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Research Article

A Research on the Generalizations of Modules Whose Submodules are Isomorphic to a Direct Summand

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Abstract

A module M is called $virtually\ semisimple$ (resp. $virtually\ extending$) if every submodule (resp. complement submodule) of M is isomorphic to a direct summand of M. It is known that virtually extending modules is a generalization of virtually semisimple modules. In this paper, the relationships between virtually extending modules and other generalizations of virtually semisimple modules are examined. Moreover, we introduce a new generalization of virtually semisimple modules; namely CH modules: We say a module M is a *c-epi-retractable* (or briefly CH *module*) if any complement submodule of M is a homomorphic image of M. CH modules contains the class of virtually extending modules and the class of epi-retractable modules. We also give some basic properties of this new module class.

Keywords: virtually semisimple module, virtually extending module, epi-retractable module, CH module

Her Alt Modülü Bir Diktoplanana İzomorf Olan Modüllerin Genellemeleri Üzerine Bir Araştırma

Öz

Eğer bir M modülünün her alt modülü (sırasıyla tamamlayıcı alt modülü), M modülünün bir dik toplananına izomorfik ise, M modülüne *sanal yarı basit (sırasıyla sanal genişleyen) modül* denir. Sanal genişleyen modüllerin, sanal yarı basit modüllerin bir genellemesi olduğu bilinmektedir. Bu yazıda, sanal genişleyen modüller ile sanal yarı basit modüllerin diğer genellemeleri arasındaki ilişkiler incelenmektedir. Ayrıca, sanal yarı basit modüllerin yeni bir genellemesini de tanıtıyoruz; yani CH modülleri: Bir M modülünün herhangi bir tamamlayıcı alt modülü, M modülünün bir homomorfik görüntüsü ise, M modülüne bir *epi-c-geri-çekilebilir modül* (ya da kısaca *CH modül*) olarak adlandırıyoruz. CH modüllerin sınıfı, sanal genişleyen modüllerin sınıfını ve epi-geri-çekilebilir modüllerin sınıfını içerir. Ayrıca bu yeni modül sınıfının bazı temel özelliklerini de veriyoruz.

Anahtar Kelimeler: sanal yarı-basit modül, sanal genişleyen modül, epi-geri-çekilebilir modül, CH modül.

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1. Introduction

Throughout this note, any ring is associative with unity and is denoted by R, any module is unital right module. Some notations, which we will use in this paper, are listed below:

$A \leq M$	: A is a submodule of M.
$A \leq^c M$	: A is a complement (closed) submodule of M.
$A \leq^{ess} M$	: A is an essential submodule of M.
$A \leq^{\oplus} M$	: A is a direct summand of M.
$A \lesssim^{\oplus} M$	: A is isomorphic to a direct summand of M.
$A \cong B$	: A is isomorphic to B.
$E(M)$	: The injective hull of M.
$Hom_R(M, N)$	: The set of all R-homomorphisms from M to N.
$End_R(M)$	: The endomorphism ring of M.

We recall some of the definitions we used throughout the article: A submodule C of M is called *closed* if for any $A \leq M$ such that $C \leq^{ess} A$ in M, we have $C = A$. A submodule C of M is called *complement of a submodule* A of M if C is maximal with respect to the property that $C \cap A = 0$. In modules, being a closed submodule is equivalent to being a complement submodule [1, 1.10].

A module M is called *semisimple* if for any $X \leq M$, we have $X \leq^{\oplus} M$ (see [1, 1.15]). Semisimple modules and rings has significant role in module and ring theory. In 2018, the authors [2] introduced and investigated a new module class, namely virtually semisimple modules: A module M is called *virtually semisimple* if for any $X \leq M$, we have $X \lesssim^{\oplus} M$. For virtually semisimple rings, they proved a generalization of the reowned Weddernburn-Artin theorem (which characterize semisimple rings). Later, this interesting module family and related concepts were studied by many algebraist. Karabacak and his co-author(s) introduced several generalizations of virtually semisimple modules: Generalized SIP and SSP modules, and virtually extending modules [3,4,5]. Virtually extending modules is a generalization of both virtually semisimple modules and extending modules: A module M is called *extending* or CS (resp. *virtually extending*) if for any $X \leq^c M$, we have $X \leq^{\oplus} M$ ($X \lesssim^{\oplus} M$). The authors proved a generalization of the Osofsky-Smith Theorem in [3].

