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Conformable Flett's theorem and Sahoo and Riedel theorem

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Abstract

Since fractional analysis has attracted considerable interest by virtue of their ability to model complex phenomena, it is crucial to investigate properties of fractional derivatives. In this research, accordingly, we first give the extension of Flett's theorem and Sahoo and Riedel theorem to conformable derivative as a variety of conformable mean value theorem.

Keywords: *Conformable derivative, Flett's theorem, Sahoo and Riedel theorem..*

Uyumlu Flett teoremi ve Sahoo ve Riedel teoremi

Öz

Karmaşık olayları modellemede kesirli analiz büyük ilgi çektiğinden, kesirli türevlerin özelliklerini araştırmak çok önemlidir. Bu çalışmada, uyumlu türev için ortalama değer teoreminin bir türü olan Flett teoremi ile Sahoo ve Riedel teoremlerini ilk kez vereceğiz.

Anahtar Kelimeler: *Uyumlu türev, Flett teoremi, Sahoo ve Riedel teoremi.*

1. Introduction and preliminaries

Fractional calculus has roots that are as old as classical calculus although it has only been recognized as a powerful tool since the 1970s, because a great number of applications emerged in finance, control theory, biological systems, complex-valued neural networks and many more [1-13]. Caputo, Riemann-Liouville, Atangana and Baleanu [14-22] present several notions of the fractional operators and these notions give impressively description representing of many real word issues. Further, Khalil et al. [23] put into place

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conformable derivative which is a depend on limit definition for real functions. This newly-operator attracts lots of researchers from many disciplines and it is significant to delve the theory of the conformable derivative in the view of their mathematical features.

Due to the prominent role played by fractional operators in modeling, as well as the aforementioned applications of fractional operators to real-world problems, we are confident we will be working on conformable derivative properties. As there is a limited amount of work in the literature on Flett's and Rolle's theorems of conformable derivatives, we will aim to obtain some results through our consideration of these theorems. For that reason, we firstly give basic concepts and fundamental theorems which we will need in the next chapters. Then, we introduce conformable Flett's and Sahoo and Riedel theorems in a detailed manner.

In 1958, Flett give a nice theorem affiliated to mean value theorem.

Theorem 1. [24] (Flett's Theorem) Let g is differentiable on $[a,b]$ and $g'(a)=g'(b)$, then a point $c\in(a,b)$ exists such that

$$g'(c)=\frac{g(c)-g(a)}{c-a}. \tag{1}$$

Also, Sahoo and Riedel give the following theorem.

Theorem 2. [25] Assume that $g:[a,b]\to\mathbb{R}$ is differentiable function, then there exists a point $c\in(a,b)$ such that

$$g(c)-g(a)=(c-a)g'(c)-\frac{1}{2}\frac{g'(b)-g'(a)}{b-a}(c-a)^2. \tag{2}$$

In the current article, we firstly recall several important definitions and then, obtain Flett's theorem and Sahoo and Riedel theorem for conformable derivative.

Definition 1. [23] Suppose that $g:[0,\infty)\to\mathbb{R}$ be a function. The "conformable derivative" of g order α is given as

$$T_\alpha(g)(x)=\lim_{h\to 0}\frac{g\left(x+hx^{1-\alpha}\right)-g(x)}{h}, \tag{3}$$

for $x>0$ and $\alpha\in(0,1)$.

If g has conformable derivative of order α, we say g is α-differentiable. Also, $T_\alpha(g)(z)$ is more often than not indicated as $g^{(\alpha)}(z)$ in order to stand for the conformable derivative.

Theorem 3. [23] If $g:[0,\infty)\to\mathbb{R}$ is α-differentiable at $x_0>0$, $\alpha\in(0,1]$, then g is continuous at x_0.

