be the component of $F$ at $A$ in the coordinate neighbourhood $U$ of $M^n$. Then the horizontal lift $F^H$ of $F$ is also a tensor field of type $(1,1)$ in $T(M^n)$ whose components $\tilde{F}^A_B$ in $\pi^{-1}(U)$ are given by

$$F^H = F^C - \gamma(\nabla F) = \begin{pmatrix} F^h_i & 0 \\ -\Gamma^h_t F^t_i + \Gamma^t_i F^h_t & F^h_i \end{pmatrix}.$$

Let $F$, $G$ be two tensor fields of type $(1,1)$ on the manifold $M$. If $F^H$ denotes the horizontal lift of $F$, we have

$$(FG)^H = F^H G^H. \tag{11}$$

Taking $F$ and $G$ identical, we get

$$(F^H)^2 = (F^2)^H. \tag{12}$$

Multiplying both sides by $F^H$ and making use of the same (12), we get

$$(F^H)^3 = (F^3)^H$$

and so on. Thus it follows that

$$(F^H)^4 = (F^4)^H,\ (F^H)^5 = (F^5)^H. \tag{13}$$

Taking horizontal lift on both sides of equation $F^5 - a^2F = 0$ we get

$$(F^5)^H - (a^2F)^H = 0$$

view of (13), we can write

$$(F^H)^5 - a^2F^H = 0. \tag{14}$$

## 2. Main Results

### 2.1. **The Nijenhuis Tensor $N_{(F^5)^C(F^5)^C}(X^C, Y^C)$ of the Complete Lift $F^5$ on Tangent Bundle $T(M^n)$.**

**Definition 1.** *Let $F$ be a tensor field of type $(1,1)$ admitting $F_a(5,1)-$structure in $M^n$. The Nijenhuis tensor of a $(1,1)$ tensor field $F$ of $M^n$ is given by*

$$N_F = [FX, FY] - F[X, FY] - F[FX, Y] + F^2[X, Y] \tag{15}$$

for any $X, Y \in \Im_0^1(M^n)$ [2,12,13]. The condition of $N_F(X,Y) = N(X,Y) = 0$ is essential to integrability condition in these structures.

The Nijenhuis tensor $N_F$ is defined local coordinates by

$$N_{ij}^k\partial_k = (F_i^s\partial_s^k F_j^k - F_j^l\partial_l F_i^k - \partial_i F_j^l F_l^k + \partial_j F_i^s F_s^k)\partial_k,$$

where $X = \partial_i$, $Y = \partial_j$, $F \in \Im_1^1(M^n)$.

**Definition 2.** *Let $X$ and $Y$ be any vector fields on a Riemannian manifold $(M^n, g)$, we have [17]*

$$\begin{aligned}
\left[X^H, Y^H\right] &= [X,Y]^H - (R(X,Y)u)^V, \qquad (16)\\
\left[X^H, Y^V\right] &= (\nabla_X Y)^V,\\
\left[X^V, Y^V\right] &= 0,
\end{aligned}$$

*where $R$ is the Riemannian curvature tensor of $g$ defined by*

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{17}$$

In particular, we have the vertical spray $u^V$ and the horizontal spray $u^H$ on $T(M^n)$ defined by

$$u^V = u^i(\partial_i)^V = u^i\partial_{\bar{i}},\ u^H = u^i(\partial_i)^H = u^i\delta_i, \tag{18}$$

where $\delta_i = \partial_i - u^j\Gamma_{ji}^s\partial_{\bar{s}}$. $u^V$ is also called the canonical or Liouville vector field on $T(M^n)$.

**Theorem 3.** *The Nijenhuis tensor $N_{(F^5)^C(F^5)^C}\left(X^C, Y^C\right)$ of the complete lift of $F^5$ vanishes if the Nijenhuis tensor of the $F$ is zero.*

*Proof.* In consequence of Definition 1 the Nijenhuis tensor of $\left(F^5\right)^C$ is given by

$$\begin{aligned} N_{(F^5)^C(F^5)^C}\left(X^C, Y^C\right) &= [\left(F^5\right)^C X^C, \left(F^5\right)^C Y^C] - \left(F^5\right)^C [\left(F^5\right)^C X^C, Y^C] \\ &\quad - \left(F^5\right)^C [X^C, \left(F^5\right)^C Y^C] + \left(F^5\right)^C \left(F^5\right)^C \left[X^C, Y^C\right] \\ &= a^4\{[(FX)^C, (FY)^C] - (F)^C [(FX)^C, Y^C] \\ &\quad - (F)^C [X^C, (FY)^C] + (F)^C (F)^C \left[X^C, Y^C\right]\} \\ &= a^4\{[FX, FY] - F[FX, Y] \\ &\quad -F[X, FY] + F^2[X, Y]\}^C \\ &= a^4 N(X, Y)^C \end{aligned}$$

