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ON THE (k+1)-DIMENSIONAL SPACE-LIKE RULED SURFACES IN THE MINKOWSKI SPACE R_1^n

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ABSTRACT

In this paper, space-like ruled surfaces in the Minkowski n-space are defined. Moreover, some results and theorems related with the Riemannian curvature K and mean curvature vector H of the (k+1)-dimensional space-like ruled surface are given.

1. INTRODUCTION

We shall assume throughout this paper all manifolds, maps, vector fields, etc... are differentiable of class C^∞. Consider a general Semi-Riemannian submanifold M of dimension (k+1) of the Minkowski space R_1^n ($n \geq 3$). If $\bar{D}$ (resp. D) is the Levi-Civita connection of R_1^n (resp. M) and if X and Y are tangent vector fileds of M, then we find by decomposing $\bar{D}_X Y$ into a tangent and normal component

$$\bar{D}_X Y = D_X Y + V(X,Y) \tag{1.1}$$

V(X,Y) is a normal vector filed on M and is symmetric in X and Y. A vector field Z of M P is called an asymptotic vector field if V(Z,Z) = 0. A curve on M is an asymptotic curve if its tangent vector field T is an asymptotic vector field along the curve [1].

Let ξ be a normal vector filed on M, then, by decomposing $\bar{D}_X \xi$ in a tangent and a normal component, we find that

$$\bar{D}_\xi x = - A_\xi(X) + D_X^\perp \xi \tag{1.2}$$

which determines, at each point, a self-adjoint linear map, where $D^\perp$ is a metric connection in the normal bundle $\chi^\perp(M)$. We use the same notation A_ξ to show the linear map and the matrix of the linear map. A normal vector filed ξ on $\chi(M)$ is called parallel on the normal bundle $\chi^{\perp}(M)$ if $D^{\perp}_X\xi = 0$ for each vector field X. A subbundle F of $\chi^{\perp}(M)$ is said to be parallel in $\chi^{\perp}(M)$ if for each vector field η of F and each vector field X of $\chi^{\perp}(M)$, $D^{\perp}_X\eta$ is again a vector field of F, [2].

Suppose that X and Y are vector fields on $\chi(M)$ while ξ is a normal vector field, then, if the standard metric tensor of R^n_1 is denoted by $\langle , \rangle$,

$$\langle \bar{D}_x Y, \xi \rangle = \langle V(X,Y), \xi \rangle = \langle A_\xi(X), Y \rangle. \quad (1.3)$$

If $\xi_1, \xi_2, \ldots, \xi_{n-k-1}$ constitute an orthonormal base field of the normal bundle $\chi^{\perp}(M)$, then we put

$$\langle V(X,Y), \xi_j \rangle = V_j(X,Y) \quad (1.4)$$

or

$$V(X,Y) = \sum_{j=1}^{n-m} V(X,Y)\xi_j.$$

The mean curvature vector H of M at the point P is given by

$$H = \sum_{j=1}^{n-k-1} \frac{\operatorname{tr} A_{\xi_j}}{\text{boy } M} \xi_j. \quad (1.5)$$

$\|H\|$ shows the mean curvature. If H = 0 at each point P of M, then M is said to be minimal, [1]. Let R^n_1 be a Minkowski space in the Levi-Civita connection D. The function,

$$\bar{R}: \chi(R^n_1) \times \chi(R^n_1) \times \chi(R^n_1) \to \chi(R^n_1)$$

given by

$$\bar{R}(X,Y)Z = \bar{D}_{[X,Y]}Z - \bar{D}_X\bar{D}_Y Z + \bar{D}_Y\bar{D}_X Z \quad (1.6)$$

is a (1,3) tensor field on $\chi(R^n_1)$ called the curvature tensor field of R^n_1. If $X,Y \in T_{R^n_1}(p)$ the linear operator

$$R_{XY}: T_{R^n_1}(p) \to T_{R^n_1}(p)$$

sending each Z to R_{XY} is called a curvature operator, [3]. The function

$$R: T_M(\rho) \times T_M(\rho) \times T_M(\rho) \times T_M(\rho) \rightarrow R$$

given by

$$R(X_1,X_2,X_3,X_4) = \langle X_1,R(X_3,X_4)X_2\rangle \tag{1.7}$$

is a covarient tensor field of order 4 on $\chi(M)$ called the Riemannian curvature tensor field of M.

