TITLE: Investigation of Exact Solutions of Perturbed Nonlinear Schrödinger`s Equation by

$\exp\left(-\Phi\left(\xi\right)\right)$-Expansion Method

AUTHORS: Mehmet EKICI

PAGES: 373-379

ORIGINAL PDF URL: https://dergipark.org.tr/tr/download/article-file/409383

# Investigation of Exact Solutions of Perturbed Nonlinear Schrödinger's Equation by $\exp(-\Phi(\xi))$-Expansion Method

## Mehmet EKİCİ [*1]

[1] Bozok University, Faculty of Science and Arts, Department of Mathematics, 66100, Yozgat

**Abstract:** This paper presents an analytic study on optical solitons of a perturbed nonlinear Schrödinger's equation (NLSE). An integration tool that is the $\exp(-\Phi(\xi))$-expansion approach is used to find exact solutions. As a consequence, hyperbolic, trigonometric and rational function solutions are extracted by this approach.

## $\exp(-\Phi(\xi))$-Açılım Metodu ile Pertürbe Edilmiş Lineer Olmayan Schrödinger Denkleminin Tam Çözümlerinin Araştırılması

**Özet:** Bu makale, pertürbe edilmiş lineer olmayan Schrödinger denkleminin optik solitonları üzerine analitik bir çalışma sunar. Tam çözümler elde etmek için $\exp(-\Phi(\xi))$-açılım metodu kullanılır. Sonuç olarak bu metot ile hiperbolik, trigonometrik ve rasyonel fonksiyon çözümler elde edilir.

## 1. Introduction

The NLSE (nonlinear Schrödinger equation) has a central importance in many natural sciences as well as engineering with numerous interpretations and applications concerning eg. nonlinear optics, protein chemistry, plasma physics and fluid dynamics. This paper will consider the perturbed NLSE which governs the dynamics of solitons in negative-index material with non-Kerr nonlinearity and third-order dispersion, and the dimensionless form of the equation is given by [1, 2]

$$
\begin{aligned}
iu_t + au_{xx} &+ bu_{xt} + cF(|u|^2)u = \\
&-i\lambda u_x - is\left(|u|^2 u\right)_x - i\mu\left(|u|^2\right)_x u - i\theta|u|^2 u_x \\
&-i\gamma u_{xxx} - \theta_1\left(|u|^2 u\right)_{xx} - \theta_2|u|^2 u_{xx} - \theta_3 u^2 u_{xx}^*, \quad (1)
\end{aligned}
$$

where $u(x,t)$ is the complex field amplitude. $a$, $b$, and $c$ are the coefficients of group velocity dispersion, spatial-temporal dispersion and non-Kerr nonlinearity, and $\lambda$, $s$, $\mu$, $\theta$ and $\gamma$ account for the inter-modal dispersion, self-steepening, Raman effect, nonlinear dispersion and third-order dispersion, respectively. The last three terms appear in the context of negative-index material [1–6]. Recently, various analytical and numerical methods have been introduced to obtain solutions of nonlinear evolution equations. Some of these methods are $F$-function method [7], exp-function method [8], Hirota's bilinear method [9], homotopy perturbation method [10], variational iteration method [11, 12], Adomian Pade approx-

imation [13], Lie group method [14], homogeneous balance method [15], inverse scattering transform method [16], Jacobi elliptic expansion method [17], sine-cosine method [18], $(G'/G)$-expansion method [19] and improved $\tan(\Phi(\xi)/2)$-expansion method [20].

The main aim of this study is to extract exact solitons to Eq. (1) using the $\exp(-\Phi(\xi))$-expansion approach [21–23]. Four kinds of nonlinearity are considered for Eq. (1). They are Kerr law, power law, parabolic law and dual-power law.