□

**Theorem 4.** *The Nijenhuis tensor $N_{(F^5)^C(F^5)^C}\left(X^C, Y^V\right)$ of the complete lift of $F^5$ vanishes if the Nijenhius tensor $F$ is zero.*

*Proof.*

$$\begin{aligned} N_{(F^5)^C(F^5)^C}\left(X^C, Y^V\right) &= [\left(F^5\right)^C X^C, \left(F^5\right)^C Y^V] - \left(F^5\right)^C [\left(F^5\right)^C X^C, Y^V] \\ &\quad - \left(F^5\right)^C [X^C, \left(F^5\right)^C Y^V] + \left(F^5\right)^C \left(F^5\right)^C \left[X^C, Y^V\right] \\ &= a^4\{[(FX)^C, (FY)^V] - (F)^C [(FX)^C, Y^V] \\ &\quad - (F)^C [X^C, (FY)^V] + \left(F^2\right)^C [X, Y]^V\} \\ &= a^4\{[FX, FY]^V - (F[FX, Y])^V \\ &\quad - (F[X, FY])^V - \left(F^2[X, Y]\right)^V\} \\ &= a^4 N(X, Y)^V \end{aligned}$$

□

**Theorem 5.** *The Nijenhuis tensor $N_{(F^5)^C(F^5)^C}\left(X^V, Y^V\right)$ of the complete lift of $F^5$ vanishes.*

*Proof.* Thus $\left[X^V, Y^V\right] = 0$ for all $X, Y \in \Im_0^1(M^n)$, easily we get

$$N_{(F^5)^C(F^5)^C}\left(X^V, Y^V\right) = 0.$$

□

## 2.2. The Purity Conditions of Sasakian Metric with Respect to $(F^5)^C$ on $T(M^n)$.

**Definition 6.** *The Sasaki metric* $^Sg$ *is a (positive definite) Riemannian metric on the tangent bundle* $T(M^n)$ *which is derived from the given Riemannian metric on* $M$ *as follows:*

$$\begin{aligned} {}^Sg\left(X^H,Y^H\right) &= g\left(X,Y\right), \\ {}^Sg\left(X^H,Y^V\right) &= {}^Sg\left(X^V,Y^H\right)=0, \\ {}^Sg\left(X^V,Y^V\right) &= g\left(X,Y\right) \end{aligned} \tag{19}$$

for all $X,Y\in\Im_0^1\left(M^n\right)$.

**Theorem 7.** *The Sasaki metric* $^Sg$ *is pure with respect to* $\left(F^5\right)^C$ *if* $\nabla F=0$ *and* $F=a^2I$ *, where I=ıdentity tensor field of type* $(1,1)$.

*Proof.* $S(\widetilde{X},\widetilde{Y})={}^S g((F^5)^C\widetilde{X},\widetilde{Y})-{}^S g(\widetilde{X},(F^5)^C\widetilde{Y})$ if $S(\widetilde{X},\widetilde{Y})=0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form $X^V,Y^V$ or $X^H,Y^H$ then $S=0$.

*i)*

$$\begin{aligned} S\left(X^V,Y^V\right) &= {}^Sg((F^5)^C X^V,Y^V)-{}^S g(X^V,(F^5)^C Y^V) \\ &= a^2\{{}^Sg((FX)^V,Y^V)-{}^S g(X^V,(FY)^V)\} \\ &= a^2\{(g\left(FX,Y\right))^V-(g\left(X,FY\right))^V\} \end{aligned}$$

*ii)*

$$\begin{aligned} S\left(X^V,Y^H\right) &= {}^Sg((F^5)^C X^V,Y^H)-{}^S g(X^V,(F^5)^C Y^H) \\ &= -a^2\ {}^Sg(X^V,(FY)^H+(\nabla_\gamma F)\,Y^H) \\ &= -a^2\ {}^Sg\left(X^V,(\nabla_\gamma F)\,Y^H\right) \\ &= -a^2\ {}^Sg(X^V,(((\nabla F)\,u)\,Y)^V) \\ &= -a^2(\ g\left(X,((\nabla F)\,u)\,Y\right)^V) \end{aligned}$$

*iii)*

$$\begin{aligned} S\left(X^H,Y^H\right) &= {}^Sg((F^5)^C X^H,Y^H)-{}^S g(X^H,(F^5)^C Y^H) \\ &= a^2\ {}^Sg((F)^C X^H,Y^H)-a^2\ {}^Sg(X^H,(F)^C Y^H) \\ &= a^2\ {}^Sg((FX)^H+(\nabla_\gamma F)\,X^H,Y^H) \\ &\quad -a^2\ {}^Sg(X^H,(FY)^H+(\nabla_\gamma F)\,Y^H) \\ &= a^2\{g\left((FX),Y\right)^V-g\left(X,(FY)\right)^V\} \end{aligned}$$

