TENSOR PRODUCT IMMERSIONS WITH TOTALLY REDUCIBLE FOCAL SET

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ABSTRACT

In [1], Carter and the author introduced the idea of an immersion $f : M \to R^n$ with *totally reducible focal set* (TRFS). Such an immersion has the property that, for all $p \in M$, the focal set with base p is a union of hyperplanes in the normal plane to $f(M)$ at $f(p)$. In this study we show that if $f : S^1 \to R^2$ and $g : S^1 \to R^3$ are two isometric immersions then the tensor product immersions $f \otimes f$ and $f \otimes g$ have TRFS property.

Keywords: Immersions, Focal set, Totally reducible focal set, Tensor product immersion

ÖZET

[1] de Carter ve yazar, $f : M \to R^n$ tamamen indirgenebilen focal cümleye (TRFS) sahip immersiyon tanımını verdi. Bu immersiyon, her $p \in M$ için, p ye bağlı focal cümle, $f(p)$ de $f(M)$ ye normal düzlemdeki hiperyüzeylerin bir birleşimidir. Bu çalışmada, eğer $f : S^1 \to R^2$ ve $g : S^1 \to R^3$ iki izometrik immersiyon ise $f \otimes f$ ve $f \otimes g$ tensor çarpım immersiyonlarınında TRFS şartını sağladıkları gösterildi.

Anahtar Kelimeler: İmmersiyonlar, Focal Cümle, Tamamen indirgenebilen focal cümle, Tensor çarpım immersiyonu

1. INTRODUCTION

Let $f: M \rightarrow R^n$ be a smooth immersion of connected smooth m-dimensional manifold without boundary into Euclidean n-space. For each $p \in M$, the focal set of f with base p is an algebraic variety. In this study we consider immersions for which this variety is a union of hyperplanes.

For $p \in M$, let U be a neighborhood of p in M such that $f|_U : U \rightarrow R^n$ is an embedding. Let $\upsilon_f(p)$ denote the (n-m)-plane which is normal to $f(U)$ at $f(p)$. Then the total space of normal bundle is $N_f = \{(p,x) \in M \times R^n \mid x \in \upsilon_f(p)\}$. The projection map $\eta_f : N_f \rightarrow R^n$ is defined by $\eta_f(p,x) = x$ and the set of focal points with base p is $\Gamma_f(p) = \{p \in R^n \mid (p,x) \text{ is a singularity of } \eta_f\}$. The focal set of f which denoted by $\Gamma_f = \bigcup_{p \in M} \Gamma_f(p)$ is the image by η_f. For each $p \in M$, $\Gamma_f(p)$ is a real algebraic variety in $\upsilon_f(p)$ which can be defined as the zeros of polynomial on $\upsilon_f(p)$ of degree $\leq m$. The focal point of f has weight (multiplicity) k if $rank(Jac\,\eta_f) = n-k$ [3].

Definition 1. The immersion $f: M \rightarrow R^n$ has totally reducible focal set (TRFS) property if for all $p \in M$, $\Gamma_f(p)$ can be defined as the zeros of real polynomial which is a product of real linear factors [1].

Thus each irreducible component of $\Gamma_f(p)$ is an affine in $\upsilon_f(p)$, and $\Gamma_f(p)$ is a union of $(n-m-1)$-planes (possible $\Gamma_f(p) = \Phi$). There are other ways of describing this property. It is shown in ([5], [7], and [8]) that f has TRFS property if and only if f has flat normal bundle, where M is thought of as a Riemannian manifold with metric g induced from R^n. We will give explicit ways of constructing immersions with TRFS property.

In calculating focal sets it is often easiest to work with distance functions. For $x \in R^n$ the distance function $L_x : M \rightarrow R$ is defined by $L_x(p) = \|x - f(p)\|^2$. Then $x \in R^n$ is a focal point of f with base p if and only if p is a degenerate critical point of L_x, where at p, $\frac{\partial L_x}{\partial p_i} = 0$ and $\left[\frac{\partial^2 L_x}{\partial p_i \partial p_j}\right]$ is singular for $i, j = 1, 2, \cdots, m$, ([6]).

In this study it has been shown that if $f: S^1 \rightarrow R^2$ and $g: S^1 \rightarrow R^3$ are two isometric immersions then the tensor product immersions $f \otimes f$ and $f \otimes g$ have TRFS property.

2. TENSOR PRODUCT IMMERSIONS

Let us recall definitions and results of [2]. Let M and N be two differentiable manifolds and $f: M \rightarrow R^n$, $g: N \rightarrow R^d$ two immersions. The direct sum and tensor product maps

$$f \oplus g : M \times N \rightarrow R^{n+d},$$
$$f \otimes g : M \times N \rightarrow R^{nd}$$

are defined by

$$(f \oplus g)(p,q) = (f(p), g(p)),$$
$$(f \otimes g)(p,q) = f(p) \otimes g(p).$$

The necessary and sufficient conditions for $f \otimes g$ to be an immersion were obtained in [3]. It is also proved there that the pairing $(\oplus, \otimes)$ determines a structure of a semiring on the set of classes of differentiable manifolds transversally immersed in Euclidean spaces, modulo orthogonal transformations. Some subsemirings were studied in [4].

