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VIBRATION ANALYSIS OF ROTATING TIMOSHENKO BEAMS WITH DIFFERENT MATERIAL DISTRIBUTION PROPERTIES

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ABSTRACT: In this study, vibration characteristics of functionally graded rotating Timoshenko beams that undergoes flapwise bending vibration are analysed. Beam models with different material distribution properties are considered. The energy expressions are derived by introducing several explanotary figures and tables. Applying the Hamilton's Principle to the derived energy expressions, governing differential equations of motion and the boundary conditions are obtained. Related formulation is coded by using MATLAB and in the solution part, the equations of motion, including the parameters for the rotary inertia, shear deformation, power law index parameter and slenderness ratio are solved using an efficient mathematical technique, called the differential transform method (DTM). Natural frequencies of the modeled beams are obtained. Increasing effects of the slenderness ratio and the rotational speed and the decreasing effect of the power-law index on the natural frequencies are investigated. Moreover, differences between the natural frequencies of the beam models with different material distribution characteristics is noticed and examined. Obtained results are distributed in several tables.

Keywords: *Differential Transform Method, Functionally Graded Beam, Rotating Timoshenko Beam*

Fonksiyonel Derecelendirilmiş Dönen Timoshenko Kirişlerin Titreşim Analizi

ÖZ: Bu çalışma kapsamında, düzlem dışı eğilme (flaplama hareketi) deplasmanı altında fonksiyonel derecelendirilmiş, dönen Timoshenko kirişlerin titreşim analizi yapılmıştır. Farklı malzeme dağılımlarına sahip kiriş modelleri incelenmiş ve enerji ifadeleri, çeşitli şekil ve tablolar kullanılarak çıkarılmış ve bu enerji denklemlerine Hamilton Prensibi uygulanarak hareket denklemleri ve sınır şartları elde edilmiştir ve ilgili denklemler, MATLAB programında kodlanmıştır. Çözüm aşamasında; dönme ataleti, kayma etkisi, malzeme dağılımı üstel fonksiyonu, kiriş narinlik oranı gibi çok çeşitli parametrelerin katıldığı hareket denklemleri ve sınır şartlarına, etkin ve hızlı bir matematiksel yöntem olan Diferansiyel Dönüşüm Yöntemi (Differential Transform Method) uygulanmıştır. Modellenen kirişlere ait doğal frekanslar hesaplanmıştır. Narinlik oranı ve dönme hızının frekanslar üzerindeki yükseltici etkileri ve malzeme dağılımı ile ilgili olarak güç indeksinin, frekanslar üzerindeki azaltıcı etkisi incelenmiştir. Ayrıca, farklı malzeme dağılım karakterlerine sahip kirişlere air frekans değerleri arasındaki farklar fark edilmiş ve incelenmiştir. Elde edilen sonuçlar, çeşitli tablolarda sunulmuştur.

Anahtar Kelimeler: *Diferansiyel Dönüşüm Yöntemi, Dönen Timoshenko Kiriş, Fonksiyonel Derecelendirilmiş Kiriş*

INTRODUCTION

The concept of functionally graded materials (FGMs) was originated from a group of material scientists in Japan as means of preparing thermal barrier materials *(Loy et al., 1999)*. FGMs are special composites that have continuous variation of material properties in one or more directions to provide designers with the ability to distribute strength and stiffness in a desired manner to get suitable structures for specific purposes in engineering and scientific fields such as design of aircraft and space vehicle structures, electronic and biomedical installations, automobile sector, defence endustries, nuclear reactors, electronics, transportation sector, etc. As a consequence, it is important to understand the static and dynamic behavior of FGMs so it has been an area of intense research in recent years. Especially, functionally graded beam (FGB) structures have become a fertile area of research since beam structures have been widely used in aeronautical, astronautical, civil, mechanical and other kind of installations. Several research papers provide a good introduction and further references on the subject *(Alshorbagy et al., 2011; Chakraborty et al., 2003; Giunta et al., 2011; Huang and Li, 2010; Kapuria et al., 2008; Lai et al., 2012; Li, 2008; Loja et al., 2012; Lu and Chan, 2005; Tai and Vo, 2012; Wattanasakulpong et al., 2012; Zhong and Yu, 2007).*

Due to the increasing application trend of FGMs, several beam theories have been developed to examine the response of FGBs. The Classical Beam Theory (CBT), i.e. Euler Bernoulli Beam Theory, is the simplest theory that can be applied to slender FGBs. The first order shear deformation theory (FSDT), i.e. Timoshenko Beam Theory, is used for the case of either short beams or high frequency applications to overcome the limitations of the CBT by accounting for the tranverse shear deformation effect. *Bhimaraddi and Chandrashekhara (1991)* derived laminated composite beam's equations of motion using the first-order shear deformation plate theory (FSDPT). *Dadfarnia (1997)* developed a new beam theory for laminated composite beams using the assumption that the lateral stresses and all derivatives with respect to lateral coordinate in the plate equations of motion are ignored.

In this study, which is an extension of the authors' previous works *(Kaya and Ozdemir Ozgumus, 2007; Kaya and Ozdemir Ozgumus, 2010; Ozdemir Ozgumus and Kaya, 2013, Ozdemir, 2016),* free vibration analysis of a rotating functionally graded Timoshenko beam that undergoes flapwise bending vibrations is performed. At the beginning of the study, expressions for both the kinetic and the potential energies are derived in a detailed way by using explanatory tables and figures. In the next step, the governing differential equations of motion are obtained applying the Hamilton's principle. In the solution part, the equations of motion, including the parameters for the rotary inertia, shear deformation, power law index parameter and slenderness ratio are solved using an efficient mathematical technique, called the differential transform method (DTM). Natural frequencies are calculated and effects of the parameters,mentioned above, are investigated. Calculated results are compared with the ones in open literature and consequently, it is observed that there is a good agreement between the results which proves the correctness and the accuracy of the DTM.