## 2. Analytical Solutions

In order to solve Eq. (1), we use the wave transformation as

$$
u(x,t) = P(\xi)e^{i\Phi(x,t)}, \quad (2)
$$

where $P(\xi)$ represents the shape of the pulse and

$$
\xi = x - vt, \quad (3)
$$

$$
\Phi(x,t) = -\kappa x + \omega t + \zeta. \quad (4)
$$

In Eq. (2), the function $\Phi(x,t)$ gives the phase component of the soliton. Then, in Eq. (4), $\kappa$, $\omega$ and $\zeta$ respectively represent the frequency, wave number and phase constant. Finally in Eq. (3), $v$ shows the velocity of the soliton. Inserting (2) into (1) and then decomposing into real and

imaginary parts yield a pair of relations. Real part gives

$$
\begin{aligned}
&(a - bv + 3\kappa\gamma)P'' - \left((1 - b\kappa)\omega + a\kappa^2 - \lambda\kappa + \gamma\kappa^3\right)P \\
&+ cF(P^2)P + \left(s\kappa + \theta\kappa - \theta_1\kappa^2 - \theta_2\kappa^2 - \theta_3\kappa^2\right)P^3 \\
&+ 6\theta_1 P(P')^2 + (3\theta_1 + \theta_2 + \theta_3)P^2 P'' = 0,
\end{aligned} \tag{5}
$$

while imaginary part leads to

$$
\begin{aligned}
&(-v - 2a\kappa + b\omega + b\kappa v + \lambda - 3\gamma\kappa^2)P' \\
&+ (3s + 2\mu + \theta - 2\kappa(3\theta_1 + \theta_2 - \theta_3))P^2 P' + \gamma P''' = 0.
\end{aligned} \tag{6}
$$

The imaginary part equation implies the relations given by

$$\gamma = 0, \tag{7}$$

$$v = -\frac{2a\kappa - b\omega - \lambda}{1 - b\kappa}, \tag{8}$$

$$3s + 2\mu + \theta - 2\kappa(3\theta_1 + \theta_2 - \theta_3) = 0. \tag{9}$$

Therefore, Eq. (5) by virtue of Eq. (7) reduces to

$$
\begin{aligned}
&(a - bv)P'' - \left((1 - b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)P + cF(P^2)P \\
&+ \left(s\kappa + \theta\kappa - \kappa^2\theta_1 - \kappa^2\theta_2 - \kappa^2\theta_3\right)P^3 \\
&+ (3\theta_1 + \theta_2 + \theta_3)P^2 P'' + 6\theta_1 P\left(P'\right)^2 = 0.
\end{aligned} \tag{10}
$$

To obtain an analytic solution, one applies the transformations $\theta_1 = 0$, $\theta_2 = -\theta_3$ and $s = -\theta$ in Eq. (10) to find

$$(a - bv)P'' - \left((1 - b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)P + cF(P^2)P = 0, \tag{11}$$

where

$$\theta - \mu - 2\theta_3\kappa = 0. \tag{12}$$

## 2.1. Kerr law

For Kerr law nonlinearity

$$F(q) = q, \tag{13}$$

Eq. (1) reduces to

$$
\begin{aligned}
&iu_t + au_{xx} + b|u|^2 u + bu_{xt} + c|u|^2 u = \\
&- i\lambda u_x - is\left(|u|^2 u\right)_x - i\mu\left(|u|^2\right)_x u - i\theta|u|^2 u_x \\
&- i\gamma u_{xxx} - \theta_1\left(|u|^2 u\right)_{xx} - \theta_2|u|^2 u_{xx} - \theta_3 u^2 u_{xx}^*,
\end{aligned} \tag{14}
$$

and Eq. (11) simplifies to

$$(a - bv)P'' - \left((1 - b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)P + cP^3 = 0. \tag{15}$$

In this subsection, the $\exp(-\Phi(\xi))$-expansion method will be used, to obtain hyperbolic, trigonometric and rational function solutions to Eq. (14). According to the homogeneous balance method, Eq. (15) has the solution as

$$P(\xi) = A_0 + A_1 \exp\left[-\Phi(\xi)\right], \tag{16}$$

where $A_0$, $A_1$ will be determined later, such that $A_1$ is non-zero constant and $\Phi = \Phi(\xi)$ satisfies the auxiliary ODE is given by

$$\Phi'(\xi) = \exp\left[-\Phi(\xi)\right] + \rho\exp\left[\Phi(\xi)\right] + \tau. \tag{17}$$