Theorem 4. [23] Assume that $\alpha\in(0,1]$ and g, h be α-differentiable at a point $x>0$. Then

1) $T_\alpha(ag+bh)=aT_\alpha(g)+bT_\alpha(h)$ for every $a,b\in\mathbb{R}$.
2) $T_\alpha(x^p)=px^{p-\alpha}$ for all $p\in\mathbb{R}$.
3) For all constant functions $g(x)=\lambda$, $T_\alpha(\lambda)=0$.
4) $T_\alpha(gh)=gT_\alpha(h)+hT_\alpha(g)$.
5) $T_\alpha\left(\dfrac{g}{h}\right)=\dfrac{hT_\alpha(g)-gT_\alpha(h)}{h^2}$.
6) If g is differentiable, $T_\alpha(h)(x)=x^{1-\alpha}\dfrac{dg}{dx}(x)$.

What is more, Khalil et al. [23] presented Rolle's theorem of conformable differentiable functions:

Theorem 5. 1(See [23] on page 68) Let a be a positive real number and $g:[a,b]\to\mathbb{R}$ be a function satisfying the following requirements

(i) The function g is continuous on $[a,b]$,
(ii) g is α-differentiable, for some $\alpha\in(0,1)$,
(iii) $g(a)=g(b)$.
Then, $c\in(a,b)$ exists like that $g^{(\alpha)}(c)=0$.

2. Conformable Flett's and Sahoo and Riedel theorems

Here, we touch on conformable Flett's and Sahoo and Riedel theorems that is a kind of Rolle's theorem of conformable derivative.

Theorem 6. (Conformable Flett's theorem) Considering that $g:[a,b]\to\mathbb{R}$ be α-differentiable function on $[a,b]$ and $g^{(\alpha)}(a)=g^{(\alpha)}(b)$. Then a point $c\in(a,b)$ exists like that

$$g^{(\alpha)}(c)=\frac{g(c)-g(a)}{\frac{1}{\alpha}c^\alpha-\frac{1}{\alpha}a^\alpha}. \tag{4}$$

Proof. While preserving generality, we may suppose that $g^{(\alpha)}(a)=g^{(\alpha)}(b)=0$. If $g^{(\alpha)}(a)=g^{(\alpha)}(b)\neq 0$, then we should use the function $g(x)-\frac{1}{\alpha}x^{\alpha}g^{(\alpha)}(a)$. Suppose that $h:[a,b]\to\mathbb{R}$ be a function of the following form

$$h(x)=\begin{cases}\dfrac{g(x)-g(a)}{\frac{1}{\alpha}x^{\alpha}-\frac{1}{\alpha}a^{\alpha}}, & x\in(a,b] \\ g^{(\alpha)}(a), & x=a\end{cases}. \tag{5}$$

It is evident that h is continuous on $[a,b]$ and $\alpha-$differentiable function on (a,b). Benefiting from Eq. (5), we get for every $x\in(a,b)$

$$\begin{aligned} h^{(\alpha)}(x) &= \frac{g^{(\alpha)}(x)\left(\frac{1}{\alpha}x^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)-\left(g(x)-g(a)\right)}{\left(\frac{1}{\alpha}x^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{2}} \\ &= -\frac{g(x)-g(a)}{\left(\frac{1}{\alpha}x^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{2}}+\frac{g^{(\alpha)}(x)}{\left(\frac{1}{\alpha}x^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)}. \end{aligned} \tag{6}$$

In order to prove the theorem, it is sufficient to gain a point $c\in(a,b)$ like that $h^{(\alpha)}(c)=0$. From the definition of $h(x)$, it is clear that $h(a)=0$.

If $h(b)=0$, Rolle's theorem for conformable differentiable functions in [23] is the basis for the proof. Suppose that $h(b)>0$. $g^{(\alpha)}(b)=0$ and we write $h(b)=\frac{g(b)-g(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}$ in the Eq. (6), we achieve

$$h^{(\alpha)}(b)=-\frac{h(b)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}<0. \tag{7}$$

Because $h^{(\alpha)}(b)<0$ and h is continuous, there is a point $x_1\in(a,b)$ such that

$$h(x_1)>h(b). \tag{8}$$

So, we get

$$h(a)<h(b)<h(x_1). \tag{9}$$

Since h is continuous on $[a,x_1]$ and considering Eq. (9), from intermediate value theorem, there is a point $x_2 \in (a,x_1)$ like that

$$h(x_2) = h(b). \tag{10}$$

Putting into practice Rolle's Theorem 5 to the function h on the interval $[x_2,b]$, we find a point $c \in (a,b)$ like that $h^{(\alpha)}(c) = 0$. If $h(b) < 0$, the proof is completed by a analogous way.