BEAM MODEL

The governing differential equations of motion are derived for the free vibration analysis of a functionally graded rotating Timoshenko beam model with a right-handed Cartesian coordinate system which is represented by Figure 1.

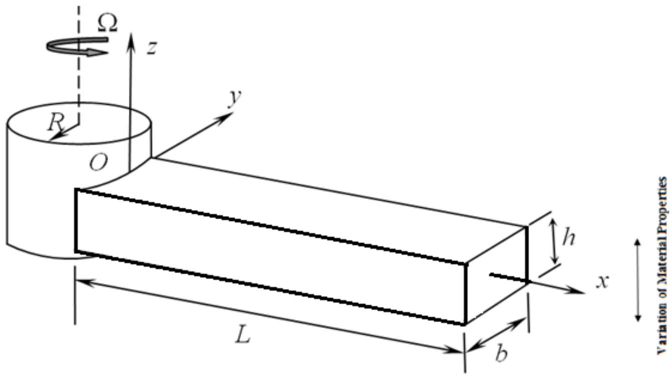

Figure 1. Rotating beam model and coordinate system

Here, a functionally graded Timoshenko beam of length L , which is fixed at point O to a rigid hub with radius R, is shown. The beam height is h and width is b with cantilever boundary condition at point O. is shown. The xyz axes represent a global orthogonal coordinate system with its origin at the root of the beam. The beam is assumed to be rotating in the counter-clockwise direction at a constant angular velocity, Ω. In the right-handed Cartesian coordinate system, the x -axis coincides with the neutral axis of the beam in the undeflected position, the z-axis is parallel to the axis of rotation, but not coincident and the y-axis lies in the plane of rotation.

FUNCTIONALLY GRADED BEAM FORMULATION

Beam Models and Related Formulation

In the present work, flapwise bending vibration analysis of a functionally graded Timoshenko beam that rotates with a constant angular velocity is carried out. The analyses are carried out for two types of FGB models. For the first type, the material properties of the beam vary such that the material at the bottom surface is ceramic and the material at the the top surface is metal as shown in Figure 2(a). For the second type, the beam consists of metallic core with ceramic surfaces as shown in Figure 2(b).

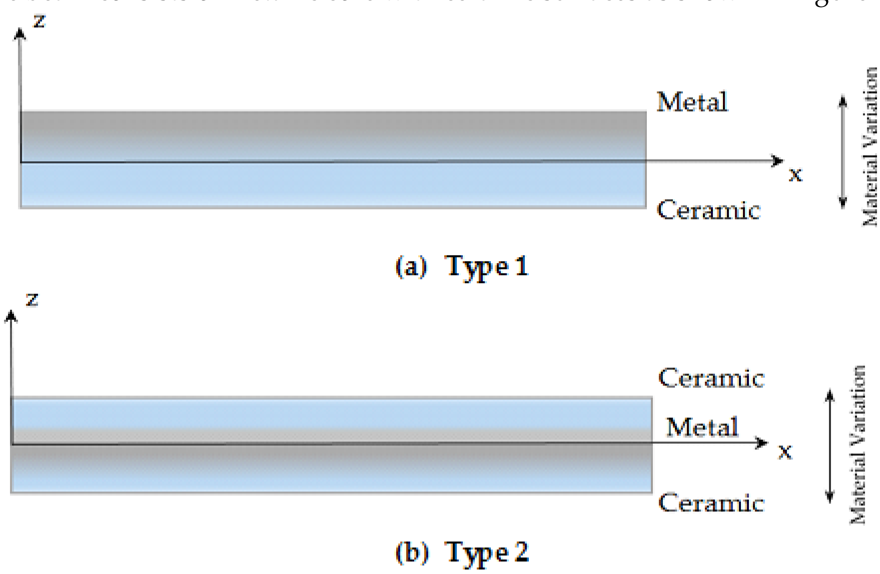

Figure 2. Fuctionally graded beam models and coordinate systems

Material properties of the beam, i.e. modulus of elasticity, E, shear modulus, G, Poisson's ratio, ν and material density, ρ are assumed to vary continuously in the thickness direction, z, as a function of the volume fraction and the properties of the the constituent materials according to a simple power law.

According to the rule of mixture, the effective material property P(z) can be expressed as follows

$$P(z) = P_c V_c + P_m V_m \tag{1}$$

where P_c and P_m are the properties of the ceramic and metal while V_c and V_m are the corresponding volume fractions. The relation between the volume fractions is given by

$$V_m + V_c = 1 \tag{2}$$

Beam of The 1st Type (Figure 1a)

The volume fraction of the top constituent of the beam, V_m, is assumed to be given by

$$V_m = \left(\frac{z}{h} + \frac{1}{2}\right)^k, \quad (k \geq 0) \tag{3}$$

Here k is the non-negative power law index parameter that dictates the material variation profile through the beam thickness.

Considering Eqns. (1)-(3), the effective material property of the 1st type beam can be rewritten as follows

$$P_1(z) = (P_m - P_c)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_c \tag{4}$$

It is evident from Eqn.(4) that

$$@\, z = h/2\,, \quad E = E_m, \nu = \nu_m, G = G_m, \rho = \rho_m \tag{5a}$$

$$@\, z = -h/2\,, \quad E = E_c, \nu = \nu_c, G = G_c, \rho = \rho_c. \tag{5b}$$

Beam of The 2nd Type (Figure 1b)

The volume fraction of the top constituent of the beam, V_t, is assumed to be given by

$$V_c = \left[\frac{z}{h}\right]^k, \quad (k \geq 0) \tag{6}$$

Considering Eqns. (1), (2) and (6), the effective material property of the 2nd type beam can be rewritten as follows

$$P_2(z) = (P_m - P_c)\left[\frac{z}{h}\right]^k + P_c \tag{7}$$

It is evident from Eqn.(7) that

$$@\, z = \pm h/2\,, \quad E = E_c, \nu = \nu_c, G = G_c, \rho = \rho_c. \tag{8a}$$

$$@\, z = 0\,, \quad E = E_m, \nu = \nu_m, G = G_m, \rho = \rho_m \tag{8b}$$

DISPLACEMENT FIELD AND STRAIN FIELD

The cross-sectional and the longitudinal views of a Timoshenko beam that undergoes extension and flapwise bending deflections are introduced in Figure 3 (a) and3 (b), respectively. Here, a reference point is chosen and is represented by P_0 before deformation and by P after deformation.