Eq. (17) has solutions in the following forms:

When $\rho \neq 0$ and $\tau^2 - 4\rho > 0$,

$$\Phi(\xi) = \ln\left(-\frac{\sqrt{\tau^2 - 4\rho}\tanh\left(\frac{\sqrt{\tau^2 - 4\rho}}{2}(\xi + C)\right) + \tau}{2\rho}\right). \tag{18}$$

When $\rho \neq 0$ and $\tau^2 - 4\rho < 0$,

$$\Phi(\xi) = \ln\left(\frac{\sqrt{4\rho - \tau^2}\tan\left(\frac{\sqrt{4\rho - \tau^2}}{2}(\xi + C)\right) - \tau}{2\rho}\right). \tag{19}$$

When $\rho = 0$, $\tau \neq 0$ and $\tau^2 - 4\rho > 0$,

$$\Phi(\xi) = -\ln\left(\frac{\tau}{\exp\left(\tau(\xi + C)\right) - 1}\right). \tag{20}$$

When $\rho \neq 0$, $\tau \neq 0$ and $\tau^2 - 4\rho = 0$,

$$\Phi(\xi) = \ln\left(-\frac{2\left(\tau(\xi + C) + 2\right)}{\tau^2(\xi + C)}\right). \tag{21}$$

When $\rho = 0$, $\tau = 0$ and $\tau^2 - 4\rho = 0$,

$$\Phi(\xi) = \ln(\xi + C). \tag{22}$$

Here $C$ is the integration constant. Inserting (16) along with (17) into Eq. (15), and equating the coefficients of $\exp(-\Phi(\xi))$ to zero, one obtains a system of algebraic equations. Solving it by Mathematica, one obtains the following results:

$$
\begin{aligned}
A_0 &= \pm\frac{\sqrt{\tau^2(bv - a)}}{\sqrt{2c}}, \\
A_1 &= \pm\frac{2\sqrt{\tau^2(bv - a)}}{\sqrt{2c}\,\tau}, \\
\omega &= \frac{-2\kappa\lambda + bv\left(4\rho - \tau^2\right) + a\left(2\kappa^2 - 4\rho + \tau^2\right)}{2b\kappa - 2},
\end{aligned} \tag{23}
$$

where $\rho$ and $\tau$ are arbitrary constants. Substituting the solution set (23) into Eq. (16), the solution formula of (15) can be written in the form

$$P(\xi) = \pm\frac{\sqrt{\tau^2(bv - a)}}{\sqrt{2c}}\left\{1 + \frac{2}{\tau}\exp\left[-\Phi(\xi)\right]\right\}. \tag{24}$$

Substituting the general solutions of (17) into Eq. (50) and inserting the result into the hypothesis (2), one recovers the hyperbolic, trigonometric and plane wave solutions to Eq. (14) as below :

When $\rho \neq 0$ and $\tau^2 - 4\rho > 0$, hyperbolic function solutions are:

$$
\begin{aligned}
u(x,t) = &\pm\frac{\sqrt{\tau^2(bv - a)}}{\sqrt{2c}} \\
&\times\left\{1 - \frac{4\rho}{\tau\sqrt{\tau^2 - 4\rho}\tanh\left(\frac{\sqrt{\tau^2 - 4\rho}}{2}(\xi + C)\right) + \tau^2}\right\} \\
&\times e^{i\left\{-\kappa x + \left(\frac{-2\kappa\lambda + bv\left(4\rho - \tau^2\right) + a\left(2\kappa^2 - 4\rho + \tau^2\right)}{2b\kappa - 2}\right)t + \theta\right\}}.
\end{aligned} \tag{25}
$$

When $\rho \neq 0$ and $\tau^2 - 4\rho < 0$, trigonometric function solutions are:

$$u(x,t) = \pm \frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}}$$

$$\times \left\{ 1 + \frac{4\rho}{\tau\sqrt{4\rho - \tau^2}\tan\left(\frac{\sqrt{4\rho - \tau^2}}{2}(\xi + C)\right) - \tau^2} \right\}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-2\kappa\lambda + bv\left(4\rho - \tau^2\right) + a\left(2\kappa^2 - 4\rho + \tau^2\right)}{2b\kappa - 2}\right)t + \theta\right\}}. \quad (26)$$

