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Canonical, Noncanonical, and Semicanonical Third Order Dynamic Equations on Time Scales

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Abstract

The notion of third order semicanonical dynamic equations on time scales is introduced so that any third order equation is either in canonical, noncanonical, or semicanonical form. Then a technique for transforming each of the two types of semicanonical equations to an equation in canonical form is given. The end result is that oscillation and other asymptotic results for canonical equations can then be applied to obtain analogous results for semicanonical equations.

Keywords: Dynamic equations third order canonical form noncanonical form semicanonical form asymptotic behavior
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1. Introduction

The study of qualitative properties of solutions of second order differential equations of the form

$$(r(t)x')' + q(t)x^{\gamma}(t) = 0, \quad t \geq t_0, \tag{1}$$

with r, $q : [t_0, \infty) \to \mathbb{R}^+$ and γ the ratio of odd positive integers, is often divided into two parts depending on whether equation (1) is in canonical form, that is,

$$\int_{t_0}^{\infty} \frac{1}{r(s)} ds = \infty,$$

or is in nonconanical form

$$\int_{t_0}^{\infty} \frac{1}{r(s)} ds < \infty.$$

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This is also the case for second order difference equations and dynamic equations on time scales. In the case of third order equations,

$$(b(t)(a(t)x')')' + q(t)x^{\gamma}(t) = 0, \tag{2}$$

with a, $b : [t_0, \infty) \to \mathbb{R}^+$ and q and γ as above, the situation is more complicated due to the presence of two coefficients in the lead term.

Here we will consider the more general setting of the third order dynamic equation

$$\left(b(t)(a(t)x^{\Delta}(t))^{\Delta}\right)^{\Delta} + q(t)x^{\gamma}(t) = 0, \quad t \in [t_0, \infty) \cap \mathbb{T}, \tag{E}$$

where $\mathbb{T}$ is a time scale with $t_0 \geq 0$, $\sup \mathbb{T} = \infty$, $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$, and a, b, $q \in C_{rd}\left((t_0, \infty)_{\mathbb{T}}, (0, \infty)\right)$. The basic notation and terminology for time scales can be found in the well-known monograph by Bohner and Peterson [2] and will be used without further mention.

We will say that equation (E) is in canonical form if

$$\int_{t_0}^{\infty} \frac{1}{a(s)} \Delta s = \int_{t_0}^{\infty} \frac{1}{b(s)} \Delta s = \infty,$$

and it is in noncanonical form if

$$\int_{t_0}^{\infty} \frac{1}{a(s)} \Delta s < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{b(s)} \Delta s < \infty.$$

If either

$$\int_{t_0}^{\infty} \frac{1}{a(s)} \Delta s < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{b(s)} \Delta s = \infty \tag{S$_1$}$$

or

$$\int_{t_0}^{\infty} \frac{1}{a(s)} \Delta s = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{b(s)} \Delta s < \infty, \tag{S$_2$}$$

then we will say that equation (E) is in semicanonical form.

In this paper we wish to show that under certain conditions, a semicanonical equation, i.e., equation (E) with either (S$_1$) or (S$_2$) holding, can be written as an equivalent equation in canonical form. One advantage of dealing with equations in canonical form is that we can apply the famous Kiguradze lemma to classify the behavior of nonoscillatory solutions, and the number of possible types is less for canonical equations than it is for noncanonical ones.

Interest in the relationship between canonical, noncanonical and semicanonical equations can be traced back to the now classic 1974 paper of Trench [10]. This has attracted the attention of other authors as can be seen, for example, from the recent papers [1, 3, 4, 5, 6, 7, 8]. The motivation for examining this classification scheme for dynamic equations on time scales stems partially from some recent results for difference equations in [9].

2. Semicanonical Equations of Type (S$_1$).

In this case, we let

$$A(t) = \int_{t}^{\infty} \frac{\Delta s}{a(s)}.$$

Here is our theorem in this case.