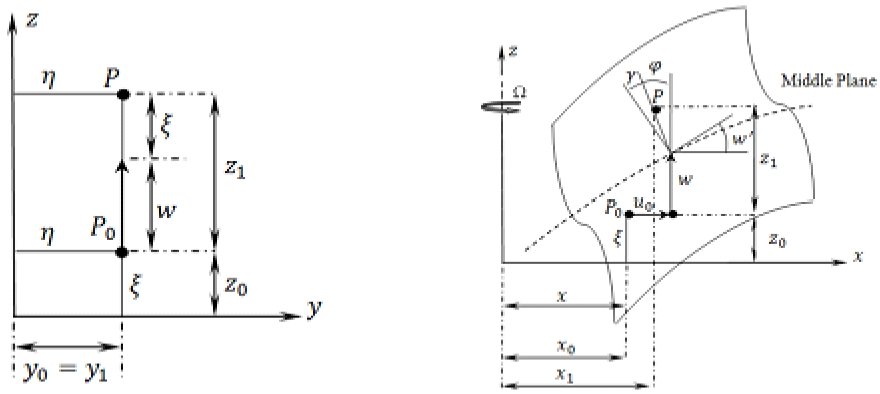

Figure 3. (a) Cross-sectional view **(b)** Longitudinal view of the rotating Timoshenko beam

Here, η is the offset of the reference point from the z- axis, ξ is the offset of the reference point from the middle plane, x is the offset of the reference point from the z-axis, u_0 is the elongation, w is the flapwise bending displacement, φ is the rotation due to bending and γ is the shear angle.

Considering Figure 3 (a) and 3 (b), coordinates of the reference point are obtained as follows

Before deflection (coordinates of P_0):

$$x_0 = x \tag{9a}$$

$$y_0 = \eta \tag{9b}$$

$$z_0 = \xi \tag{9c}$$

After deflection (coordinates of P):

$$x_1 = x + u_0 + \xi\varphi \tag{10a}$$

$$y_1 = \eta \tag{10b}$$

$$z_1 = w + \xi \tag{10c}$$

The position vectors of the reference point are represented by $\vec{r_0}$ and $\vec{r_1}$ before and after deflection, respectively. Therefore, $d\vec{r_0}$ and $d\vec{r_1}$ can be written as follows

$$d\vec{r_0} = dx\vec{\imath} + d\eta\vec{\jmath} + d\xi\vec{k} \tag{11a}$$

$$d\vec{r_1} = [(1 + u_0' + \xi\varphi')]dx\vec{\imath} + d\eta\vec{\jmath} + (w'dx + d\xi)\vec{k} \tag{11b}$$

where $(\quad)'$ denotes differentiation with respect to the spanwise coordinate, x .

The classical strain tensor ε_{ij} may be obtained by using the following equilibrium equation given by *Eringen (1980)*

$$d\vec{r_1}.d\vec{r_1} - d\vec{r_0}.d\vec{r_0} = 2[dx \quad d\eta \quad d\xi]\left[\varepsilon_{ij}\right]\begin{bmatrix} dx \\ d\eta \\ d\xi \end{bmatrix} \tag{12}$$

where

$$\left[\varepsilon_{ij}\right] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{x\eta} & \varepsilon_{x\xi} \\ \varepsilon_{\eta x} & \varepsilon_{\eta\eta} & \varepsilon_{\eta\xi} \\ \varepsilon_{\xi x} & \varepsilon_{\xi\eta} & \varepsilon_{\xi\xi} \end{bmatrix} \tag{13}$$

Substituting Eqn. (11a) and Eqn. (11b) into Eqn.(12), the components of the strain tensor ε_{ij} are obtained as follows

$$\varepsilon_{xx} = u_0' + \frac{(u_0')^2}{2} + \frac{(w')^2}{2} + u_0'\varphi'\xi + \varphi'\xi + \frac{(\varphi')^2}{2}\xi^2 \tag{14a}$$

$$\gamma_{x\eta} = 0 \tag{14b}$$

$$\gamma_{x\xi} = (w' + \varphi) + \varphi\varphi'\xi - u_0'\varphi \tag{14c}$$

where ε_{xx}, $\gamma_{x\eta}$ and $\gamma_{x\xi}$ are the axial strain and the shear strains, respectively.

In this work; only ε_{xx}, $\gamma_{x\eta}$ and $\gamma_{x\xi}$ are used in the calculations because as noted by *Hodges and Dowell (1974)* for long slender beams, the axial strain ε_{xx} is dominant over the transverse normal strains, $\varepsilon_{\eta\eta}$ and $\varepsilon_{\xi\xi}$. Moreover, the shear strain $\gamma_{\eta\xi}$ is two order smaller than the other shear strains, $\gamma_{x\xi}$ and $\gamma_{x\eta}$. Therefore, $\varepsilon_{\eta\eta}$, $\varepsilon_{\xi\xi}$ and $\gamma_{\eta\xi}$ are neglected.