When $\rho = 0$, $\tau \neq 0$ and $\tau^2 - 4\rho > 0$, hyperbolic function solutions are:

$$u(x,t) = \pm \frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}} \left\{ 1 + \frac{2}{\exp(\tau(\xi + C)) - 1} \right\}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-2\kappa\lambda + bv\left(4\rho - \tau^2\right) + a\left(2\kappa^2 - 4\rho + \tau^2\right)}{2b\kappa - 2}\right)t + \theta\right\}}. \quad (27)$$

When $\rho \neq 0$, $\tau \neq 0$ and $\tau^2 - 4\rho = 0$, rational function solutions are:

$$u(x,t) = \pm \frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}} \left\{ 1 - \frac{\tau(\xi + C)}{\tau(\xi + C) + 2} \right\}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-2\kappa\lambda + bv\left(4\rho - \tau^2\right) + a\left(2\kappa^2 - 4\rho + \tau^2\right)}{2b\kappa - 2}\right)t + \theta\right\}}. \quad (28)$$

When $\rho = 0$, $\tau = 0$ and $\tau^2 - 4\rho = 0$, plane wave solutions are:

$$u(x,t) = \pm \frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}} \left\{ 1 + \frac{2}{\tau(\xi + C)} \right\}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-2\kappa\lambda + bv\left(4\rho - \tau^2\right) + a\left(2\kappa^2 - 4\rho + \tau^2\right)}{2b\kappa - 2}\right)t + \theta\right\}}. \quad (29)$$

### 2.2. Power law

In this case,

$$F(q) = q^n, \quad (30)$$

for power law nonlinear medium. Based on this nonlinearity, (1) reduces to

$$iu_t + au_{xx} + b|u|^2 u + bu_{xt} + c|u|^{2n}u =$$
$$- i\lambda u_x - is\left(|u|^2 u\right)_x - i\mu\left(|u|^2\right)_x u - i\theta|u|^2 u_x$$
$$- i\gamma u_{xxx} - \theta_1\left(|u|^2 u\right)_{xx} - \theta_2|u|^2 u_{xx} - \theta_3 u^2 u_{xx}^*, \quad (31)$$

so that (11) simplifies to

$$(a - bv)P'' - \left((1 - b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)P + cP^{2n+1} = 0. \quad (32)$$

Balancing $P''$ with $P^{2n+1}$ in Eq. (32) gives $N = \frac{1}{n}$. To obtain an analytic solution, one employs the transformation

$$P = U^{\frac{1}{2n}}, \quad (33)$$

in Eq. (32) to find

$$(a - bv)\left((1 - 2n)(U')^2 + 2nUU''\right)$$
$$- 4n^2\left((1 - b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)U^2 + 4cn^2U^3 = 0. \quad (34)$$

In this subsection, the $\exp(-\Phi(\xi))$-expansion method will be utilized, to obtain hyperbolic, trigonometric and rational function solutions to Eq. (31). According to the homogeneous balance method, Eq. (34) has the solution as

$$U(\xi) = A_0 + A_1 \exp\left[-\Phi(\xi)\right] + A_2\left(\exp\left[-\Phi(\xi)\right]\right)^2, \quad (35)$$

where $A_i$ $(i = 0, 1, 2)$ will be determined later, such that $A_2$ is non-zero constant and $\Phi = \Phi(\xi)$ satisfies Eq. (17). Inserting (35) along with (17) into Eq. (34), and equating the coefficients of $\exp(-\Phi(\xi))$ to zero, one obtains a system of algebraic equations. Solving it by Mathematica, one obtains the following results:

$$A_0 = -\frac{\rho(1+n)(a - bv)}{cn^2},$$
$$A_1 = -\frac{\tau(1+n)(a - bv)}{cn^2},$$
$$A_2 = -\frac{(1+n)(a - bv)}{cn^2}, \quad (36)$$
$$\omega = \frac{-4n^2\kappa\lambda + a\left(4n^2\kappa^2 + 4\rho - \tau^2\right) + bv\left(\tau^2 - 4\rho\right)}{4n^2(b\kappa - 1)},$$

where $\rho$ and $\tau$ are arbitrary constants. Substituting the solution set (36) into Eq. (35), the solution formula of Eq. (34) can be written as

$$U(\xi) = -\frac{(1+n)(a - bv)}{cn^2}$$
$$\times \left\{ \rho + \tau \exp\left[-\Phi(\xi)\right] + \left(\exp\left[-\Phi(\xi)\right]\right)^2 \right\}. \quad (37)$$