□

**Definition 8.** *Let $\varphi \in \Im_1^1(M^n)$, and $\Im(M^n) = \sum_{r,s=0}^{\infty} \Im_s^r(M^n)$ be a tensor alebra over R. A map $\phi_\varphi \mid_{r+s)0}: \overset{*}{\Im}(M^n) \to \Im(M^n)$ is called as Tachibana operatör or $\phi_\varphi$operatör on $M^n$ if*

$a)$ $\phi_\varphi$ is linear with respect to constant coefficient,

$b)$ $\phi_\varphi : \overset{*}{\Im}(M^n) \to \Im_{s+1}^r(M^n)$ for all $r$ and $s$,

$c)$ $\phi_\varphi(K \overset{C}{\otimes} L) = (\phi_\varphi K) \otimes L + K \otimes \phi_\varphi L$ for all $K, L \in \overset{*}{\Im}(M^n)$,

$d)$ $\phi_{\varphi X} Y = -(L_Y \varphi)X$ for all $X, Y \in \Im_0^1(M^n)$, where $L_Y$ is the Lie derivation with respect to $Y$ (see [3,5,8]),

$e)$

$$\begin{aligned}(\phi_{\varphi X}\eta)Y &= (d(\imath_Y\eta))(\varphi X) - (d(\imath_Y(\eta o\varphi)))X + \eta((L_Y\varphi)X) \\ &= \phi X(\imath_Y\eta) - X(\imath_{\varphi Y}\eta) + \eta((L_Y\varphi)X)\end{aligned}$$

for all $\eta \in \Im_1^0(M^n)$ and $X, Y \in \Im_0^1(M^n)$, where $\imath_Y\eta = \eta(Y) = \eta \overset{C}{\otimes} Y, \overset{*}{\Im}{}_s^r(M^n)$ the module of all pure tensor fields of type $(r, s)$ on $M^n$ with respect to the affinor field, $\overset{C}{\otimes}$ is a tensor product with a contraction $C$ [2,4,12](see [13] for applied to pure tensor field).

**Remark 9.** *If $r = s = 0$, then from $c), d)$ and $e)$ of Definition8 we have $\phi_{\varphi X}(\imath_Y\eta) = \phi X(\imath_Y\eta) - X(\imath_{\varphi Y}\eta)$ for $\imath_Y\eta \in \Im_0^0(M^n)$, which is not well-defined $\phi_\varphi-$operator. Different choices of $Y$ and $\eta$ leading to same function $f = \imath_Y\eta$ do get the same values. Consider $M^n = R^2$ with standard coordinates $x, y$. Let $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the function $f = 1$. This may be written in many different ways as $\imath_Y\eta$. Indeed taking $\eta = dx$, we may choose $Y = \frac{\partial}{\partial_x}$ or $Y = \frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y}$. Nov the right-hand side of $\phi_{\varphi X}(\imath_Y\eta) = \phi X(\imath_Y\eta) - X(\imath_{\varphi Y}\eta)$ is $(\phi X)1 - 0 = 0$ in the first case, and $(\phi X)1 - Xx = -Xx$ in the second case. For $X = \frac{\partial}{\partial_x}$, the latter expression is $-1 \neq 0$. Therefore, we put $r + s > 0$ [12].*

**Remark 10.** *From $d)$ of Definition8 we have*

$$\phi_{\varphi X} Y = [\varphi X, Y] - \varphi[X, Y].$$

By virtue of

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

for any $f, g \in \Im_0^0(M^n)$, we see that $\phi_{\varphi X}Y$ is linear in $X$, but not $Y$ [12].

**Theorem 11.** *Let $\phi_\varphi$ be the Tachibana operator and the structure $\left(F^5\right)^C - a^2F^C = 0$ defined by Definition 8 and (10), respectively. If $L_YF = 0$, then all results with respect to $\left(F^5\right)^C$ is zero, where $X, Y \in \Im_0^1(M)$, the complete lifts $X^C, Y^C \in \Im_0^1(T(M))$ and the vertical lift $X^V, Y^V \in \Im_0^1(T(M))$.*

$$i)\ \phi_{(F^5)^C X^C} Y^C = -a^2\left((L_YF)X\right)^C$$

$$\begin{aligned}
ii)\ \phi_{(F^5)^C X^C} Y^V &= -a^2\left(\left(L_Y F\right)X\right)^V \\
iii)\ \phi_{(F^5)^C X^V} Y^C &= -a^2\left(\left(L_Y F\right)X\right)^V \\
iv)\ \phi_{(F^5)^C X^V} Y^V &= 0
\end{aligned}$$