In order to obtain simpler expressions for the strain components given by Eqns. (10a)-(10c), higher order terms can be neglected so an order of magnitude analysis is performed by using the ordering scheme, taken from *Hodges and Dowell (1974)*and introduced in Table 1.

Table 1. Ordering scheme for the Timoshenko beam model

Term	Order
w'	$O(\varepsilon)$
φ	$O(\varepsilon)$
$w'+\varphi$	$O(\varepsilon^2)$
u_0'	$O(\varepsilon^2)$
φ'	(ε^2)

Hodges and Dowell (1974) used the formulation for an Euler-Bernoulli beam so in this study, their formulation is modified for a Timoshenko beam and a new expression, $w'+\varphi=O(\varepsilon^2)$ is added to their ordering scheme as a contribution to literature.

Considering Table 1, Eqns. (14a)-(14c) are simplified as follows

$$\varepsilon_{xx}=u_0'+\frac{(u_0')^2}{2}+\frac{(w')^2}{2}+\varphi'\xi \tag{15a}$$

$$\gamma_{x\eta}=0 \tag{15b}$$

$$\gamma_{x\xi}=w'+\varphi \tag{15c}$$

Potential Energy

The potential energy expression is given by

$$U=\frac{1}{2}\int_0^L\int_A\left(\sigma_{xx}\varepsilon_{xx}+\tau_{x\xi}\gamma_{x\xi}\right)dA\,dx=\frac{b}{2}\int_0^L\int_{-h/2}^{h/2}\left(\sigma_{xx}\varepsilon_{xx}+\tau_{x\xi}\gamma_{x\xi}\right)d\xi\,dx \tag{16}$$

The axial force, N, the bending moment, M and the shear force, Q that act on a laminate at the midplane are expressed as follows *(Kollar and Springer, 2003)*

$$N=b\int_{-h/2}^{h/2}\sigma dz \tag{17a}$$

$$M=b\int_{-h/2}^{h/2}z\sigma dz \tag{17b}$$

$$Q=b\int_{-h/2}^{h/2}\tau dz \tag{17c}$$

Substituting Eqns. (15a)-(15c) into Eqn. (16) and considering Eqns. (17a)-(17c), the following expression is obtained.

$$U=\frac{1}{2}\int_0^L\left\{N_x\left[u_0'+\frac{(w')^2}{2}\right]+M_x\varphi'+Q_{xz}(w'+\varphi\,)\right\} \tag{18}$$

where

$$N_x=\bar{A}_{11}u_0'+\bar{B}_{11}\varphi' \tag{19a}$$

$$M_x=\bar{B}_{11}u_0'+\bar{D}_{11}\varphi' \tag{19b}$$

$$Q=\bar{A}_{55}\gamma_{x\xi} \tag{19c}$$

Here the stiffness coefficients are obtained as follows

$$[\bar{A}_{11}\quad \bar{B}_{11}\quad \bar{D}_{11}]=\int_A E(z)\,[1\quad z\quad z^2]\,dA \tag{20a}$$

$$\bar{A}_{55}=K\int_A G(z)\,dA \tag{20b}$$

where K is defined as the shear correction factor that takes the value of $K={}^5/_6$ for rectangular cross sections.

Substituting Eqns. (19a)-(19c) into Eqn. (18) gives

$$U=\frac{1}{2}\int_0^L\{\bar{A}_{11}(u_0')^2+2\bar{B}_{11}u_0'\varphi'+\bar{D}_{11}(\varphi')^2+\bar{A}_{55}(w'+\varphi\,)\}dx \tag{21}$$

The uniform strain, ε_0, and the associated axial displacement, u_0, due to the centrifugal force, $T(x)$, are related to each other as follows:

$$u_0' = \varepsilon_0 = \frac{T(x)}{\bar{A}_{11}} \tag{22}$$

where the centrifugal force, $T(x)$, is expressed by

$$T(x) = \int_x^L \rho A(R + x)\Omega^2\, dx \tag{23}$$

Here ρ is the material density and A is the cross-sectional area.

Referring to the Eqns. (17)-(18), variation of Eqn. (17) is obtained as follows.

$$\delta U = \int_0^L \{(\bar{A}_{11}u_0' + \bar{B}_{11}\varphi')\delta u_0' + (\bar{B}_{11}u_0' + \bar{D}_{11}\varphi')\delta\varphi' + \bar{A}_{55}(w' + \varphi)(\delta w' + \delta\varphi)\}dx \tag{24}$$

Kinetic Energy

The position vector of point P shown in Figure 3 is given by

$$\vec{r} = (x + u_0 + \xi\varphi)\vec{\imath} + w\vec{k} \tag{25a}$$

The velocity vector of this point due to rotation of the beam is obtained as follows

$$\vec{V} = \frac{\partial\vec{r}}{\partial t} + \Omega\vec{k} \times \vec{r} = (\dot{u}_0 + \xi\dot{\varphi})\vec{\imath} + (R + x + u_0 + \xi\varphi)\Omega\vec{\jmath} + \dot{w}\vec{k} \tag{25b}$$

Hence, the velocity components are

$$V_x = \dot{u}_0 + \xi\dot{\varphi} \tag{26a}$$

$$V_y = (R + x + u_0 + \xi\varphi)\Omega \tag{27b}$$

$$V_z = \dot{w} \tag{28c}$$

The kinetic energy expression is given by

$$K = \frac{1}{2}\int_0^L \int_A \rho\left(V_x^2 + V_y^2 + V_z^2\right) dA\, dx = \frac{b}{2}\int_0^L \int_{-h/2}^{h/2} \rho\left(V_x^2 + V_y^2 + V_z^2\right) d\xi\, dx \tag{29}$$