At the beginning of the study, we provide some equivalent definitions for virtually extending modules (Theorem 2). Then the relationships among the generalizations of virtually semisimple modules are examined. We proved in Propositions 3 and 4 that any virtually extending UC module has *GSIP*, any virtually extending module with *CSP* has *GSSP*. Then, a new module class, which are called CH modules, is introduced and its basic properties are examined. A module M is called CH if any complement submodule of M is a homomorphic image of M. An example is given in Example 9 that the CH condition is not inherited by direct summands. After giving this example, we ensure some results on the CH condition to be inherited by direct summands and direct sums (Propositions 10, 11 and 12). We also give some results about that when CS, virtually extending and CH modules coincide (Proposition 13, 14 and 15). In Theorem 17, a characterization of quasi-projective virtually extending modules is given by using CH modules: A module M is virtually extending and quasi-projective if and only if it is a CH module and all of its complement submodules are M-projective. In Theorem 18, it is proved that if a module M is morphic and $CH,$ then M is finitely generated if and only if M is CF (i.e., any closed submodule is finitely generated).

2. Results

We begin the paper by giving some equivalent definitions for virtually extending modules.

Theorem 2. The next statements are equivalent for a module M:

a. M is virtually extending.
b. For any $X \leq M$, there exists a $Z \leq^c M$ such that $X \leq^{ess} Z$ and $Z \lesssim^{\oplus} M$.
c. For any $X, Y \leq M$ with $X \cap Y = 0$, there exists a $Z \leq^c M$ such that $Y \leq Z$, $X \cap Z = 0$ and $Z \lesssim^{\oplus} M$.
d. For given $e^2 = e \in End(E(M))$, there exists $d^2 = d \in End(M)$ such that $eE(M) \cap M \cong dM$.

Proof. $(a) \Rightarrow (c)$ Let M be virtually extending, $X, Y \leq M$ with $X \cap Y = 0$. There exists a $Z \leq M$ satisfying that $Y \leq Z$ and Z is complement of X in M. By (a), $Z \lesssim^{\oplus} M$.

$(c) \Rightarrow (a)$ Let $Y \leq^c M$. There exists a $X \leq M$ satisfying that Y is complement of X in M. By (c), there exists a $Z \leq^c M$ satisfying that $Y \leq Z$, $X \cap Z = 0$ and $Z \lesssim^{\oplus} M$. Since $Y \leq^c M$, then $Y = Z$.

$(a) \Rightarrow (b)$ Let $X \leq M$. There exists a $Z \leq^c M$ satisfying that $X \leq^{ess} Z$. By (a), $Z \lesssim^{\oplus} M$.

$(b) \Rightarrow (d)$ Suppose (b) holds. Then $eE(M) \cap M \leq^{ess} C \leq^c M$ such that $C \cong D \leq^{\oplus} M$. It implies that $eE(M) = E(C)$ and so, $C \leq eE(M) \cap M$. Thus $C = eE(M) \cap M$. Now, by (b), there exists a $d^2 = d \in End(M)$ satisfying that $C = eE(M) \cap M \cong dM = D$.

$(d) \Rightarrow (a)$ Let $A \leq M$. There exists a $N \leq M$ such that $A \leq^{ess} N \leq^c M$. Then, we have

$$A \leq^{ess} N \leq^{ess} E(N) \leq^{\oplus} E(M).$$

Then, there exists a $e^2 = e \in End(E(M))$ satisfying that $eE(M) = E(N)$. Since $N \leq^{ess} E(N)$ and $M \leq^{ess} M$ and by [6, Lemma 1.1(2)], we have $N \cap M = N \leq^{ess} E(N) \cap M$. Now

$$N \leq^{ess} E(N) \cap M \leq M.$$

Since $N \leq^c M$, we have $N = E(N) \cap M$, and hence $N = eE(M) \cap M$. By (d), there exist $d^2 = d \in End(M)$ satisfying that $eE(M) \cap M \cong dM$. Therefore, M is virtually extending.

A module M is said to have *GSIP* if for any $X, Y \leq^{\oplus} M$, we have $X \cap Y \lesssim^{\oplus} M$ [4]. M is called UC if and only if for any $X, Y \leq^c M$, we have $X \cap Y \leq^c M$ [7].

Now, we give a result showing that virtually extending modules are related to modules having *GSIP*.