Consequently, one recovers the hyperbolic, trigonometric and plane wave solutions to Eq. (31) in the forms:

When $\rho \neq 0$ and $\tau^2 - 4\rho > 0$, hyperbolic function solutions are:

$$u(x,t) = \left\{ -\frac{(1+n)(a - bv)}{cn^2} \right.$$

$$\times \left( \rho - \frac{2\rho\tau}{\sqrt{\tau^2 - 4\rho}\tanh\left(\frac{\sqrt{\tau^2 - 4\rho}}{2}(\xi + C)\right) + \tau} \right.$$

$$\left. + \frac{4\rho^2}{\left[\sqrt{\tau^2 - 4\rho}\tanh\left(\frac{\sqrt{\tau^2 - 4\rho}}{2}(\xi + C)\right) + \tau\right]^2} \right)^{\frac{1}{2n}} \right\}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-4n^2\kappa\lambda + a\left(4n^2\kappa^2 + 4\rho - \tau^2\right) + bv\left(\tau^2 - 4\rho\right)}{4n^2(b\kappa - 1)}\right)t + \theta\right\}}. \quad (38)$$

When $\rho \neq 0$ and $\tau^2 - 4\rho < 0$, trigonometric function solutions are:

$$u(x,t) = \left\{ -\frac{(1+n)(a - bv)}{cn^2} \right.$$

$$\times \left( \rho + \frac{2\rho\tau}{\sqrt{4\rho - \tau^2}\tan\left(\frac{\sqrt{4\rho - \tau^2}}{2}(\xi + C)\right) - \tau} \right.$$

$$+\frac{4\rho^2}{\left[\sqrt{4\rho-\tau^2}\tan\left(\frac{\sqrt{4\rho-\tau^2}}{2}(\xi+C)\right)-\tau\right]^2}\Bigg)\Bigg\}^{\frac{1}{2n}}$$

$$\times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (39)$$

When $\rho=0$, $\tau\neq0$ and $\tau^2-4\rho>0$, hyperbolic function solutions are:

$$u(x,t)=\Bigg\{-\frac{(1+n)(a-bv)}{cn^2}$$

$$\times\left(\rho+\frac{\tau^2}{\exp(\tau(\xi+C))-1}+\frac{\tau^2}{[\exp(\tau(\xi+C))-1]^2}\right)\Bigg\}^{\frac{1}{2n}}$$

$$\times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (40)$$

When $\rho\neq0$, $\tau\neq0$ and $\tau^2-4\rho=0$, rational function solutions are:

$$u(x,t)=\Bigg\{-\frac{(1+n)(a-bv)}{cn^2}$$

$$\times\left(\rho-\frac{\tau^3(\xi+C)}{2(\tau(\xi+C)+2)}+\frac{\tau^4(\xi+C)^2}{[2(\tau(\xi+C)+2)]^2}\right)\Bigg\}^{\frac{1}{2n}}$$

$$\times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (41)$$

When $\rho=0$, $\tau=0$ and $\tau^2-4\rho=0$, plane wave solutions are:

$$u(x,t)=\Bigg\{-\frac{(1+n)(a-bv)}{cn^2}\left(\rho+\frac{\tau}{\xi+C}+\frac{1}{(\xi+C)^2}\right)\Bigg\}^{\frac{1}{2n}}$$

$$\times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (42)$$

### 2.3. Parabolic law

In this case,

$$F(q)=q+\eta q^2, \quad (43)$$

where $\eta$ is a real-valued constant. Based on this nonlinearity, (1) reduces to