Substituting the velocity components into Eq. (29) and taking the variation of the kinetic energy give

$$\delta K = \int_0^L \{I_1[\dot{u}_0\delta\dot{u}_0 + \dot{w}\delta\dot{w} + (R + x + u_0)\Omega^2\,\delta u_0] + I_2[\dot{u}_0\delta\dot{\varphi} + \dot{\varphi}\delta\dot{u}_0 + (R + x + u_0)\delta\varphi + \varphi\Omega^2\,\delta u_0] + I_3(\dot{\varphi}\delta\dot{\varphi} + \varphi\Omega^2\,\delta\varphi)\}dx \tag{30}$$

where I_1, I_2 and I_3 are the inertial characteristics of the beam given by

$$[I_1 \quad I_2 \quad I_3] = \int_A \rho(z)\,[1 \quad z \quad z^2]\, dA \tag{31}$$

Equations of Motion and The Boundary Conditions

The Hamilton's Principle is expressed as follows

$$\int_{t1}^{t2} \delta(U - K)dt = 0 \tag{32}$$

Substituting Eqn.(24) and Eqn.(30) into Eqn. (32) gives the equations of motion and the boundary conditions as follows

Equations of Motion:

$$\bar{A}_{11}u_0'' + \bar{B}_{11}\varphi'' + I_1\Omega^2 u_0 + I_2\Omega^2\varphi = I_1\ddot{u}_0 + I_2\ddot{\varphi} \tag{33a}$$

$$(w'T)' + (\bar{A}_{55} - \bar{E}_{11})w'' + \bar{A}_{55}\varphi' = I_1\ddot{w} \tag{33b}$$

$$\bar{D}_{11}\varphi'' + \bar{B}_{11}u_0'' + I_2\Omega^2 u_0 + I_3\Omega^2\varphi - \bar{A}_{55}(w' + \varphi) = I_2\ddot{u}_0 + I_3\ddot{\varphi} \tag{33c}$$

where the expression of the centrifugal force, T, given by Eq. (23) can be rewritten as follows

$$T(x) = \int_x^L I_1(R + x)\Omega^2\, dx \tag{34}$$

Clamped-Free Boundary Conditions:

$$@\, x = 0 \qquad u_0(0,t) = w(0,t) = \varphi(0,t) = 0 \tag{35a}$$

$$@\, x = L \qquad \bar{A}_{11}u_0'(L,t) + \bar{B}_{11}\varphi'(L,t) = 0 \tag{35b}$$

$$Tw'(L,t) + \bar{A}_{55}[w'(L,t) + \varphi(L,t)] = 0 \tag{35c}$$

$$\bar{D}_{11}\varphi(L,t) + \bar{B}_{11}u_0'(L,t) - \bar{A}_{55}(w' + \varphi) = 0 \tag{35d}$$

The boundary condition expressed by Eq. (35c) can be written in a simpler form by noting that the centrifugal force is zero at the free end of the beam, $T(L) = 0$.

$\bar{A}_{55}[w'(L,t)+\varphi(L,t)] = 0$ (36)

In order to investigate the free vibration of the beam model considered in this study, a sinusoidal variation of u_0, w and φ with a circular natural frequency, ω, is assumed and the functions are approximated as

$u_0(x,t) = \bar{u}(x)e^{i\omega t}$ (37a)
$w(x,t) = \bar{w}(x)e^{i\omega t}$ (37b)
$\varphi(x,t) = \bar{\varphi}(x)e^{i\omega t}$ (37c)

Substituting Eqns. (37a)-(37c) into the equations of motion, i.e. Eqns.(33a)-(33c), and into the boundary conditions, i.e. Eqns.(35a)-(35d), the following dimensionless equations are obtained as follows

Equations of motion:

$\gamma^2\tilde{u}^{**} + \alpha^2\tilde{\varphi}^{**} + (\beta^2+\lambda^2)\,(\tilde{u}+\mu^2\tilde{\varphi}) = 0$ (38a)
$\left\{\beta^2\left[\sigma(1-\bar{x})+\frac{1}{2}\,(1-\bar{x}^2)\right]+\frac{1}{\tau^2}-e_0\right\}\tilde{w}^{**} - \beta^2(\sigma+\bar{x})\tilde{w}^{*} + \lambda^2\tilde{w} + \frac{1}{\tau^2}\tilde{\varphi}^{*} = 0$ (38b)
$\alpha^2\tau^2\tilde{u}^{**} + \tau^2\tilde{\varphi}^{**} + \mu^2\tau^2(\beta^2+\lambda^2)\tilde{u} + [r^2\tau^2(\beta^2+\lambda^2)-1]\tilde{\varphi} - \tilde{w}^{*} = 0$ (38c)

Clamped-Free Boundary Conditions:

@ $\bar{x} = 0$ $\quad \tilde{u}(0,t) = \tilde{w}(0,t) = \tilde{\varphi}(0,t) = 0$ (39a)
@ $\bar{x} = 1$ $\quad \gamma^2\tilde{u}^{*}(L,t) + \alpha^2\tilde{\varphi}^{*}(L,t) = 0$ (39b)
$\left(\frac{1}{\tau^2}-e_0\right)\tilde{w}^{*}(L,t) + \frac{1}{\tau^2}\tilde{\varphi}(L,t) = 0$ (39c)
$\alpha^2\tilde{u}^{*}(L,t) + \tilde{\varphi}^{*}(L,t) = 0$ (39d)

Here, the dimensionless parameters given below in Table 2, are used to be able to simplify the equations of motion, boundary conditions and to make comparisons with open literature.