Theorem 2.1. *If*

$$\int_{t_0}^{\infty} \frac{A^{\sigma}(s)}{b(s)} \Delta s = \infty, \tag{C_1}$$

then the operator $Px(t) = \left(b(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}$ *can be written as the canonical operator*

$$Px(t) = \left(\frac{b(t)}{A^{\sigma}(t)}\left(a(t)A(t)A^{\sigma}(t)\left(\frac{x(t)}{A(t)}\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} \tag{3}$$

Proof. Recalling that $\sigma(t)$ is the forward jump operator and differentiating, we obtain

$$\begin{aligned}
&\frac{b(t)}{A^{\sigma}(t)}\left(a(t)A(t)A^{\sigma}(t)\left(\frac{x(t)}{A(t)}\right)^{\Delta}\right)^{\Delta} \\
&\quad= \frac{b(t)}{A^{\sigma}(t)}\left\{a(t)A(t)A^{\sigma}(t)\left[\frac{x^{\Delta}(t)A(t) - x(t)A^{\Delta}(t)}{A(t)A^{\sigma}(t)}\right]\right\}^{\Delta} \\
&\quad= \frac{b(t)}{A^{\sigma}(t)}\left\{a(t)x^{\Delta}(t)A(t) - a(t)x(t)A^{\Delta}(t)\right\}^{\Delta} \\
&\quad= \frac{b(t)}{A^{\sigma}(t)}\left\{(a(t)x^{\Delta}(t))^{\Delta}A^{\sigma}(t) + a(t)x^{\Delta}(t)\left(-\frac{1}{a(t)}\right) + x^{\Delta}(t)\right\} \\
&\quad= b(t)(a(t)x^{\Delta}(t))^{\Delta}
\end{aligned}$$

since $A^{\Delta}(t) = -\frac{1}{a(t)}$. Now

$$\int_{t_0}^{\infty}\frac{\Delta s}{a(s)A(s)A^{\sigma}(s)} = \int_{t_0}^{\infty}\left(\frac{1}{A(s)}\right)^{\Delta}\Delta s = \lim_{t\to\infty}\frac{1}{A(t)} - \frac{1}{A(t_0)} = \infty,$$

so this together with condition (C_1) proves that the operator (3) is in canonical form. □

3. Semicanonical Equations of Type $(\mathbf{S_2})$.

Now we let

$$B(t) = \int_{t}^{\infty}\frac{\Delta s}{b(s)}.$$

Our result in this case is the following.

Theorem 3.1. *If*

$$\int_{t_0}^{\infty}\frac{B(s)}{a(s)}\Delta s = \infty, \tag{C_2}$$

then the operator $Qx(t) = \left(b(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}$ *can be written as the canonical operator*

$$Qx(t) = \frac{1}{B^{\sigma}(t)}\left(b(t)B(t)B^{\sigma}(t)\left(\frac{a(t)}{B(t)}x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}. \tag{4}$$

Proof. Once again by a straightforward differentiation,

$$\begin{aligned}
&\left(b(t)B(t)B^\sigma(t)\left(\frac{a(t)}{B(t)}x^\Delta(t)\right)^\Delta\right)^\Delta\\
&=\left[b(t)B(t)B^\sigma(t)\left(\frac{B(t)(a(t)x^\Delta(t))^\Delta-a(t)x^\Delta(t)\left(-\frac{1}{b(t)}\right)}{B(t)B^\sigma(t)}\right)\right]^\Delta\\
&=\left[b(t)B(t)(a(t)x^\Delta(t))^\Delta+a(t)x^\Delta(t)\right]^\Delta\\
&=(b(t)(a(t)x^\Delta(t))^\Delta)^\Delta B^\sigma(t)+b(t)(a(t)x^\Delta(t))^\Delta\left(-\frac{1}{b(t)}\right)+(a(t)x^\Delta(t))^\Delta\\
&=B^\sigma(t)(b(t)(a(t)x^\Delta(t))^\Delta)^\Delta.
\end{aligned}$$

Since

$$\int_{t_0}^\infty \frac{\Delta s}{b(s)B(s)B^\sigma(s)}=\int_{t_0}^\infty\left(\frac{1}{B(s)}\right)^\Delta\Delta s=\lim_{t\to\infty}\frac{1}{B(t)}-\frac{1}{B(t_0)}=\infty,$$

this together with (C_2) shows that the operator $B^\sigma(t)Qx(t)$ is in canonical form. □