Proposition 3. If a module M is both virtually extending and UC, then M has *GSIP*.

Proof. Let $X, Y \leq^{\oplus} M$. Clearly, $X, Y \leq^{c} M$. Since M is *UC*, we have $X \cap Y \leq^{c} M$. Now, $X \cap Y \lesssim^{\oplus} M$ because M is virtually extending. It means that M has *GSIP*.

A module M is said to have *GSSP* if for any pair of $X, Y \leq^{\oplus} M$, we have $X + Y \lesssim^{\oplus} M$ [5]. M is said to have *closed sum property (CSP)* if for any $X, Y \leq^{c} M$, we have $X + Y \leq^{c} M$ [8].

Now, we give a result showing that virtually extending modules are related to modules having *GSSP*.

Proposition 4. If a module M is virtually extending and has *CSP*, then M has *GSSP*.

Proof. Let $X, Y \leq^{\oplus} M$. Clearly, $X, Y \leq^{c} M$. Since M has *CSP*, we have $X + Y \leq^{c} M$. Now, $X + Y \lesssim^{\oplus} M$ because M is virtually extending. It means that M has *GSSP*.

In the next section, we introduce a new generalization of virtually semisimple modules.

Khuri [9] calls a module M *retractable* if for every $X \leq M$, there exists a $\tau \in End_R(M)$ satisfying that $\tau(M) \subseteq X$ (i.e., $Hom(M, X) \neq 0$).

Ghorbani and Vedadi [10] call a module M *epi-retractable* if for every $X \leq M$, there exists a $\tau \in End_R(M)$ satisfying that $\tau(M) = X$.

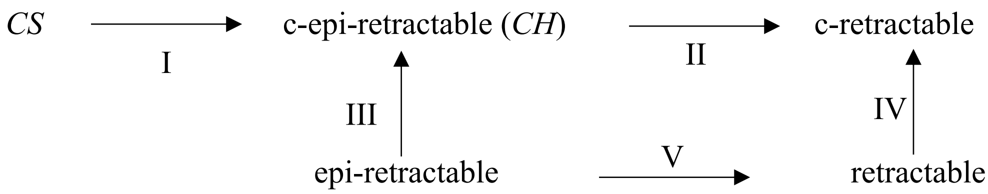
Chatters and Khuri [11] call a module M *c-retractable* if for every $X \leq^{c} M$, there exists a $\tau \in End_R(M)$ satisfying that $\tau(M) \subseteq X$ (i.e., $Hom(M, X) \neq 0$).

Now, we introduce c-epi-retractable modules which is a generalization of epi-retractable modules:

Definition 5. We call a module M *c-epi-retractable* if for any $X \leq^{c} M$, there exists a $\tau \in End_R(M)$ satisfying that $\tau(M) = X$.

As it can be seen from the definition, since the complement submodules are the homomorphic image of the module M, we will briefly call this module class as CH *modules*, where, "C" is the first letter of the word "complement" and "H" is the first letter of the word "homomorphic".

First, we should state that any CS module is a CH module. More generally we have the following hierarchy:



In the next theorem, we give some equivalent conditions for CH modules:

Theorem 6. The next statements are equivalent for a module M:

a. M is a CH module.
b. For any $X \leq M$, there exists a $Z \leq^c M$ such that $X \leq^{ess} Z$ and Z is homomorphic image of M.
c. For any $X, Y \leq M$ with $X \cap Y = 0$, there exists a $Z \leq^c M$ such that $Y \leq Z$, $X \cap Z = 0$ and Z is homomorphic image of M.
d. There exist epimorphisms $M \to N$ and $N \to M$ for some CH module N.
e. There exists an epimorphism $M/X \to M$ for some CH factor module M/X.

Proof. $(a) \Rightarrow (b)$ Let $X \leq M$. There exists a $Z \leq^c M$ satisfying that $X \leq^{ess} Z$. By (a), Z is homomorphic image of M.

$(b) \Rightarrow (a)$ Let $X \leq^c M$. By (b), there exists a $Z \leq^c M$ satisfying that $X \leq^{ess} Z$ and Z is homomorphic image of M. Thus $X = Z$, and hence M is CH.

$(a) \Rightarrow (c)$ Assume M is CH. Let $X, Y \leq M$ with $X \cap Y = 0$. There exists a complement Z of X in M satisfying that $Y \leq Z$. Then Z is homomorphic image of M by the hypothesis.