*Proof. i)*

$$\begin{aligned}
\phi_{(F^5)^C X^C} Y^C &= -(L_{Y^C}\left(F^5\right)^C)X^C \\
&= a^2\{-L_{Y^C}\left(FX\right)^C + \left(F\right)^C L_{Y^C} X^C\} \\
&= -a^2\left(\left(L_Y F\right)X\right)^C
\end{aligned}$$

*ii)*

$$\begin{aligned}
\phi_{(F^5)^C X^C} Y^V &= -(L_{Y^V}\left(F^5\right)^C)X^C \\
&= -L_{Y^V}\left(F^5\right)^C X^C + \left(F^5\right)^C L_{Y^V} X^C \\
&= a^2\{-L_{Y^V}\left(FX\right)^C + \left(F\right)^C L_{Y^V} X^C\} \\
&= -a^2\left(\left(L_Y F\right)X\right)^V
\end{aligned}$$

*iii)*

$$\begin{aligned}
\phi_{(F^5)^C X^V} Y^C &= -(L_{Y^C}\left(F^5\right)^C)X^V \\
&= -L_{Y^C}\left(F^5\right)^C X^V + \left(F^5\right)^C L_{Y^C} X^V \\
&= a^2\{-L_{Y^C}\left(FX\right)^V + \left(F\right)^C L_{Y^C} X^V\} \\
&= -a^2\left(\left(L_Y F\right)X\right)^V
\end{aligned}$$

*iv)*

$$\begin{aligned}
\phi_{(F^5)^C X^V} Y^V &= -(L_{Y^V}\left(F^5\right)^C)X^V \\
&= -L_{Y^V}\left(F^5\right)^C X^V + \left(F^5\right)^C L_{Y^V} X^V \\
&= 0
\end{aligned}$$

□

**Theorem 12.** *If $L_Y F = 0$ for $Y \in M$ , then its complete lift $Y^C$ to the tangent bundle is an almost holomorfic vector field with respect to the structure $\left(F^5\right)^C - a^2 F^C = 0$.*

*Proof. i)*

$$\begin{aligned}
(L_{Y^C}\left(F^5\right)^C)X^C &= L_{Y^C}\left(F^5\right)^C X^C - \left(F^5\right)^C L_{Y^C} X^C \\
&= a^2\{L_{Y^C}\left(FX\right)^C - \left(F\right)^C L_{Y^C} X^C\} \\
&= a^2\left(\left(L_Y F\right)X\right)^C
\end{aligned}$$

*ii*)

$$\begin{aligned}(L_{Y^C}\left(F^5\right)^C)X^V &= L_{Y^C}\left(F^5\right)^C X^V - \left(F^5\right)^C L_{Y^C}X^V \\ &= a^2\{L_{Y^C}\left(FX\right)^V - \left(F\right)^C L_{Y^C}X^V\} \\ &= a^2\left(\left(L_Y F\right)X\right)^V\end{aligned}$$

□

### 2.3. **The Structure** $\left(F^5\right)^H - a^2F^H = 0$ **on Tangent Bundle** $T\left(M^n\right)$**.**

**Theorem 13.** *The Nijenhuis tensor* $N_{(F^5)^H(F^5)^H}\left(X^H, Y^H\right)$ *of the horizontal lift of* $F^5$ *vanishes if the Nijenhuis tensor of the* $F$ *is zero and* $\{-(\hat{R}\left(FX, FY\right)u) + (F(\hat{R}\left(FX, Y\right)u)) + (F\left(R\left(X, FY\right)u\right)) - ((F)^2\left(\hat{R}\left(X, Y\right)u\right))\}^V = 0.$

*Proof.*

$$\begin{aligned}N_{(F^5)^H(F^5)^H}\left(X^H, Y^H\right) &= [\left(F^5\right)^H X^H, \left(F^5\right)^H Y^H] - \left(F^5\right)^H [\left(F^5\right)^H X^H, Y^H] \\ &\quad - \left(F^5\right)^H [X^H, \left(F^5\right)^H Y^H] + \left(F^5\right)^H \left(F^5\right)^H \left[X^H, Y^H\right] \\ &= a^4\{([FX, FY] - (F)\left[FX, Y\right] \\ &\quad - (F)\left[X, FY\right] - (F)\left(F\right)\left[X, Y\right])^H \\ &\quad -(\hat{R}\left(FX, FY\right)u)^V + (F(\hat{R}\left(FX, Y\right)u))^V \\ &\quad +(F(\hat{R}\left(X, FY\right)u))^V - ((F)^2\left(\hat{R}\left(X, Y\right)\right)u)^V\} \\ &= a^4\{(N_{FF}\left(X, Y\right))^H - (\hat{R}\left(FX, FY\right)u)^V \\ &\quad +(F(\hat{R}\left(FX, Y\right)u))^V + (F(\hat{R}\left(X, FY\right)u))^V \\ &\quad -((F)^2\left(\hat{R}\left(X, Y\right)u\right))^V\}.\end{aligned}$$

□

If $N_{FF}\left(X, Y\right) = 0$ and $\{-\hat{R}\left(FX, FY\right)u + (F(\hat{R}\left(FX, Y\right)u)) + (F(\hat{R}\left(X, FY\right)u)) - ((F)^2\left(\hat{R}\left(X, Y\right)u\right))\}^V = 0$, then we get $N_{(F^5)^H(F^5)^H}\left(X^H, Y^H\right) = 0$. The theorem is proved.