The function given by (1.7), at each point P, is called the Riemannian curvature and we denote

$$K(P) = \langle X,R(X,Y)Y\rangle. \tag{1.8}$$

If V is the second fundemental form of Semi-Riemannian manifold M, then we obtain

$$\langle X,R(X,Y)Y\rangle = \langle V(X,Y),V(X,Y)\rangle - \langle V(X,X),V(Y,Y)\rangle. \tag{1.9}$$

A two-dimensional subspace π of the tangent space $T_M(\rho)$ is called a tangent plane to M at P. For tangent vectors $X_p,Y_p \in T_M(\rho)$ defined by

$$K(X_p,Y_p) = \frac{\langle R(X_p,Y_p)X_p,Y_p\rangle}{\langle X_p,X_p\rangle\langle Y_p,Y_p\rangle - \langle X_p,Y_p\rangle^2} \tag{1.10}$$

is caled the sectional curvature of M at P, [3].

2. (k+1)-DIMENSIONAL RULED SURFACE IN R_1^n

Let $\{e_1(s), e_2(s), ..., e_k(s)\}$ be a system of orthonormal vector fields, which is defined for each point of a space-like curve α in the n-dimensional Minkowski space R_1^n. This system spannes a k-dimensional subspace of the tangent space $T_{R_1^n}(\alpha(s))$ at each point. This subspace is denoted by $E_k(s)$, that is

$$E_k(s) = Sp\{e_1(s), e_2(s), \ldots , e_k(s)\}.$$

We get a (k+1)-dimensional surface in R_1^n if the subspace $E_k(s)$ moves along the curve α. We call this space a (k+1)-dimensional generalized space-like ruled surface in the n-dimensional Minkowski Space R_1^n . A parametrization of this ruled surface is

$$\phi(s,u_1,..,u_k) = \alpha(s) + \sum_{i=1}^{k} u_i e_i(s) \tag{2.1}$$

Throughout this paper $E_k(s) = Sp\{e_1(s), e_2(s), ..., e_k(s)\}$ denotes a subspace which is a space-like subspace, α is a space-like curve which is an orthogonal trajectory of the k-dimensional generating space $E_k(s)$ $(k \geq 1)$. We denote this ruled surface by M. If we take the partial derivate of ϕ we get

$$\phi_s = \alpha(s) + \sum_{i=1}^{k} u_i e_i(s) \ ,$$

$$\phi_{u_i} = e_i(s) \ , \ 1 \leq i \leq k \ .$$

Throughout our paper we assume that the system

$$\left\{ \alpha(s) + \sum_{i=1}^{k} u_i e_i(s) \ , e_1, ..., e_k \right\}$$

is linear independent.

Let $\{e_0, e_1, ..., e_k\}$ be an orthonormal base of $\chi(M)$ (i.e. e_0 is the unit tangent vector of the orthonogal trajectories of the generating spaces). Suppose that timelike subspace $\{\xi_1, \xi_2, ..., \xi_{n-k-1}\}$ is an orthonormal base field of $T_M^{\perp}(p)$. Then $\{e_0, e_1, ..., e_k, \xi_1, \xi_2, ..., \xi_{n-k-1}\}$ is a base field of $T_{R_1^n}(p)$ at the point $P \in R_1^n$. Then we have

$$\langle e_0,e_0\rangle = 1 \ , \langle e_i,e_0\rangle = 0 \ , \langle e_i,e_j\rangle = \delta_{ij} = \begin{cases} 0 \ , & i \neq j \\ 1 \ , & i = j \end{cases} \ , \ \langle \xi_j,\xi_j\rangle = \varepsilon_j = \pm 1 \ . \qquad (2.2)$$

Then M is said to be m-developable if

$$\text{rank } [e_0, e_1, ..., e_k, \overline{D}_{e_0} e_1, ..., \overline{D}_{e_0} e_k] = 2k - m \qquad (2.3)$$

at each point $P \in M$. If $m = -1$, then the space-like ruled surface M is called non-developable; if $m = k-1$, then M is said to be total developable, [4].