$(c) \Rightarrow (a)$ Let $Y \leq^c M$. There exists $X \leq M$ such that Y is complement of X in M. By the hypothesis, there exists a $Z \leq^c M$ satisfying that $Y \leq Z$, $X \cap Z = 0$ and Z is homomorphic image of M. Then $Y = Z$, and M is CH.

$(a) \Rightarrow (d)$ Clear.

$(d) \Rightarrow (e)$ Suppose that there exists a CH module N and epimorphisms $f: M \to N$, $g: N \to M$. Say $X = Ker(f)$. Then, f induces an isomorphism $\bar{f}: M/X \to N$. Thus, M/X is a CH module.

$(e) \Rightarrow (a)$ Let $C \leq^c M$. By our assumption, there exists an isomorphism $\bar{f}: M/Y \to M$ for some submodule Y of M with $X \subseteq Y$. Let $\bar{f}(A/Y) = C$ for some $A \leq^c M$. Since $A \leq^c M$ and by [7, Corollary 2(ii)], we have $A/X \leq^c M/X$. Since M/X is CH, then there exists an epimorphism $h: M/X \to A/X$. Consider $g: A/X \to A/Y$ with $g(a + X) = a + Y$, and the canonical epimorphism $\pi: M \to M/X$. Then, the map $\bar{f}gh\pi: M \to C$ is an epimorphism, and hence M is CH.

Example 7. Let M be an R-module as in [12, Example 2.6 or 2.7]. The authors show that M is CS (and hence CH) but not retractable (and hence not epi-retractable). Thus the reverse implication of (III) in above diagram doesn't hold, in general. Another example can also be given: the $\mathbb{Z}$-module $\mathbb{Q}$ which is CH but not retractable (see [10, Remark 2.12]).

As prior studies have indicated that the direct sum of CS (uniform) modules need not to be a CS module [13]. The next is a well-known example of this. In addition, the next example is a CH module.

Example 8. Let p be a prime number. The $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ is a CH module because every finitely generated module over a PID is epi-retractable (see [10, Example 2.4(3)]). But it is not CS (see [13, p. 56]). Hence the reverse implication of (I) in above diagram doesn't hold, in general.

As prior studies have indicated that the CS property is inherited by direct summands [13], but the next example shows that the CH condition is not inherited by direct summands.

Example 9. [10, Remark 2.12] Let $\mathbb{F}$ be a free $\mathbb{Z}$-module with an infinite countable basic set and A be any countable $\mathbb{Z}$-module which is not CH. Then $M_{\mathbb{Z}} = \mathbb{F} \oplus A$ is epi-retractable by [10, Remark 2.12], and hence it is CH.

In the next results, we give some conditions which ensure that direct summands of CH modules are again CH.

Proposition 10. Let M be a CH module. Then

a. M/F is a CH module for any fully invariant complement submodule F of M.
b. If $M = M_1 \oplus M_2$ such that $Hom_R(M_1, M_2) = 0$, then M_2 is CH.

Proof. (a) Let F be a fully invariant complement submodule of M, and C/F be any complement submodule of M/F. Since $F \leq^c M$ and $C/F \leq^c M/F$, we have $C \leq^c M$ by [7, Corollary 2(iii)]. Then there is an epimorphism $\epsilon: M \to C$. Now, $\epsilon(F) \subseteq F$ by our assumption, and hence $\bar{\epsilon}: M/F \to C/F$ with $\bar{\epsilon}(m + F) = \epsilon(m) + F$ is an epimorphism. It means that M/F is CH.

(b) Note that $End_R(M) = \begin{bmatrix} End_R(M_1) & Hom_R(M_2, M_1) \\ 0 & End_R(M_2) \end{bmatrix}$. Thus $End_R(M) \begin{bmatrix} M_1 \\ 0 \end{bmatrix} \subseteq \begin{bmatrix} M_1 \\ 0 \end{bmatrix}$. It implies that $M_1 \oplus 0$ is a fully invariant submodule of M. Now, we get result applying (a).

More generally, we can give the following result.

Proposition 11. Let $M = \oplus_{i \in I} M_i$ be an R-module with $End_R(M)$ is abelian. Then M is CH if and only if each M_i is CH.

Proof. First we note that $End_R(M)$ is abelian if and only if any direct summand of M is fully invariant in M (see [14, Theorem 4.4]).