Where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$ defined by $\hat{\nabla}_X Y = \nabla_Y X + [X, Y]$ (see [17] p.88-89).

**Theorem 14.** *The Nijenhuis tensor* $N_{(F^5)^H(F^5)^H}\left(X^H, Y^V\right)$ *of the horizontal lift of* $F^5$ *vanishes if the Nijenhuis tensor of the* $F$ *is zero and* $\nabla F = 0$.

*Proof.*

$$\begin{aligned}N_{(F^5)^H(F^5)^H}\left(X^H, Y^V\right) &= [\left(F^5\right)^H X^H, \left(F^5\right)^H Y^V] - \left(F^5\right)^H [\left(F^5\right)^H X^H, Y^V] \\ &\quad - \left(F^5\right)^H [X^H, \left(F^5\right)^H Y^V] + \left(F^5\right)^H \left(F^5\right)^H \left[X^H, Y^V\right] \\ &= a^4\{[FX, FY]^V - (F\left[FX, Y\right])^V - (F\left[X, FY\right])^V \\ &\quad +((F)^2\left[X, Y\right])^V + (\nabla_{FY}FX)^V - (F\left(\nabla_Y FX\right))^V\end{aligned}$$

$$
\begin{aligned}
&\quad - (F(\nabla_{FY}X))^V + ((F)^2 \nabla_Y X)^V\} \\
&= a^4\{(N_{FF}(X,Y))^V + (\nabla_{FY}F)X - (F((\nabla_Y F)X))^V\}
\end{aligned}
$$

□

**Theorem 15.** *The Nijenhuis tensor $N_{(F^5)^H(F^5)^H}\left(X^V, Y^V\right)$ of the horizontal lift of $F^5$ vanishes.*

*Proof.* Because of $\left[X^V, Y^V\right] = 0$ for $X, Y \in M$, easily we get

$$N_{(F^5)^H(F^5)^H}\left(X^V, Y^V\right) = 0.$$

□

**Theorem 16.** *The Sasakian metric ${}^S g$ is pure with respect to $\left(F^5\right)^H$ if $F = a^2 I$, where $I$ =identity tensor field of type $(1,1)$.*

*Proof.* $S(\widetilde{X}, \widetilde{Y}) = {}^S g(\left(F^5\right)^H \widetilde{X}, \widetilde{Y}) - {}^S g(\widetilde{X}, \left(F^5\right)^H \widetilde{Y})$ if $S(\widetilde{X}, \widetilde{Y}) = 0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form $X^V, Y^V$ or $X^H, Y^H$ then $S = 0$.

*i)*

$$
\begin{aligned}
S\left(X^V, Y^V\right) &= {}^S g(\left(F^5\right)^H X^V, Y^V) - {}^S g(X^V, \left(F^5\right)^H Y^V) \\
&= a^2\{{}^S g((FX)^V, Y^V) - {}^S g(X^V, (FY)^V)\} \\
&= a^2\{(g(FX, Y))^V - (g(X, FY))^V\}
\end{aligned}
$$

*ii)*

$$
\begin{aligned}
S\left(X^V, Y^H\right) &= {}^S g(\left(F^5\right)^H X^V, Y^H) - {}^S g(X^V, \left(F^5\right)^H Y^H) \\
&= -a^2\, {}^S g(X^V, (FY)^H) \\
&= 0
\end{aligned}
$$

*iii)*

$$
\begin{aligned}
S\left(X^H, Y^H\right) &= {}^S g(\left(F^5\right)^H X^H, Y^H) - {}^S g(X^H, \left(F^5\right)^H Y^H) \\
&= a^2\ \{({}^S g(FX)^H, Y^H) - {}^S g(X^H, (FY)^H)\} \\
&= a^2\ \{(g(FX), Y)^V - (g(X, (FY)^H))^V\}
\end{aligned}
$$

□

**Theorem 17.** *Let $\phi_\varphi$ be the Tachibana operator and the structure $\left(F^5\right)^H - a^2 F^H = 0$ defined by Definition 8 and (14), respectively. if $L_Y F = 0$ and $F = a^2 I$, then all results with respect to $\left(F^5\right)^H$ is zero, where $X, Y \in \Im_0^1(M)$, the horizontal lifts $X^H, Y^H \in \Im_0^1(T(M^n))$ and the vertical lift $X^V, Y^V \in \Im_0^1(T(M^n))$*