Denote of $\overline{D}$ the Levi-Civita connection of the Minkowski space R_1^n. For the orthonormal base $\{e_1, ..., e_k\}$ of the generating space $E_k(s)$, we observe that

$$\overline{D}_{e_i} e_j = 0 \ , \ 1 \leq i \ , j \leq k \ .$$

Hence, if V denotes the second fundamental form of R_1^n, we must have

$$V(e_j,e_j) = 0 \ , \ 1 \leq i \ , j \leq k \ . \qquad (2.4)$$

Let $X = \sum_{i=1}^{k} a_i e_i + a e_0$ and $Y = \sum_{i=1}^{k} b_i e_i + b e_0$ be two vector fields of $\chi(M)$. So we find that

$$V(X,Y) = \sum_{i=1}^{k} (a_i b + b_i a) V(e_0,e_i) + ab V(e_0,e_0) \ . \tag{2.5}$$

The normal subbundle of $\chi^{\perp}(M)$ spanned by the normal fields $V(e_0,e_i)$, $1 \leq i \leq k$ is denoted by F.

Theorem 2.1. M is m-developable iff the normal subbundle F is (k-m-1)-dimensional.

Proof. Suppose that we have (2.3). Because of (1.1) we can write

$$\bar{D}_{e_0} e_i = D_{e_0} e_i + V(e_0,e_i) \ , \ 1 \leq i \leq k \ .$$

But $D_{e_0} e_i$ is a linear combination of the vector fields $\{e_0, e_1, ..., e_k\}$ and so we may replace the fields $\bar{D}_{e_0} e_i$ by $V(e_0,e_i)$ in (2.3). Now, the tangent space spanned by $e_0, e_1, ..., e_k$ is at each point normal to F and thus we find k+1+dim F = 2k-m or dim F = k-m-1, which completes the proof of the theorem.

From (2.2) we observe that $\bar{D}_{e_i} e_0 \perp e_0$ and $\bar{D}_{e_i} e_0 \perp e_j$. This means that $\bar{D}_{e_i} e_0$ is a normal vector field or

$$\bar{D}_{e_i} e_0 = V(e_i, e_0) \ , \ 1 \leq i \leq k \ . \tag{2.6}$$

Suppose that $\{\xi_1, \xi_2, ..., \xi_{n-k-1}\}$ is an orthonormal base field of the normal bundle $\chi^{\perp}(M)$, then we have the following Weingarten equations

$$\begin{aligned}
\bar{D}_{e_0} \xi_j &= a^j_{00} e_0 + \sum_{r=1}^{k} a^j_{or} e_r + \sum_{s=1}^{n-k-1} b^j_{os} \xi_s \ , \ 1 \leq j \leq n-k-1 \ , \\
\bar{D}_{e_1} \xi_j &= a^j_{10} e_0 + \sum_{r=1}^{k} a^j_{1r} e_r + \sum_{s=1}^{n-k-1} b^j_{1s} \xi_s \\
\cdots\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots \\
\bar{D}_{e_k} \xi_j &= a^j_{k0} e_0 + \sum_{r=1}^{k} a^j_{kr} e_r + \sum_{s=1}^{n-k-1} b^j_{ks} \xi_s \ .
\end{aligned} \tag{2.7}$$

These equation together with (2.4) and (1.3) yield

$$\begin{aligned}
a^j_{or} &= a^j_{ro} \\
a^j_{ir} &= 0 \qquad 1 \leq j \leq n-k-1 \ , \ 1 \leq i,r \leq k \ .
\end{aligned} \tag{2.8}$$

So the matrix of A_{ξ_j} has the form

$$A_{\xi_j} = -\begin{bmatrix} a^j_{00} & a^j_{01} & \cdots & a^j_{0k} \\ a^j_{01} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a^j_{0k} & 0 & \cdots & 0 \end{bmatrix} \quad (2.9)$$

and this means det $A_{\xi_j} = 0$ if $k \geq 2$, from which we have:

Corollary 2.2. If $k \geq 2$, then the Lipschitz-Killing curvature of M is zero at each point in each normal direction.