If $n = m+1$, $G_f(p)$ consists of a finite number of points so, trivially, any immersion $f: M^m \to R^{m+1}$ has TRFS property. Thus especially an immersion $f: S^1 \to R^2$ has TRFS property. Also every immersions $f: S^1 \to R^n$, $n \geq 3$, has TRFS property [1].

The following results are well known.

Theorem 1. [1] Let $f: M \to R^n$ and $g: N \to R^d$ be immersions with TRFS property. Then $f \times g: M \times N \to R^{n+d}$ defined by $(f \times g)(p,q) = (f(p), g(p))$ has TRFS property.

Theorem 2. [1] If $f: M \to R^n$ has TRFS property and $g: M \to R^{n+k}$ is defined by $g(p) = (f(p), t) \in R^n x R^k$. Then g has TRFS property.

We prove the following results.

Theorem 3. If $f: S^1 \to R^2$ is an isometric immersion then the tensor product immersion $f \otimes f: S^1 \times S^1 \to R^4$ has TRFS property.

Proof. The tensor product immersion $h = f \otimes f: S^1 \times S^1 \to R^4$ is defined by

$$h(\theta,\varphi) = (f \otimes f)(\theta,\varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta\cos\varphi, \sin\theta\sin\varphi),$$

$(\theta, \varphi \in R \bmod 2\pi)$. Let $x \in R^4$ and $L_x(\theta,\varphi) = \sum_{i=1}^{4} (x_i - h_i(\theta,\varphi))^2$. Then

$$\tfrac{\partial L_x}{\partial \theta} = x_1 \sin\theta\cos\varphi + x_2 \sin\theta\sin\varphi - x_3 \cos\theta\cos\varphi - x_4 \cos\theta\sin\varphi = 0, \qquad (1)$$
$$\tfrac{\partial L_x}{\partial \varphi} = x_1 \cos\theta\sin\varphi - x_2 \cos\theta\cos\varphi + x_3 \sin\theta\sin\varphi - x_4 \sin\theta\cos\varphi = 0, \qquad (2)$$

and

$$A = \tfrac{\partial^2 L_x}{\partial \theta^2} = \tfrac{\partial^2 L_x}{\partial \varphi^2} = x_1 \cos\theta\cos\varphi + x_2 \cos\theta\sin\varphi + x_3 \sin\theta\cos\varphi + x_4 \sin\theta\sin\varphi,$$
$$B = \tfrac{\partial^2 L_x}{\partial \theta \partial \varphi} = -x_1 \sin\theta\sin\varphi + x_2 \sin\theta\cos\varphi + x_3 \cos\theta\sin\varphi - x_4 \cos\theta\cos\varphi,$$

and

$$\det H = A^2 - B^2 = 0. \qquad (3)$$

Thus $A^2 - B^2 = (A-B)(A+B) = 0$

If $A - B = 0$ then

$$\begin{aligned} &x_1(\cos\theta\cos\varphi + \sin\theta\sin\varphi) + x_2(\cos\theta\sin\varphi - \sin\theta\cos\varphi) \\ &+x_3(\sin\theta\cos\varphi - \cos\theta\sin\varphi) + x_4(\sin\theta\sin\varphi + \cos\theta\cos\varphi) = 0. \end{aligned} \qquad (4)$$

If $A + B = 0$ then

$$
\begin{aligned}
&x_1\left(\cos\theta\cos\varphi-\sin\theta\sin\varphi\right)+x_2\left(\cos\theta\sin\varphi+\sin\theta\cos\varphi\right)\\
&+x_3\left(\sin\theta\cos\varphi+\cos\theta\sin\varphi\right)+x_4\left(\sin\theta\sin\varphi-\cos\theta\cos\varphi\right)=0.
\end{aligned}
\tag{5}
$$

Therefore using (1), (2) and (4) we get

$$
{\Gamma^1}_h(\theta,\varphi)=\left\{\left(x_1=\lambda x_4, x_2=-\lambda x_4, x_3=x_4, x_4\right)\,\middle|\,\lambda=\tfrac{(\tan\theta\tan\varphi-1)}{\tan\theta+\tan\varphi}, \tan\theta\neq-\tan\varphi\right\} \tag{6}
$$

and using (1), (2) and (5) we get

$$
{\Gamma^2}_h(\theta,\varphi)=\left\{\left(x_1=-\mu x_4, x_2=\mu x_4, x_3=-x_4, x_4\right)\,\middle|\,\mu=\tfrac{(\tan\theta\tan\varphi-1)}{\tan\theta-\tan\varphi}, \tan\theta\neq\tan\varphi\right\} \tag{7}
$$

Thus from (6) and (7) we get

$$\Gamma_h={\Gamma^1}_h(\theta,\varphi)\cup{\Gamma^2}_h(\theta,\varphi).$$

So h has TRFS property.