*i)* $\phi_{(F^5)^H X^H} Y^H = -a^2\{-((L_Y F)X)^H + (\hat{R}(Y, FX)u)^V - (F(\hat{R}(Y,X)u))^V\}$,

*ii)* $\phi_{(F^5)^H X^H} Y^V = a^2\{-((L_Y F)X)^V + ((\nabla_Y F)X)^V\}$,

$$iii)\ \phi_{(F^5)^H X^V} Y^H = a^2\{-((L_Y F)X)^V - (\nabla_{FX} Y)^V + (F(\nabla_X Y))^V\},$$
$$iv)\ \phi_{(F^5)^H X^V} Y^V = 0,$$

*Proof.* *i)*

$$\begin{aligned}
\phi_{(F^5)^H X^H} Y^H &= -(L_{Y^H}\left(F^5\right)^H)X^H \\
&= -L_{Y^C}\left(F^5\right)^H X^H + \left(F^5\right)^H L_{Y^H} X^H \\
&= -a^2[Y,FX]^H + a^2\gamma\hat{R}[Y,FX] \\
&\quad +a^2(F[Y,X])^H - a^2(F)^H(\hat{R}(Y,X)u)^V \\
&= -a^2\{-((L_Y F)X)^H + (\hat{R}(Y,FX)u)^V \\
&\quad -(F(\hat{R}(Y,X)u))^V\}
\end{aligned}$$

*ii)*

$$\begin{aligned}
\phi_{(F^5)^H X^H} Y^V &= -(L_{Y^V}\left(F^5\right)^H)X^H \\
&= -L_{Y^V}\left(F^5 X\right)^H + \left(F^5\right)^H L_{Y^V} X^H \\
&= -a^2[Y,FX]^V + a^2(\nabla_Y FX)^V \\
&\quad +a^2(F[Y,X])^V - a^2(F(\nabla_Y X))^V \\
&= a^2\{-((L_Y F)X)^V + ((\nabla_Y F)X)^V\}
\end{aligned}$$

*iii)*

$$\begin{aligned}
\phi_{(F^5)^H X^V} Y^H &= -(L_{Y^H}\left(F^5\right)^H)X^V \\
&= -L_{Y^H}\left(F^5 X\right)^V + \left(F^5\right)^H L_{Y^H} X^V \\
&= a^2[Y,FX]^V - a^2(\nabla_{FX} Y)^V \\
&\quad +a^2(F[Y,X])^H + a^2(F(\nabla_X Y))^V \\
&= a^2\{-((L_Y F)X)^V - (\nabla_{FX} Y)^V + (F(\nabla_X Y))^V\}
\end{aligned}$$

*iv)*

$$\begin{aligned}
\phi_{(F^5)^H X^V} Y^V &= -(L_{Y^V}\left(F^5\right)^H)X^V \\
&= -a^2 L_{Y^V}(FX)^V + a^2(F)^H L_{Y^V} X^V \\
&= 0
\end{aligned}$$

□

2.4. **The Structure $(F^5)^H - a^2F^H = 0$ on Cotangent Bundle.** In this section, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of $F_a(5,1)-$structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of $F_a(5,1)-$structure in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of the structure.

Let $F$, $G$ be two tensor fields of type $(1,1)$ on the manifold $M$. If $F^H$ denotes the horizontal lift of $F$, we have [17]

$$F^H G^H + G^H F^H = (FG + GF)^H$$

Taking $F$ and $G$ identical, we get

$$(F^H)^2 = (F^2)^H \tag{20}$$

Multiplying both sides by $F^H$ and making use of the same (20), we get

$$(F^H)^3 = (F^3)^H$$

and so on. Thus it follows that

$$(F^H)^4 = (F^4)^H \tag{21}$$

and so on. Thus

$$(F^H)^5 = (F^5)^H \tag{22}$$

Since $F$ gives on $M$ the $F_a(5,1)$−structure, we have

$$F^5 - a^2 F = 0. \tag{23}$$

Taking horizontal lift, we obtain

$$(F^5)^H - a^2 F^H = 0. \tag{24}$$

In view of (22), we can write

$$(F^H)^5 - a^2 F^H = 0. \tag{25}$$

**Theorem 18.** *The Nijenhuis tensor $N_{(F^5)^H(F^5)^H}(X^H, Y^H)$ of the horizontal lift $F^5$ vanishes if $F = a^2 I$ on $M$.*