Corollary 2.3. The matrix A_{ξ_j} of the shape operator of M is of the form (2.9) and is symmetric.

Because of the equations (2.7), we get

$$a^j_{0i} = \langle \bar{D}_{e_i}\xi_j, e_0\rangle = -\langle \xi_j, \bar{D}_{e_i}e_0\rangle \quad (2.10)$$

and from (2.6) together with (2.10) we receive

$$\bar{D}_{e_i}e_0 = V(e_i,e_0) + \sum_{j=1}^{n-k-1} \varepsilon_j\langle \xi_j, \bar{D}_{e_i}e_0\rangle\, \xi_j = -\sum_{j=1}^{n-k-1} \varepsilon_j a^j_{0i}\xi_j \,. \quad (2.11)$$

Theorem 2.4. Let M be a (k+1)-dimensional space-like ruled surface of $\mathbb{R}^n_1$. Then the Riemannian curvature of M in the two-dimensional direction spanned by e_1 and e_0 is given by

$$K(e_i, e_0) = \langle \bar{D}_{e_i}e_0, \bar{D}_{e_i}e_0\rangle, \quad 1 \leq i \leq k \,.$$

Proof: Let R be the Riemannian curvature tensor field of M. From (1.10) and (2.2), we find

$$K(e_i,e_0) = \langle R(e_i,e_0)e_i,e_0\rangle \,. \quad (2.12)$$

If we connect (2.12) with (1.9) and (2.4), then we get

$$K(e_i,e_0) = \langle V(e_i,e_0),\, V(e_i,e_0)\rangle$$

or

$$K(e_i, e_0) = \langle \bar{D}_{e_i}e_0, \bar{D}_{e_i}e_0\rangle \,. \quad (2.13)$$

From (2.11) and (2.13) we receive the following corollary.

Corollary 2.5. The Riemannian curvature of M in the two-dimensional direction spanned by e_i and e_0 can be written with the entries of the Matrix A_{ξ_j} as follows

$$K(e_i,e_0) = \sum_{j=1}^{n-k-1} \varepsilon_j\left(a_{0i}^{j}\right)^2 , \quad 1 \le i \le k , \quad \varepsilon_j = \langle\xi_j,\xi_j\rangle = \pm 1 . \tag{2.14}$$

It is easy to see that (1.10) and (2.4) gives

$$K(e_i,e_j) = 0 , \; 1 \le i , j \le k . \tag{2.15}$$

Theorem 2.6. Let M be a (k+1)-dimensional space-like ruled surface in R_1^n and e_0 be the tangent vector field of the base curve of M. The mean curvature is

$$H = \varepsilon_j \frac{V(e_0,e_0)}{k+1} , \quad \varepsilon_j = \langle\xi_j,\xi_j\rangle = \pm 1 .$$

Proof: From (1.5) we known that

$$H = \sum_{j=1}^{n-k-1} \frac{trA_{\xi_j}}{k+1}\xi_j . \tag{2.16}$$

Using (1.4), we can write

$$V(e_0,e_0) = \sum_{j=1}^{n-k-1} \xi_j\langle\bar{D}_{e_0}e_0,\xi_j\rangle\xi_j , \quad \varepsilon_j = \langle\xi_j,\xi_j\rangle = \mp 1$$

Because of the last equation and equation (2.7), we get

$$V(e_0,e_0) = -\sum_{j=1}^{n-k-1} \xi_j \left(a_{00}^{j}\right) \xi_j \tag{2.17}$$

For the matrix A_{ξ_j} given (2.9) we find

$$trA_{\xi_j} = - a_{00}^{j} . \tag{2.18}$$

If we substitute (2.17) and (2.18) in (2.16), we observe that

$$H = \varepsilon_j \frac{V(e_0,e_0)}{k+1} , \quad \varepsilon_j = \langle\xi_j,\xi_j\rangle = \mp 1$$

From Theorem 2.6 we have immediately:

Corollary 2.7. The space-like ruled surface M is minimal iff each orthogonal trajectory of the generating spaces is an asymptotic line of M.