Table 2. Dimensionless parameters

$\bar{x} = \frac{x}{L}$	$\tilde{w} = \frac{\bar{w}}{L}$	$\tilde{u} = \frac{\bar{u}}{L}$	$\tilde{\varphi} = \bar{\varphi}$
$\sigma = \frac{R}{L}$	$\gamma^2 = \frac{\bar{A}_{11}L^2}{\bar{D}_{11}}$	$\alpha^2 = \frac{\bar{B}_{11}L}{\bar{D}_{11}}$	$\beta^2 = \frac{I_1L^4\Omega^2}{\bar{D}_{11}}$
$r^2 = \frac{I_3}{I_1L^2}$	$\tau^2 = \frac{\bar{D}_{11}}{\bar{A}_{55}L^2}$	$\lambda^2 = \frac{I_1L^4\omega^2}{\bar{D}_{11}}$	$\mu^2 = \frac{I_2}{I_1L}$

where β is the rotational speed parameter, λ is the frequency parameter, r is the inverse of the slenderness ratio and σ is the hub radius parameter.

DIFFERENTIAL TRANSFORM METHOD

The Differential Transform Method (DTM) is a transformation technique based on the Taylor series expansion and is a useful tool to obtain analytical solutions of the differential equations. In this method, certain transformation rules are applied and the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions and the solution of these algebraic equations gives the desired solution of the problem.

Consider a function $f(x)$ which is analytic in a domain D and let $x = x_0$ represent any point in D. The function $f(x)$ is then represented by a power series whose center is located at x_0. The differential transform of the function $f(x)$ is given by

$$F[k]=\frac{1}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{40}$$

where $f(x)$ is the original function and $F[k]$ is the transformed function. The inverse transformation is defined as

$$f(x)=\sum_{k=0}^{\infty}(x-x_0)^k F[k] \tag{41}$$

Combining Eqn. (40) and Eqn. (41), we get

$$f(x)=\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{42}$$

Considering Eqn. (42), it is noticed that the concept of differential transform is derived from Taylor series expansion. However, the method does not evaluate the derivatives symbolically.

In actual applications, the function $f(x)$ is expressed by a finite series and Eqn. (42) can be written as follows

$$f(x)=\sum_{k=0}^{m}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{43}$$

which means that the rest of the series

$$f(x)=\sum_{k=m+1}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{44}$$

is negligibly small. Here, the value of m depends on the convergence of the natural frequencies.

Theorems that are frequently used in the transformation procedure are introduced in Table 3 and theorems that are used for boundary conditions are introduced in Table 4.

Table 3. DTM theorems used for equations of motion

Original Function	Transformed Function
$f(x)=g(x)\pm h(x)$	$F[k]=G[k]\pm H[k]$
$f(x)=\lambda g(x)$	$F[k]=\lambda G[k]$
$f(x)=g(x)h(x)$	$F[k]=\sum_{l=0}^{k}G[k-l]H[l]$
$f(x)=\frac{d^n g(x)}{dx^n}$	$F[k]=\frac{(k+n)!}{k!}G[k+n]$
$f(x)=x^n$	$F[k]=\delta(k-n)=\begin{cases}0 & if \quad k\neq n\\ 1 & if \quad k=n\end{cases}$

Table 4. DTM theorems used for boundary conditions

$x=0$		$x=1$	
Original B.C.	**Transformed B.C.**	**Original B.C.**	**Transformed B.C.**
$\frac{df(0)}{dx}=0$	$F(0)=0$	$f(1)=0$	$\sum_{k=0}^{\infty}F(k)=0$
$\frac{df}{dx}(0)=0$	$F(1)=0$	$\frac{df}{dx}(1)=0$	$\sum_{k=0}^{\infty}kF(k)=0$
$\frac{d^2f}{dx^2}(0)=0$	$F(2)=0$	$\frac{d^2f}{dx^2}(1)=0$	$\sum_{k=0}^{\infty}k(k-1)F(k)=0$
$\frac{d^3f}{dx^3}(0)=0$	$F(3)=0$	$\frac{d^3f}{dx^3}(1)=0$	$\sum_{k=0}^{\infty}(k-1)(k-2)kF(k)=0$

After applying the differential transform method to Eqns. (38a)-(39d), the transformed equations of motion and boundary conditions are obtained as follows

Equations of motion:

$$\gamma^2(k+1)(k+2)U[k+2]+\alpha^2(k+1)(k+2)\varphi[k+2]+(\beta^2+\lambda^2)\,(U[k]+\mu^2\varphi[k])=0 \quad (45a)$$

$$\left[\beta^2\left(\sigma+\frac{1}{2}\right)+\frac{1}{\tau^2}\right](k+1)(k+2)W[k+2]-\beta^2\sigma(k+1)^2W[k+1]+\left[\lambda^2-\frac{1}{2}\beta^2k(k+1)\right]W[k]+\frac{1}{\tau^2}(k+1)\,\varphi[k+1]=0 \quad (45b)$$

$$\alpha^2(k+1)(k+2)U[k+2]+(k+1)(k+2)\varphi[k+2]+\mu^2(\beta^2+\lambda^2)U[k]+\left[r^2(\beta^2+\lambda^2)-\frac{1}{\tau^2}\right]\varphi[k]-\frac{1}{\tau^2}(k+1)W[k+1]=0 \quad (45c)$$

Clamped-Free Boundary Conditions:

$$@\ \bar{x}=0 \qquad U[k]=W[k]=\varphi[k]=0 \quad (46a)$$

$$@\ \bar{x}=1 \qquad \gamma^2\sum_{k=0}^{\infty}kU[k]+\alpha^2\sum_{k=0}^{\infty}k\varphi[k]=0 \quad (46b)$$

$$\left(\frac{1}{\tau^2}-e_0\right)\sum_{k=0}^{\infty}kW[k]+\frac{1}{\tau^2}\sum_{k=0}^{\infty}\varphi[k]=0 \quad (47c)$$

$$\alpha^2\sum_{k=0}^{\infty}kU[k]+\sum_{k=0}^{\infty}k\varphi[k]-f_0=0 \quad (48d)$$

RESULTS AND DISCUSSIONS

In numerical analysis, effects of several parameters, i.e. material distribution property, angular velocity, Ω, , slenderness ratio, ***L/h*** and power law index parameter, ***k***, on the natural frequencies of a FG Timoshenko beam are investigated and the results are presented in related tables. In order to validate the calculated results, comparisons with the studies in open literature are made. It is believed that the tabulated results can be used as references by the other researchers to validate their results.