4. Discussion

In order to develop some insight and intuition about the results obtained in Sections 2 and 3 above, let us begin by considering a differential equation ($\mathbb{T}=\mathbb{R}$) in which the coefficients are powers of t, namely,

$$(t^\beta(t^\alpha x'(t))')'+q(t)x^\gamma(t)=0,\quad t\geq 1. \tag{E}$$

First note that equation (E) is in canonical form if

$$\int_{t_0}^\infty\frac{ds}{a(s)}=\int_{t_0}^\infty\frac{ds}{s^\alpha}=\infty$$

and

$$\int_{t_0}^\infty\frac{ds}{b(s)}=\int_{t_0}^\infty\frac{ds}{s^\beta}=\infty.$$

That is, for equation (E) to be in canonical form, we must have $\alpha\leq 1$ and $\beta\leq 1$. On the other hand, (E) is in noncanonical form provided $\alpha>1$ and $\beta>1$. Finally, for the semicanonical cases, (S_1) holds provided $\alpha>1$ and $\beta\leq 1$, and for (S_2) to hold, we need $\alpha\leq 1$ and $\beta>1$.

By Theorem 2.1, if $\alpha+\beta\leq 2$, then equation (E) is semicanonical, but the equation involving the operator P given in (3), namely,

$$(t^{\alpha+\beta-1}(t^{2-\alpha}(t^{\alpha-1}x(t))')')',$$

where $\alpha>1$, $\beta\leq 1$, and $\alpha+\beta\leq 2$ (e.g., $\alpha=\frac{3}{2}$, $\beta=\frac{1}{2}$) is in canonical form.

By Theorem 3.1, if $\alpha+\beta\leq 2$, $\alpha\leq 1$, and $\beta>1$, then equation (E) is semicanonical, but the operator $B(t)Qx(t)$ is in canonical form. This means that the equation becomes

$$(t^{2-\beta}(t^{\alpha+\beta-1}x'(t))')')'+t^{1-\beta}q(t)x^\gamma(t)=0,$$

where $\alpha\leq 1$, $\beta>1$, and $\alpha+\beta\leq 2$ (e.g., $\alpha=\frac{1}{2}$, $\beta=\frac{3}{2}$) is in canonical form.

As a consequence of Theorem 2.1, we have the following result.

Theorem 4.1. *Under conditions (S_1) and (C_1), the semicanonical equation* (E) *has a solution $x(t)$ if and only if the canonical equation*

$$(c(t)(d(t)y(t))^\Delta)^\Delta)^\Delta+A^\gamma(t)q(t)y^\gamma(t)=0 \tag{5}$$

has the solution $y(t)=x(t)/A(t)$ where $c(t)=b(t)/A(\sigma(t))$ and $d(t)=a(t)A(t)A(\sigma(t))$.

We thus see that if (S_1) and (C_1) hold, then equation (E) has an eventually positive solution if and only if (5) has an eventually positive solution. One advantage here is that for equation (E), the positive solutions are of one of the three types

(i) $x^\Delta(t) > 0$ and $(a(t)x^\Delta(t))^\Delta > 0$,

(ii) $x^\Delta(t) < 0$ and $(a(t)x^\Delta(t))^\Delta > 0$,

(iii) $x^\Delta(t) < 0$ and $(a(t)x^\Delta(t))^\Delta < 0$,

while for equation (5), positive solutions are of one of only the two types

(iv) $y^\Delta(t) < 0$ and $(d(t)y^\Delta(t))^\Delta > 0$,

(v) $y^\Delta(t) > 0$ and $(a(t)x^\Delta(t))^\Delta > 0$.

That is, there is one fewer class of solutions that needs to be eliminated when trying to prove an oscillation result.