*Proof.* The Nijenhuis tensor $N(X^H, Y^H)$ for the horizontal lift of $F^5$ is given by

$$\begin{aligned} N_{(F^5)^H,(F^5)^H}(X^H, Y^H) &= [(F^5)^H X^H, (F^5)^H Y^H] - (F^5)^H[(F^5)^H X^H, Y^H] \\ &\quad -(F^5)^H[X^H, (F^5)^H Y^H] + (F^5)^H (F^5)^H [X^H, Y^H] \\ &= a^4\{[(F)^H X^H, (F)^H Y^H] - (F)^H[(F)^H X^H, Y^H] \\ &\quad -(F)^H[X^H, (F)^H Y^H] + (F)^H (F)^H [X^H, Y^H]\} \\ &= a^4\{\{[FX, FY] - F[(FX), Y] - F[X, FY] \\ &\quad +F^2[X, Y]\}^H + \gamma\{R(FX, FY) - R((FX), Y)F \\ &\quad -R(X, FY)F^2 + R(X, Y)F^2\}\} \end{aligned}$$

Let us suppose that $F = a^2 I$ on $M$. Thus, the equation becomes

$$\begin{aligned} N_{(F^5)^H,(F^5)^H}(X^H, Y^H) &= a^4\{\{[X, Y] - [X, Y] - [X, Y] + [X, Y]\}^H \\ &\quad +\gamma\{R(X, Y) - R(X, Y) - R(X, Y) + R(X, Y)\}. \end{aligned}$$

Therefore, it follows

$$N_{(F^5)^H,(F^5)^H}(X^H,Y^H)=0$$

□

**Theorem 19.** *The Nijenhuis tensor $N_{(F^5)^H(F^5)^H}(X^H,\omega^V)$ of the horizontal lift $F^5$ vanishes if $\nabla F=0$.*

*Proof.*

$$\begin{aligned}
N_{(F^5)^H,(F^5)^H}(X^H,\omega^V) &= [(F^5)^H X^H,(F^5)^H\omega^V]-(F^5)^H[(F^5)^H X^H,\omega^V] \\
&\quad -(F^5)^H[X^H,(F^5)^H\omega^V]+(F^5)^H(F^5)^H[X^H,\omega^V] \\
&= a^4\{(\nabla_{FX}(\omega\circ F))^V-((\nabla_{FX})\circ F)^V \\
&\quad -((\nabla_X(\omega\circ F))\circ F)^V+((\nabla_X\omega)\circ F^2)^V\} \\
&= a^4\{(\omega\circ(\nabla_{FX}F)-(\omega\circ(\nabla_X F)F\}^V
\end{aligned}$$

where $F\in\Im_1^1(M)$, $X\in\Im_0^1(M)$, $\omega\in\Im_1^0(M)$. The theorem is proved. □

**Theorem 20.** *The Nijenhuis tensor $N_{(F^5)^H,(F^5)^H}(\omega^V,\theta^V)$ of the horizontal lift $F^5$ vanishes.*

*Proof.* Because of $[\omega^V,\theta^V]=0$ and $\omega\circ F\in\Im_1^0(M^n)$ on $T^*(M^n)$, the equation becomes

$$N_{(F^5)^H,(F^5)^H}(\omega^V,\theta^V)=0.$$

□

**Theorem 21.** *Let $(F^5)^H$ be a tensor field of type $(1,1)$ on $T^*(M^n)$. If the Tachibana operator $\phi_\varphi$ applied to vector and covector fields according to horizontal lifts of $F^5$ defined by (25) on $T^*(M^n)$, then we get the following results.*

$$\begin{aligned}
i)\ \phi_{(F^5)^H X^H}Y^H &= a^2\{-((L_Y F)X)^H-(pR(Y,FX))^V \\
&\quad +((pR(Y,X))F)^V\}, \\
ii)\ \phi_{(F^5)^H X^H}\omega^V &= a^2\{(\nabla_{FX}\omega)^V-((\nabla_X\omega)\circ F)^V\}, \\
iii)\ \phi_{(F^5)^H\omega^V}X^H &= -a^2(\omega\circ(\nabla_X F))^V, \\
iv)\ \phi_{(F^5)^H\omega^V}\theta^V &= 0,
\end{aligned}$$

where horizontal lifts $X^H,Y^H\in\Im_0^1(T^*(M^n))$ of $X,Y\in\Im_0^1(M^n)$ and the vertical lift $\omega^V,\theta^V\in\Im_0^1(T^*(M^n))$ of $\omega,\theta\in\Im_1^0(M^n)$ are given, respectively.