Material Properties

As in the work of *Şimsek (2010),* Aluminum (Al) is used as the metal and Alumina (Al_2O_3) is used as the ceramic material. The material properties of the FG beam are introduced in Table 5.

Table 5. Material properties of the FG beam

Property	Aluminum (Al)	Alumina (Al_2O_3)
Elasticity Modulus, E	70 GPa	380 GPa
Material Density, ρ	2702 kg/m^3	3960 kg/m^3
Poisson's Ratio, ν	0.3	0.3

Variation of the fundamental natural frequency of a nonrotating C-F Functionally Graded Timoshenko Beam according to the power law exponent for L/h=20 is introduced in Table 6. The dimensionless frequency values are given by the formula, $\boldsymbol{\lambda} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$.

The calculated results of a nonrotating FG Timoshenko beam are compared with the ones given by *Şimsek (2010)*, a very good agreement between the results is observed

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Table 6. Dimensionless fundamental frequencies of a nonrotating C-F FG timoshenko beam

Frequency	**Power Law Exponent (k)**							
$\lambda = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$	**0**	**0.2**	**0.5**	**1**	**2**	**5**	**10**	**Full Metal**
Fundamental	1.94955	1.81407	1.66026	1.50103	1.36966	1.30373	1.26493	1.01297
Şimsek (2010)	***1.94957***	***1.81456***	***1.66044***	***1.50104***	***1.36968***	***1.30375***	***1.26495***	***1.01297***

In Table 7, variation of the dimensionless natural frequencies of a C-F Functionally Graded Timoshenko Beam with respect to the power law exponent, k, the slenderness ratio, L/h and the rotational velocity parameter, β is introduced for the beam model where the material properties vary such that the material at the bottom surface is ceramic and the material at the the top surface is metal as shown in Figure 2(a).

Table 7. Variation of the dimensionless natural frequencies of a C-F functionally graded rotating timoshenko beam with respect to the power law exponent, k, the slenderness ratio, L/h and the rotational velocity parameter, β where $P_1(z) = (P_m - P_c)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_c$

	K=0		**K=1**		**K=2**		**K=5**	
L/h	**β=0**	**β=5**	**β=0**	**β=5**	**β=0**	**β=5**	**β=0**	**β=5**
3	1.80329	3.34825	1.39885	2.7451	1.27348	2.58828	1.19956	2.41723
	8.21514	8.63351	6.44425	7.11864	5.77913	6.34967	5.22620	5.47542
	9.06941	10.3185	7.7081	8.47941	7.05971	7.91179	6.19459	7.28635
	17.9802	20.8083	14.3438	16.6267	12.9094	15.0887	11.6647	13.7449
4	1.86385	3.43576	1.44141	2.8159	1.31344	2.65917	1.24242	2.4888
	9.42868	11.4397	7.39468	9.02089	6.68493	8.18294	6.15895	7.32402
	12.0925	11.7692	10.1889	10.0519	9.27327	9.25019	8.07187	8.28298
	21.5877	24.1181	17.1413	19.2664	15.4804	17.5649	14.1093	16.1152
5	1.89441	3.48336	1.46276	2.85363	1.33353	2.69678	1.26419	2.52732
	10.2025	12.1629	7.97167	9.67831	7.23018	8.90268	6.72424	8.2538
	15.1157	14.8582	12.7061	12.5178	11.5406	11.3768	10.0258	9.89435
	24.2839	26.62	19.1875	21.1936	17.3683	19.3554	15.9509	17.851

In Table 8, variation of the dimensionless natural frequencies of a C-F Functionally Graded Timoshenko Beam with respect to the power law exponent, k, the slenderness ratio, L/h and the rotational velocity parameter, β is introduced for the beam model that consists of metallic core with ceramic surfaces as shown in Figure 2(b) where $P_2(z) = (P_m - P_c)\left[\frac{z}{h}\right]^k + P_c$.

Table 8. Variation of the dimensionless natural frequencies of a C-F Functionally Graded Rotating Timoshenko Beam with respect to the power law exponent, k, the slenderness ratio, L/h and the rotational velocity parameter, β where σ = 0.

	K=0		K=1		K=2		K=5	
L/h	**β=0**	**β=5**	**β=0**	**β=5**	**β=0**	**β=5**	**β=0**	**β=5**
3	1.80329	3.34825	1.42441	2.66318	1.17129	2.18738	0.958251	1.78056
	8.21514	8.63351	6.23496	6.07854	5.16473	5.09202	4.34608	4.52523
	9.06941	10.3185	6.4741	8.00631	5.40984	6.60798	4.76066	5.47231
	17.9802	20.8083	13.4681	15.9219	11.181	13.1726	9.49716	11.0175
4	1.86385	3.43576	1.48112	2.73909	1.21655	2.24861	0.991071	1.82755
	9.42868	11.4397	7.25552	8.33957	5.99512	6.97792	4.99596	6.07073
	12.0925	11.7692	8.63213	8.92615	7.21313	7.35805	6.34754	6.17294
	21.5877	24.1181	16.3358	18.5139	13.537	15.3046	11.4162	12.7746
5	1.89441	3.48336	1.51012	2.78166	1.23963	2.28276	1.00766	1.85319
	10.2025	12.1629	7.93067	9.5414	6.54094	7.85694	5.41208	6.4584
	15.1157	14.8582	10.7902	10.5576	9.01641	8.82937	7.93443	7.79545
	24.2839	26.62	18.5478	20.5329	15.3449	16.9584	12.856	14.1078

In this study, formulation of a rotating functionally graded Timoshenko beam that undergoes flapwise bending vibration is derived by introducing several explanotary figures and tables. Applying the Hamilton's Principle to the obtained energy expressions, governing differential equations of motion and the boundary conditions are derived. In the solution part, the equations of motion, including the parameters for the rotary inertia, shear deformation, power law index parameter, material distribution, slenderness ratio and rotational speed are solved using an efficient mathematical technique, called the Differential Transform Method (DTM). Natural frequencies are calculated and effects of the parameters,mentioned above, are investigated.