In view of Theorem 3.1, we can write equation (E) as the canonical equation

$$\left(b(t)B(t)B^\sigma(t)\left(\frac{a(t)}{B(t)}x^\Delta(t)\right)^\Delta\right)^\Delta + B^\sigma(t)q(t)x^\gamma(t) = 0. \tag{6}$$

Similar to Theorem 4.1 we have the following result.

Theorem 4.2. *Under conditions (S_2) and (C_2), $x(t)$ is a solution of equation* (E) *if and only if it is a solution of equation* (6).

Remark 4.3. *Clearly, the entire discussion above holds if in equation* (E) *the term $x^\gamma(t)$ involves a delay such as $x^\gamma(g(t))$ where $g(t) \le t$ and $g(t) \to \infty$ as $t \to \infty$.*

What the results in this paper allow us to us to do, and this is an important consequence, is to apply known oscillation criteria for canonical equations to obtain an oscillation result for a semicanonical equation.

To illustrate the application of the results here to difference equations ($\mathbb{T} = \mathbb{N}$), consider

$$\Delta\left(\frac{1}{n+1}\Delta\left(n(n+1)\Delta x(n)\right)\right) + q(n)x^\gamma(n) = 0, \quad n \ge n_0, \tag{D_1}$$

which we see is in semicanonical form ((S_1) holds). Here $A(n) = \frac{1}{n}$ and condition (C_1) becomes

$$\sum_{s=n_0}^{\infty} \frac{A(s+1)}{b(s)} = \sum_{s=n_0}^{\infty} 1 = \infty.$$

The equation involving the operator P becomes

$$\Delta(\Delta(\Delta(nx(n))) + q(n)x^\gamma(n) = 0, \quad n \ge n_0, \tag{7}$$

which is clearly in canonical form, and can be written as

$$\Delta(\Delta(\Delta y)) + \frac{1}{n^\gamma}q(n)y^\gamma(n) = 0. \tag{8}$$

By Theorem 4.1, $x(n)$ is a solution of (D_1) if $y(n) = nx(n)$ is a solution of (8).

The equation

$$\Delta\left(n(n+1)\Delta\left(\frac{1}{n}\Delta x(n)\right)\right) + q(n)x^\gamma(n) = 0, \quad n \ge n_0, \tag{D_2}$$

is also semicanonical ((S$_2$) holds) and the transformed equation is

$$(n+1)\Delta(\Delta(\Delta x(n))) + q(n)x^{\gamma}(n) = 0, \quad n \geq n_0, \tag{9}$$

which is a canonical equation. Here, in view of Theorem 4.2, $x(n)$ is a solution of (D_2) if and only if it is a solution of (9).

In conclusion, to demonstrate how the results in this paper can be utilized, consider the simple case of the semicanonical differential equation ((S$_2$) holds)

$$(t^2 x''(t))' + q(t)x^{\gamma}(t) = 0, \quad t \geq t_0. \tag{10}$$

Here, $b(t) = t^2$ and $a(t) = 1$, so $B(t) = \frac{1}{t}$ and $Qx(t) = (tx'(t))''$ which is in canonical form. Then any conditions that ensure that a solution $x(t)$ of the canonical equation

$$(tx'(t))'' + \frac{1}{t}q(t)x^{\gamma}(t) = 0, \quad t \geq t_0, \tag{11}$$

oscillates or possesses some other asymptotic property, implies that $x(t)$ is a solution of (10) with that same behavior. For example, by [1, Theorem 3.1], if

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \frac{q(s)}{s}(\tau(s) - \ln(\tau(s)) - 1)ds > \frac{1}{e},$$

then any positive nonoscillatory solution of (11) belongs to the class

$$\{x > 0 : x' < 0,\ (tx')' > 0\}$$

and the same is true of any positive nonoscillatory solution of (10). Applications to difference equations can be found in [9].

As a final remark, let us point out that in the case of the time scale being the real numbers so that we are talking about differential equations, our result in Theorem 2.1 in which condition (S$_1$) holds, agrees exactly with what can be obtained from Trench [10, Lemma 1]. As pointed out in [6], the coefficients obtained from [10, Lemma 2] are too complicated to make easy comparisons to results such as our Theorem 3.1 above or others.

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