Now, we present conformable Sahoo and Riedel theorem where the condition $g^{(\alpha)}(a) = g^{(\alpha)}(b)$ of conformable Rolle's theorem is removed.

Theorem 7.2 (Conformable Sahoo and Riedel theorem) Suppose that $g:[a,b] \to \mathbb{R}$ be α -differentiable function on $[a,b]$, then there is a point $c \in (a,b)$ like that

$$g(c) - g(a) = \left(\frac{1}{\alpha}c^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right)g^{(\alpha)}(c) - \frac{\alpha}{\alpha+1}\frac{g^{(\alpha)}(b) - g^{(\alpha)}(a)}{\left(\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right)^{\alpha}}\left(\frac{1}{\alpha}c^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right)^{\alpha+1}. \tag{11}$$

Proof. Let $\Psi:[a,b] \to \mathbb{R}$ be a function of the following form:

$$\Psi(x) = g(x) - \frac{1}{1+\alpha}\frac{g^{(\alpha)}(b) - g^{(\alpha)}(a)}{\left(\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right)^{\alpha}}\left(\frac{1}{\alpha}x^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right)^{\alpha+1}. \tag{12}$$

Simple observation reveals that Ψ is α -differentiable on $[a,b]$ and we achieve

$$\Psi^{(\alpha)}(x) = g^{(\alpha)}(x) - \frac{g^{(\alpha)}(b) - g^{(\alpha)}(a)}{\left(\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right)^{\alpha}}\left(\frac{1}{\alpha}x^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right)^{\alpha}. \tag{13}$$

Clearly, $\Psi^{(\alpha)}(a) = \Psi^{(\alpha)}(b) = g^{(\alpha)}(a)$. From the Conformable Flett's Theorem 6 for the function Ψ, we find

$$\Psi(c) - \Psi(a) = \left(\frac{1}{\alpha}c^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right)\Psi^{(\alpha)}(c), \tag{14}$$

for some $c \in (a,b)$. Benefiting from Eq. (14),

$$\left[g(c)-\frac{1}{1+\alpha}\frac{g^{(\alpha)}(b)-g^{(\alpha)}(a)}{\left(\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha}}\left(\frac{1}{\alpha}c^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha+1}\right]$$

$$-\left[g(a)-\frac{1}{1+\alpha}\frac{g^{(\alpha)}(b)-g^{(\alpha)}(a)}{\left(\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha}}\left(\frac{1}{\alpha}a^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha+1}\right]$$

$$= \left(\frac{1}{\alpha}c^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)\left[g^{(\alpha)}(c)--\frac{g^{(\alpha)}(b)-g^{(\alpha)}(a)}{\left(\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha}}\left(\frac{1}{\alpha}c^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha}\right]. \tag{15}$$

So, we obtain

$$g(c)-g(a)-\frac{1}{1+\alpha}\frac{g^{(\alpha)}(b)-g^{(\alpha)}(a)}{\left(\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha}}\left(\frac{1}{\alpha}c^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha+1}$$

$$= g^{(\alpha)}(c)\left(\frac{1}{\alpha}c^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)-\frac{g^{(\alpha)}(b)-g^{(\alpha)}(a)}{\left(\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha}}\left(\frac{1}{\alpha}c^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha+1}. \tag{16}$$

Thus, we accomplish

$$g(c)-g(a)=\left(\frac{1}{\alpha}c^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)g^{(\alpha)}(c)-\frac{\alpha}{\alpha+1}\frac{g^{(\alpha)}(b)-g^{(\alpha)}(a)}{\left(\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha}}\left(\frac{1}{\alpha}c^{\alpha}-\frac{1}{\alpha}a^{\alpha}\right)^{\alpha+1}. \tag{17}$$

3. Conclusions

A fractional analysis has attracted considerable interest because of their capability to model complex phenomena, which is why it is crucial to investigate properties of fractional derivatives in order to gain a better understanding of these phenomena. In this research paper, some findings are presented related to important characteristics of conformable derivatives, which is a type of these operators that are used frequently.

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