Considering the calculated results, the following conclusions are reached:

a. As the slenderness ratio L/h increases, the natural frequencies increase;

b. As the rotational speed parameter increases, the natural frequencies increase due to the stiffening effect of the centrifugal force.

c. The natural frequencies decrease as the value of the power-law exponent, k, increases for both types of beam models.

d. When Table 7 and Table 8 are compared, it is noticed that natural frequencies of the first beam model , i.e. the material properties vary such that the material at the bottom surface is ceramic and the material at the the top surface is metal, are higher than the natural frequencies of the second beam model, i.e. consists of metallic core with ceramic surfaces.

In Figure 4, convergence of the first five natural frequencies with respect to the number of terms, N, used in DTM application is introduced where L/h=3 and k=1. To evaluate up to the fifth natural frequency to fourth-digit precision, it was necessary to take 45 terms. During the calculations, it is noticed that when the rotational speed parameter is increased, the number of the terms has to be increased to achieve the same accuracy. Additionally, here it is seen that higher modes appear when more terms are taken into account in DTM application. Thus, depending on the order of the required mode, one must try a few values for the term number at the beginning of the calculations in order to find the adequate number of terms. For instance, only N=100 is enough for the results given in Table 7 and Table 8.

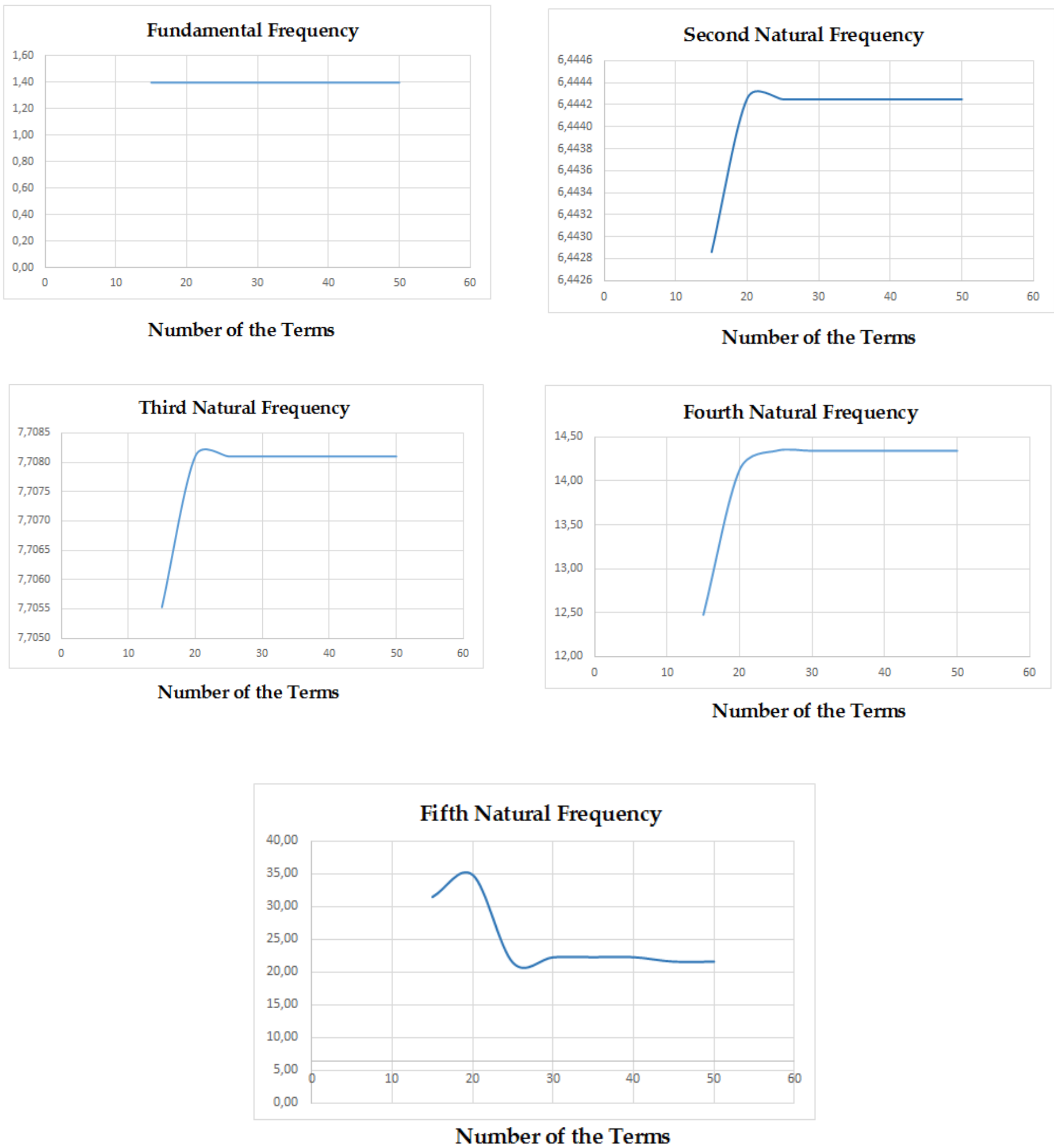

Figure 4. Convergence of the first five natural frequencies with respect to the number of terms, *N*

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