$$iu_t+au_{xx}+b|u|^2u+bu_{xt}+c(|u|^2+\eta|u|^4)u=$$
$$-i\lambda u_x-is\left(|u|^2u\right)_x-i\mu\left(|u|^2\right)_x u-i\theta|u|^2u_x$$
$$-i\gamma u_{xxx}-\theta_1\left(|u|^2u\right)_{xx}-\theta_2|u|^2u_{xx}-\theta_3u^2u_{xx}^*, \quad (44)$$

so that equation (11) simplifies to

$$(a-bv)P''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)P+cP^3+c\eta P^5=0. \quad (45)$$

Balancing $P''$ with $P^5$ gives $N=\frac{1}{2}$. To obtain an analytic solution, one employs the transformation

$$P=U^{\frac{1}{2}}, \quad (46)$$

in Eq. (45) to find

$$(a-bv)\left(2UU''-(U')^2\right)-4\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)U^2$$
$$+4cU^3+4c\eta U^4=0. \quad (47)$$

In this subsection, the $\exp(-\Phi(\xi))$-expansion method will be implemented, to obtain hyperbolic, trigonometric and rational function solutions to Eq. (44). According to the homogeneous balance method, Eq. (47) has the solution as

$$U(\xi)=A_0+A_1\exp[-\Phi(\xi)], \quad (48)$$

where $A_0$, $A_1$ will be determined later, such that $A_1$ is nonzero constant and $\Phi=\Phi(\xi)$ satisfies Eq. (17). Inserting (48) along with (17) into Eq. (47), and equating the coefficients of $\exp(-\Phi(\xi))$ to zero, one obtains a system of algebraic equations. Solving it by Mathematica, one recovers the following results:

$$A_0=-\frac{3\left(4\eta\rho-\eta\tau^2+\sqrt{\eta^2\tau^2(\tau^2-4\rho)}\right)}{8\eta^2(4\rho-\tau^2)},$$
$$A_1=\frac{3\tau}{4\sqrt{\eta^2\tau^2(\tau^2-4\rho)}},$$
$$\omega=\frac{-4\kappa\lambda-a\tau^2+4a\left(\kappa^2+\rho\right)+bv\left(\tau^2-4\rho\right)}{4b\kappa-4}, \quad (49)$$
$$c=\frac{4}{3}\eta(a-bv)\left(4\rho-\tau^2\right),$$

where $\rho$ and $\tau$ are arbitrary constants. Substituting the solution set (49) into Eq. (48), the solution formula of Eq. (47) can be written as

$$U(\xi)=-\frac{3\left(4\eta\rho-\eta\tau^2+\sqrt{\eta^2\tau^2(\tau^2-4\rho)}\right)}{8\eta^2(4\rho-\tau^2)}$$
$$+\frac{3\tau}{4\sqrt{\eta^2\tau^2(\tau^2-4\rho)}}\exp[-\Phi(\xi)]. \quad (50)$$

Consequently, one obtains the hyperbolic, trigonometric and plane wave solutions to Eq. (44) in the following forms:

When $\rho\neq0$ and $\tau^2-4\rho>0$, hyperbolic function solutions are:

$$u(x,t)=\Bigg\{-\frac{3\left(4\eta\rho-\eta\tau^2+\sqrt{\eta^2\tau^2(\tau^2-4\rho)}\right)}{8\eta^2(4\rho-\tau^2)}$$

$$-\frac{3\tau}{4\sqrt{\eta^2\tau^2(\tau^2-4\rho)}}$$

$$\times\left(\frac{2\rho}{\sqrt{\tau^2-4\rho}\tanh\left(\frac{\sqrt{\tau^2-4\rho}}{2}(\xi+C)\right)+\tau}\right)\Bigg\}^{\frac{1}{2}}$$

$$\times e^{i\left\{-\kappa x+\left(\frac{-4\kappa\lambda-a\tau^2+4a\left(\kappa^2+\rho\right)+bv\left(\tau^2-4\rho\right)}{4b\kappa-4}\right)t+\theta\right\}}. \quad (51)$$