($\Rightarrow$:) It is immediate by Proposition 10(b).

($\Leftarrow$:) Assume each M_i is CH and let $C \leq^c M$. Then by [15, Lemma 2.1], $C = \oplus_{i \in I}(C \cap M_i)$. Clearly, $C \cap M_i \leq^c M_i$ for each $i \in I$. Since each M_i is CH, for any $i \in I$ there exists $f_i \in End_R(M_i)$ such that $f_i(M_i) = C \cap M_i$. Now we can define the epimorphism

$$f \coloneqq \bigoplus_{i\in I} f_i : \bigoplus_{i\in I} M_i \ \rightarrow \ \bigoplus_{i\in I} (C \cap M_i)$$

Consequently,

$$f(M) = \bigoplus_{i\in I} f_i(M) = \bigoplus_{i\in I} f_i\left(\bigoplus_{i\in I} M_i\right) = \bigoplus_{i\in I} (C \cap M_i) = C,$$

as desired.

Proposition 12. Let $M = \oplus_{i\in I} M_i$ be a UC R-module. Then M is CH if and only if each M_i is CH.

Proof. ($\Rightarrow$:) Let M be both CH and UC. Suppose $D \leq^{\oplus} M$ and $A \leq D$. Since M is CH, by Theorem 6, there exists a $C \leq^{c} M$ such that $A \leq^{ess} C$ and C is homomorphic image of M. On the other hand, there exists a $K \leq^{c} D$ such that $A \leq^{ess} K$. Then, clearly, $C, K \leq^{c} D$. Since M is UC, we have $C = K$. It means that D is CH.

($\Leftarrow$:) Assume each M_i is CH and let $C \leq^{c} M$. Then, by [6, Lemma 6], $C \cap M_i \leq^{c} M_i$ for all $i \in I$. Since each M_i is CH, for any $i \in I$ there exists $f_i \in End_R(M_i)$ such that $f_i(M_i) = C \cap M_i$. Now we can define the epimorphism

$$f \coloneqq \bigoplus_{i\in I} f_i : \bigoplus_{i\in I} M_i \ \rightarrow \ \bigoplus_{i\in I} (C \cap M_i)$$

The proof follows by the argument which we use in the proof of Proposition 11.

Now, in the next three results, we show the relationship between CH modules and some other known module classes.

The authors [16] call a module M is *Rickart* if $Ker(\epsilon) \leq^{\oplus} M$ for every $\epsilon \in End_R(M)$. Clearly, the following implications is true for an R-module M:

M is $CS \Rightarrow M$ is virtually extending $\Rightarrow M$ is CH.

The module in Example 8 is a CH module but not virtually extending. On the other hand an example of virtually extending module which is not CS is given in [3, Example 2.1].

The next result illustrates that the class of virtually extending modules and the class of CH modules coincide when the module is Rickart:

Proposition 13. Let M be a Rickart module. Then M is virtually extending if and only if M is CH.

Proof. ($\Rightarrow$:) Clear.

($\Leftarrow$:) Let $X \leq^{c} M$. Since M is CH, there exists an epimorphism $\epsilon: M \rightarrow X$. Let $i: X \rightarrow M$ be the inclusion map. We have, $Ker(i\epsilon) = Ker(\epsilon) \leq^{\oplus} M$ because of Rickartness. Then $M/Ker(\epsilon) \cong Im(\epsilon)$. Therefore, $Im(\epsilon) = X \lesssim^{\oplus} M$, as desired.

Corollary 14. Let R be right hereditary ring. Then every projective CH right R-module M is virtually extending.

Proof. Let $X \leq^c M$. Since M is CH, there exists an epimorphism $\epsilon: M \to X$. Since R is right hereditary, X is projective, and hence $Ker(\epsilon) \leq^{\oplus} M$. Therefore, $Im(\epsilon) = X \lesssim^{\oplus} M$, as desired.

The authors [17] call a module M is *dual Rickart* if $Im(\epsilon) \leq^{\oplus} M$ for every $\epsilon \in End_R(M)$.

Proposition 15. The next statements are equivalent for a dual Rickart module M:

a. M is CS.
b. M is virtually extending.
c. M is CH.

Proof. $(\boldsymbol{a}) \Rightarrow (\boldsymbol{b}) \Rightarrow (\boldsymbol{c})$ Clear.