Teorem 2.8. If the (k+1)-dimensional m-developable space-like ruled surface M is minimal, then M is necessarily a submanifold of an R_1^{2k-m}.

Proof: Because of Theorem 2.1, we already know that the codimension of M is at least k-m-1 we have two cases:

1) First, suppose that the normal subbundle F is zero-dimensional. Thus

$V(e_0,e_i) = 0$, $1 \leq i \leq k$.

Because of the second fundemental form V is symmetric, we find

$V(e_i,e_0) = 0$, $1 \leq i \leq k$.

If we substitute $V(e_0,e_0) = 0$ and $V(e_i,e_0) = 0$, $1 \leq i \leq k$ in (2.5), we get

$V(X,Y) = 0$.

This says that the space-like ruled surface M must be totally geodesic, i.e. M is part of a (k+1)-dimensional linear space.

2) Next assume that the normal subbundle F is not zero. Consider an orthonormal base field $\xi_1, \xi_2, ..., \xi_{n-k-1}$ of $\chi^{\perp}(M)$ such that $\xi_1, \xi_2, ..., \xi_{k-m-1}$ is a base field of the normal subbundle F. Consider the equations (2.7) in this case. Since $\langle V(e_i,e_0),\xi_j\rangle = -a^j_{0i}$, $1 \leq i \leq k$, $1 \leq j \leq n\text{-}k\text{-}1$ we have immediately

$$a^j_{0i} = 0 , \quad 1 \leq i \leq k, \quad k\text{-}m \leq j \leq n - k - 1 . \tag{2.19}$$

But $H = 0$ and hence tr $A_{\xi_j} = 0$, $1 \leq j \leq n\text{-}k\text{-}1$ and so we get

$$A_{\xi_{k-m}} = ... = A_{\xi_{n-k-1}} = 0 . \tag{2.20}$$

If we set $V(X,Y) = \sum_{j=1}^{n-k-1} V_j(X,Y)\xi_j$ for each two vector fields X and Y of M, then we find

$$V_{k-m}(X,Y) = ... = V_{n-k-1}(X,Y) = 0 . \tag{2.21}$$

If $\bar{R}$ is the curvature tensor of R^n_1 and if X, Y, Z are vector fields of $\chi(M)$, then the Codazzi equation says

$$\left(\bar{R}(X,Y)Z\right)^{\perp} = \sum_{j=1}^{n-k-1}\left\{\left(D_Y V_j\right)(X,Z) - \left(D_X V_j\right)(Y,Z)\right\}\xi_j + \sum_{j=1}^{n-k-1} V_j(X,Z)D^{\perp}_Y\xi_j - \sum_{j=1}^{n-k-1} V_j(Y,Z)D^{\perp}_X\xi_j . \tag{2.22}$$

Put

$$D^{\perp}_{e_i}\xi_{\ell} = \sum_{h=1}^{n-k-1} C^h_{i\ell}\xi_h + \sum_{r=k-m}^{n-k-1} C^r_{i\ell}\xi_r, \quad 1 \leq \ell \leq k\text{-}m\text{-}1,\ 1 \leq i \leq k \tag{2.23}$$

Then, from (2.21) and (2.22), we have

$$\left(\bar{R}(e_i,e_0)e_s\right)^{\perp} = \sum_{\ell=1}^{k-m-1}\left\{\left(D_{e_0}V_\ell\right)(e_i,e_s) - \left(D_{e_i}V_\ell\right)(e_0,e_s)\right\}\xi_\ell$$

$$- \sum_{\ell=1}^{k-m-1} V_\ell(e_0,e_s)D^{\perp}_{e_i}\xi_\ell + \sum_{\ell=1}^{k-m-1} V_\ell(e_0,e_s)D^{\perp}_{e_0}\xi_\ell = 0, \quad 1 \le i,s \le k . \qquad (2.24)$$