When $\rho\neq0$ and $\tau^2-4\rho<0$, trigonometric function solutions are:

$$u(x,t) = \left\{ -\frac{3\left(4\eta\rho - \eta\tau^2 + \sqrt{\eta^2\tau^2\left(\tau^2 - 4\rho\right)}\right)}{8\eta^2\left(4\rho - \tau^2\right)} \right.$$

$$+ \frac{3\tau}{4\sqrt{\eta^2\tau^2\left(\tau^2 - 4\rho\right)}}$$

$$\left. \times \left(\frac{2\rho}{\sqrt{4\rho - \tau^2}\tan\left(\frac{\sqrt{4\rho - \tau^2}}{2}\left(\xi + C\right)\right) - \tau}\right)\right\}^{\frac{1}{2}}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-4\kappa\lambda - a\tau^2 + 4a\left(\kappa^2 + \rho\right) + bv\left(\tau^2 - 4\rho\right)}{4b\kappa - 4}\right)t + \theta\right\}}.$$

(52)

When $\rho = 0$, $\tau \neq 0$ and $\tau^2 - 4\rho > 0$, hyperbolic function solutions are:

$$u(x,t) = \left\{ -\frac{3\left(4\eta\rho - \eta\tau^2 + \sqrt{\eta^2\tau^2\left(\tau^2 - 4\rho\right)}\right)}{8\eta^2\left(4\rho - \tau^2\right)} \right.$$

$$\left. + \frac{3\tau}{4\sqrt{\eta^2\tau^2\left(\tau^2 - 4\rho\right)}}\left(\frac{\tau}{\exp\left(\tau\left(\xi + C\right)\right) - 1}\right)\right\}^{\frac{1}{2}}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-4\kappa\lambda - a\tau^2 + 4a\left(\kappa^2 + \rho\right) + bv\left(\tau^2 - 4\rho\right)}{4b\kappa - 4}\right)t + \theta\right\}}.$$

(53)

When $\rho \neq 0$, $\tau \neq 0$ and $\tau^2 - 4\rho = 0$, rational function solutions are:

$$u(x,t) = \left\{ -\frac{3\left(4\eta\rho - \eta\tau^2 + \sqrt{\eta^2\tau^2\left(\tau^2 - 4\rho\right)}\right)}{8\eta^2\left(4\rho - \tau^2\right)} \right.$$

$$\left. - \frac{3\tau}{4\sqrt{\eta^2\tau^2\left(\tau^2 - 4\rho\right)}}\left(\frac{\tau^2\left(\xi + C\right)}{2\left(\tau\left(\xi + C\right) + 2\right)}\right)\right\}^{\frac{1}{2}}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-4\kappa\lambda - a\tau^2 + 4a\left(\kappa^2 + \rho\right) + bv\left(\tau^2 - 4\rho\right)}{4b\kappa - 4}\right)t + \theta\right\}}.$$

(54)

When $\rho = 0$, $\tau = 0$ and $\tau^2 - 4\rho = 0$, plane wave solutions are:

$$u(x,t) = \left\{ -\frac{3\left(4\eta\rho - \eta\tau^2 + \sqrt{\eta^2\tau^2\left(\tau^2 - 4\rho\right)}\right)}{8\eta^2\left(4\rho - \tau^2\right)} \right.$$

$$\left. + \frac{3\tau}{4\sqrt{\eta^2\tau^2\left(\tau^2 - 4\rho\right)}}\left(\frac{1}{\xi + C}\right)\right\}^{\frac{1}{2}}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-4\kappa\lambda - a\tau^2 + 4a\left(\kappa^2 + \rho\right) + bv\left(\tau^2 - 4\rho\right)}{4b\kappa - 4}\right)t + \theta\right\}}.$$

(55)