*Proof.* *i)*

$$\phi_{(F^5)^H X^H}Y^H = -(L_{Y^H}(F^5)^H)X^H$$

$$\begin{aligned} &= -L_{Y^H}(F^5)^H X^H + (F^5)^H L_{Y^H} X^H \\ &= a^2\{-((L_Y F)X)^H - (pR(Y,FX))^V \\ &\quad +((pR(Y,X))F)^V\} \end{aligned}$$

*ii)*

$$\begin{aligned} \phi_{(F^5)^H X^H}\omega^V &= -(L_{\omega^V}(F^5)^H)X^H \\ &= -L_{\omega^V}(F^5)^H X^H + (F^5)^H L_{\omega^V} X^H \\ &= -a^2 L_{\omega^V}(FX)^H - a^2(F)^H(\nabla_X\omega)^V \\ &= a^2\{(\nabla_{FX}\omega)^V - ((\nabla_X\omega)\circ F)^V\}, \end{aligned}$$

*iii)*

$$\begin{aligned} \phi_{(F^5)^H\omega^V} X^H &= -(L_{X^H}(F^5)^H)\omega^V \\ &= -a^2(\nabla_X(\omega\circ F))^V + a^2((\nabla_X\omega)\circ F)^V \\ &= -a^2(\omega\circ(\nabla_X F))^V \end{aligned}$$

*iv)*

$$\begin{aligned} \phi_{(F^5)^H\omega^V}\theta^V &= -(L_{\theta^V}(F^5)^H)\omega^V \\ &= -L_{\theta^V}(F^5)^H\omega^V + (F^5)^H L_{\theta^V}\omega^V \\ &= 0 \end{aligned}$$

□

**Definition 22.** *A Sasakian metric ${}^S g$ is defined on $T^*(M^n)$ by the three equations*

$$ {}^S g(\omega^V,\theta^V) = (g^{-1}(\omega,\theta))^V = g^{-1}(\omega,\theta)o\pi, \tag{26}$$

$$ {}^S g(\omega^V, Y^H) = 0, \tag{27}$$

$$ {}^S g(X^H, Y^H) = (g(X,Y))^V = g(X,Y)\circ\pi. \tag{28}$$

For each $x \in M^n$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T^*_x(M^n)$ by

$$ g^{-1}(\omega,\theta) = g^{ij}\omega_i\theta_j, \tag{29}$$

where $X, Y \in \Im^1_0(M^n)$ and $\omega,\theta \in \Im^0_1(M^n)$. Since any tensor field of type $(0,2)$ on $T^*(M^n)$ is completely determined by its action on vector fields of type $X^H$ and $\omega^V$ (see [17], p.280), it follows that ${}^S g$ is completely determined by equations (26), (27) and (28).

**Theorem 23.** *Let $(T^*(M^n), {}^S g)$ be the cotangent bundle equipped with Sasakian metric ${}^S g$ and a tensor field $(F^5)^H$ of type $(1,1)$ defined by (25). Sasakian metric ${}^S g$ is pure with respect to $(F^5)^H$ if $F = a^2 I$ ($I$ = identity tensor field of type $(1,1)$).*

*Proof.* We put

$$S(\tilde{X},\tilde{Y}) = ^S g((F^5)^H\tilde{X},\tilde{Y}) - ^S g(\tilde{X},(F^5)^H\tilde{Y}).$$

If $S(\tilde{X},\tilde{Y}) = 0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form $\omega^V, \theta^V$ or $X^H, Y^H$, then $S = 0$. By virtue of $(F^5)^H - a^2F^H = 0$ and (26), (27), (28), we get

*i)*

$$\begin{aligned} S(\omega^V,\theta^V) &= {}^S g((F^5)^H\omega^V,\theta^V) - ^S g(\omega^V,(F^5)^H\theta^V) \\ &= {}^S g((a^2F)^H\omega^V,\theta^V) - ^S g(\omega^V,(a^2F)^H\theta^V) \\ &= a^2({}^S g((\omega\circ F)^V,\theta^V) - ^S g(\omega^V,(\theta\circ F)^V)). \end{aligned}$$

*ii)*

$$\begin{aligned} S(X^H,\theta^V) &= {}^S g((F^5)^H X^H,\theta^V) - ^S g(X^H,(F^5)^H\theta^V) \\ &= {}^S g((a^2F)^H X^H,\theta^V) - ^S g(X^H,(a^2F)^H\theta^V) \\ &= a^2({}^S g((FX)^H,\theta^V) - ^S g(X^H,(\omega\circ F)^V)) \\ &= 0. \end{aligned}$$

*iii)*

$$\begin{aligned} S(X^H,Y^H) &= {}^S g((F^5)^H X^H,Y^H) - ^S g(X^H,(F^5)^H Y^H) \\ &= {}^S g((a^2F)^H X^H,Y^H) - ^S g(X^H,(a^2F)^H Y^H) \\ &= a^2({}^S g((FX)^H,Y^H) - ^S g(X^H,(FY)^H)). \end{aligned}$$

Thus, $F = a^2 I$, then ${}^S g$ is pure with respect to $(F^5)^H$. □

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