But $V(e_i,e_s) = 0$, $1 \le i,s \le k$ and so we find from (2.23) and (2.24)

$$\sum_{\ell=1}^{k-m-1} C^r_{i\ell}V_\ell(e_0,e_s) = 0, \quad 1 \le i,s \le k, \ k\text{-}m \le r \le n\text{-}k\text{-}1 . \qquad (2.25)$$

Now, fix in this expression i and r and let s be variable, then we find a system of k homogeneous linear equations with k-m-1 unknows $C^r_{i\ell}$. The matrix of this system is

$$[V_\ell(e_0,e_s)], \ 1 \le \ell \le k\text{-}m\text{-}1, \ 1 \le s \le k .$$

and its rank is at each point of M equal to k-m-1 because space-like ruled surface M is m-developable. So, it is easy to see that (2.25) gives

$$C^r_{i\ell} = 0, \quad 1 \le i \le k, \ 1 \le \ell \le k\text{-}m\text{-}1, \ k\text{-}m \le r \le n\text{-}k\text{-}1 . \qquad (2.26)$$

We also have

$$\left(\bar{R}(e_0,e_i)e_0\right)^{\perp} = \sum_{\ell=1}^{k-m-1}\left\{\left(D_{e_i}V_\ell\right)(e_0,e_0) - \left(D_{e_0}V_\ell\right)(e_i,e_0)\right\}\xi_\ell$$

$$+ \sum_{\ell=1}^{k-m-1} V_\ell(e_0,e_0)D^{\perp}_{e_i}\xi_\ell - \sum_{l=1}^{k-m-1} V_\ell(e_i,e_0)D^{\perp}_{e_0}\xi_\ell = 0. \qquad (2.27)$$

But $V(e_0,e_0) = 0$, and if we put

$$D^{\perp}_{e_0}\xi_\ell = \sum_{h=1}^{k-m-1} C^h_\ell\xi_h + \sum_{r=k-m}^{n-k-1} C^r_\ell\xi_r, \quad 1 \le \ell \le k\text{-}m\text{-}1$$

we find from (2.27)

$$\sum_{\ell=1}^{k-m-1} C^r_\ell V_\ell(e_i,e_0), \quad 1 \le i \le k, \quad k\text{-}m \le r \le n\text{-}k\text{-}1 .$$

This gives analogously

$$C^r_\ell = 0, \quad 1 \le \ell \le k\text{-}m\text{-}1, \quad k\text{-}m \le r \le n\text{-}k\text{-}1 . \qquad (2.28)$$

Now, equation (2.26) together with equation (2.28) says that for each unit normal field η in F and for each vector field X of M, $D^{\perp}_X\eta$ has no component in the complementary subbundle $F^{\perp}$ i.e. the normal subbundle F is parallel. If we identify all the tangent spaces of R^n_1 with R^n_1 itself, then, since F is parallel and because of equation (2.20), we see that the (2k-m)-dimensional subspaces of R^n_1 spanned at each point of M by the tangent space and the normal space F, are independent of the choice of the point P of M, which was to be proved.

Theorem 2.9. If the mean curvature vector $H \neq 0$ of the (k+1)-dimensional m-developable space-like ruled surface M is at each point of M a vector of the normal subbundle F, then M is neccessarily a submanifold of an R_1^{2k-m}.

Proof: Take again an orthonormal base field $\xi_1, \xi_2, ..., \xi_{n-k-1}$ such that $\xi_1, \xi_2, ..., \xi_{k-m-1}$ is a base field of F. Then, since $\varepsilon_j(k+1)H = V(e_0,e_0) \in F$ we have again

$$V_{k-m}(X,Y) = ... = V_{n-k-1}(X,Y) = 0$$

for each two vector fields X and Y of $\chi(M)$.

Next, if we have (2.23), then we find from (2.24) again (2.26). Moreover, since the vector fields $D^{\perp}_{e_i}\xi_{\ell}$, $1 \leq i \leq k$, $1 \leq \ell \leq k\text{-}m\text{-}1$ have no components in the complementary subbundle $F^{\perp}$, we find because of (2.27) again (2.28) and this completes the proof.

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