$(\boldsymbol{c}) \Rightarrow (\boldsymbol{a})$ Let $X \leq^c M$. Since M is CH, there exists an epimorphism $\epsilon: M \to X$. We have $Im(\epsilon) = X \leq^{\oplus} M$ because of dual Rickartness. Hence, M is CS.

M is called $C2$ if for any $X \leq M$ with $X \lesssim^{\oplus} M$, we have $X \leq^{\oplus} M$. M is called *continuous* if it is CS and $C2$ [18].

Corollary 16. Any dual Rickart CH module is continuous.

Proof. By [17, Proposition 2.21], M is $C2$. Now, it is clear by Proposition 15.

Let M and P be R-modules. P is M-projective if and only if $Hom_R(P, -)$ is exact with respect to all exact sequences $0 \to K \to M \to N \to 0$. If P is P-projective, then P is also called *quasi-projective* (see [19, p.148]).

In the next theorem, we give a characterization of quasi-projective virtually extending modules with using CH modules.

Theorem 17. A module M is virtually extending and quasi-projective if and only if it is a CH module and all of its complement submodules are M-projective.

Proof. $(\Rightarrow:)$ M is CH because any virtually extending module is CH. Since M is virtually extending then any complement submodule of M is isomorphic to a direct summand of M. Thus by [19, Proposition 18.1], all complements submodules of M are M-projective.

$(\Leftarrow:)$ Let $X \leq^c M$. Since M is CH, there exists an epimorphism $\epsilon: M \to X$. Since X is M-projective, we have $Ker(\epsilon) \leq^{\oplus} M$ by [18, Lemma 4.30]. For some $K \leq M$, $M = Ker(\epsilon) \oplus K$. Then $K \cong M/Ker(\epsilon) \cong Im(\epsilon) = X$, i.e, $X \lesssim^{\oplus} M$. Thus, M is virtually extending. Again by [19, Proposition 18.1], M is quasi-projective.

The authors [20] call a module M *morphic* if for any $\epsilon \in End_R(M)$, $M/Im(\epsilon) \cong Ker(\epsilon)$; or equivalently, if for any $X, Y \leq M$ with $M/X \cong Y$ then $M/Y \cong X$.

Nguyen V. Dung [21] call a module M CF if any closed (complement) submodule is finitely generated. Now, it is proved that if a module M is morphic and CH then M is finitely generated if and only if M is CF.

Theorem 18. The next statements are equivalent for a morphic module M:

1) Every closed submodule of M is isomorphic to an image of M (i.e, M is CH).
2) For any $X \leq^c M$, M/X is isomorphic to a submodule of M.

In this case, the next statements hold:

a) If $X, Y \leq^c M$ then $M/X \cong M/Y$ if and only if $X \cong Y$.
b) M is finitely generated if and only if M is CF.

Proof. $(1) \Rightarrow (2)$: Let $X \leq^c M$. By (1), there is a $Z \leq M$ satisfying that $M/Z \cong X$. Since M is morphic, we have $M/X \cong Z$. So, (2) holds.

$(2) \Rightarrow (1)$: Let $X \leq^c M$. By (2), there is a $Z \leq M$ satisfying that $M/X \cong Z$. Since M is morphic, we have $M/Z \cong X$. So, (1) holds.

(a) $(\Rightarrow:)$ Let $X, Y \leq^c M$ with $M/X \cong M/Y$. By (2), there exists a $Z \leq M$ such that $M/X \cong M/Y \cong Z$. Since M is morphic, we have $X \cong Y \cong M/Z$.

$(\Leftarrow:)$ Let $X, Y \leq^c M$ with $X \cong Y$. By (2), there is a $Z \leq M$ such that $M/X \cong Z$, and there is a $T \leq M$ such that $M/Y \cong T$. Since M is morphic, we have $M/Z \cong X \cong Y \cong M/T$. Now, we have $M/Z \cong Y$. Now again, since M is morphic, $M/Y \cong Z$. On the other hand, we have just said above that $M/X \cong Z$. Thus, $M/Y \cong Z \cong M/X$.

(b) $(\Rightarrow:)$ Let $X \leq^c M$. By (1), there is a $Z \leq M$ such that $M/Z \cong X$. Thus, X is finitely generated because M is finitely generated. Hence M is CF.

$(\Leftarrow:)$ Clear.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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