Remark. If $f:S^1\to R^2$ then, by Theorem 1, $f\times f:S^1\times S^1\to R^4$ has TRFS property. But in this case $\Gamma_{f\times f}=\left\{(0,0,a,b)\,\middle|\,a,b\in R\right\}\cup\left\{(c,d,0,0)\,\middle|\,c,d\in R\right\}$.

Theorem 4. If $f:S^1\to R^2$ and $g:S^1\to R^3$ are two isometric immersions then the tensor product immersion $f\otimes g:S^1\times S^1\to R^6$ has TRFS property.

Proof. Let $f:S^1\to R^2$ and $g:S^1\to R^3$ be defined by $f(q)=(\cos q,\sin q)$ and $g(j\,)=(\cos j\,,\sin j\,,k), k\,Î\,R$, respectively. The tensor product immersion $h=f\otimes g:S^1\times S^1\to R^6$ is defined by

$$h(\theta,\varphi)=(f\otimes g)(\theta,\varphi)=(\cos\theta\cos\varphi,\cos\theta\sin\varphi,\sin\theta\cos\varphi,\sin\theta\sin\varphi,k\cos\varphi,k\sin\varphi),$$

$(\theta,\varphi\in R \bmod 2\pi)$. Let $x\in R^6$ and $L_x(\theta,\varphi)=\sum_{i=1}^{6}(x_i-h_i(\theta,\varphi))^2$. Then

$$\tfrac{\partial L_x}{\partial\theta}=x_1\sin\theta\cos\varphi+x_2\sin\theta\sin\varphi-x_3\cos\theta\cos\varphi-x_4\cos\theta\sin\varphi=0, \tag{8}$$

$$\tfrac{\partial L_x}{\partial\varphi}=x_1\cos\theta\sin\varphi-x_2\cos\theta\cos\varphi+x_3\sin\theta\sin\varphi-x_4\sin\theta\cos\varphi+x_5k\sin\varphi-x_6k\cos\varphi=0, \tag{9}$$

and

$$
\begin{aligned}
A_{11}&=\tfrac{\partial^2L_x}{\partial\theta^2}=x_1\cos\theta\cos\varphi+x_2\cos\theta\sin\varphi+x_3\sin\theta\cos\varphi+x_4\sin\theta\sin\varphi,\\
A_{12}&=\tfrac{\partial^2L_x}{\partial\theta\partial\varphi}=-x_1\sin\theta\sin\varphi+x_2\sin\theta\cos\varphi+x_3\cos\theta\sin\varphi-x_4\cos\theta\cos\varphi,\\
A_{22}&=\tfrac{\partial^2L_x}{\partial\varphi^2}=x_1\cos\theta\cos\varphi+x_2\cos\theta\sin\varphi+x_3\sin\theta\cos\varphi+x_4\sin\theta\sin\varphi+x_5k\cos\varphi+x_6k\sin\varphi,
\end{aligned}
$$

and $\det H=\det\left(A_{ij}\right)=0$. $\qquad$ (10)

From (8), (9) and (10) we get either

$$
{\Gamma^1}_h(\theta,\varphi)=\left\{\begin{array}{l}\left(x_1=-\mu x_2, x_2=x_2, x_3=-\lambda\mu x_2, x_4=\lambda x_2, x_5=x_5, x_3=\mu x_5-\left(\tfrac{1+\mu}{k\cos\theta}\right)x_2\right)\\ \mid \lambda=\tan\theta\,,\ \mu=\tan\varphi\,,\ \cos\theta\neq0\,,\ \cos\varphi\neq0\end{array}\right\},
$$

or

$$
{\Gamma^2}_h(\theta,\varphi)=\left\{\left(x_1=0, x_2=0, x_3=0, x_4=x_4, x_5=x_5, x_6=-\tfrac{x_4}{k}\right)\,\middle|\,\theta=\mp\tfrac{\pi}{2}, \varphi=0\right\},
$$

or

$$\Gamma^3{}_h(\theta,\varphi)=\left\{\left(x_1=0, x_2=0, x_3=x_3, x_4=x_4, x_5=\tfrac{x_3}{k}, x_6=\tfrac{x_4}{k}\right) \Big|\ \theta=\mp\tfrac{\pi}{2}, \varphi\in R \bmod 2\pi\right\}.$$

Therefore, $\Gamma_h=\Gamma^1{}_h(\theta,\varphi)\cup\Gamma^2{}_h(\theta,\varphi)\cup\Gamma^3{}_h(\theta,\varphi)$. So h has TRFS property.

Remark. If $f:S^1\to R^2$ and $g:S^1\to R^3$ then, by Theorem 1, $f\times g:S^1\times S^1\to R^5$ has TRFS property and using Theorem 2, $k:S^1\times S^1\to R^6$ also has TRFS property. But in this case focal set of k is different then above result.

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