## 2.4. Dual-Power law

In this case,

$$F(q) = q^n + \eta q^{2n}, \tag{56}$$

where $\eta$ is a real-valued constant. Based on this nonlinearity, (1) reduces to

$$iu_t + au_{xx} + b|u|^2 u + bu_{xt} + c(|u|^{2n} + \eta|u|^{4n})u =$$
$$- i\lambda u_x - is\left(|u|^2 u\right)_x - i\mu\left(|u|^2\right)_x u - i\theta|u|^2 u_x$$
$$- i\gamma u_{xxx} - \theta_1\left(|u|^2 u\right)_{xx} - \theta_2|u|^2 u_{xx} - \theta_3 u^2 u_{xx}^*, \tag{57}$$

so that equation (11) simplifies to

$$(a - bv)P'' - \left((1 - b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)P$$
$$+ cP^{2n+1} + c\eta P^{4n+1} = 0. \tag{58}$$

Balancing $P''$ with $P^{4n+1}$ gives $N = \frac{1}{2n}$. To obtain an analytic solution, one employs the transformation

$$P = U^{\frac{1}{2n}}, \tag{59}$$

in Eq. (58) to find

$$(a - bv)\left((1 - 2n)(U')^2 + 2nUU''\right)$$
$$- 4n^2\left((1 - b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)U^2$$
$$+ 4cn^2 U^3 + 4cn^2\eta U^4 = 0. \tag{60}$$

In this subsection, the $\exp(-\Phi(\xi))$-expansion method will be applied, to obtain hyperbolic, trigonometric and rational function solutions to Eq. (57). According to the homogeneous balance method, Eq. (60) has the solution as

$$U(\xi) = A_0 + A_1 \exp\left[-\Phi(\xi)\right], \tag{61}$$

where $A_0$, $A_1$ will be determined later, such that $A_1$ is non-zero constant and $\Phi = \Phi(\xi)$ satisfies Eq. (17). Inserting (61) along with (17) into Eq. (60), and equating the coefficients of $\exp(-\Phi(\xi))$ to zero, one obtains a system of algebraic equations. Solving it by Mathematica, one recovers the following results:

$$A_0 = \frac{\ell - cn^2(1 + n)(a - bv)\left(4\rho - \tau^2\right)}{4c^2 n^4}, \quad A_1 = \frac{\ell}{2c^2 n^4 \tau},$$
$$\omega = \frac{-4n^2\kappa\lambda + a\left(4n^2\kappa^2 + 4\rho - \tau^2\right) + bv\left(\tau^2 - 4\rho\right)}{4n^2(b\kappa - 1)},$$
$$\eta = \frac{cn^2(1 + 2n)}{(1 + n)^2(a - bv)\left(4\rho - \tau^2\right)},$$

(62)

where $\rho$, $\tau$ are arbitrary constants, and $\ell$ is given by

$$\ell = \sqrt{c^2 n^4 (1 + n)^2 (a - bv)^2 \tau^2\left(\tau^2 - 4\rho\right)}. \tag{63}$$

Substituting the solution set (62) into Eq. (61), the solution formula of Eq. (60) can be written as

$$U(\xi) = \frac{\ell - cn^2(1 + n)(a - bv)\left(4\rho - \tau^2\right)}{4c^2 n^4}$$
$$+ \frac{\ell}{2c^2 n^4 \tau}\exp\left[-\Phi(\xi)\right]. \tag{64}$$

Consequently, one obtains the hyperbolic, trigonometric and plane wave solutions to Eq. (57) as follows:

When $\rho \neq 0$ and $\tau^2 - 4\rho > 0$, hyperbolic function

solution is:

$$u(x,t) = \left\{ \frac{\ell - cn^2(1+n)(a-bv)\left(4\rho - \tau^2\right)}{4c^2n^4} \right.$$

$$\left. -\frac{\ell}{2c^2n^4\tau}\left(\frac{2\rho}{\sqrt{\tau^2-4\rho}\tanh\left(\frac{\sqrt{\tau^2-4\rho}}{2}(\xi+C)\right)+\tau}\right)\right\}^{\frac{1}{2n}}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-4n^2\kappa\lambda + a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \qquad (65)$$

When $\rho \neq 0$ and $\tau^2 - 4\rho < 0$, trigonometric function solutions are:

$$u(x,t) = \left\{ \frac{\ell - cn^2(1+n)(a-bv)\left(4\rho - \tau^2\right)}{4c^2n^4} \right.$$

$$\left. +\frac{\ell}{2c^2n^4\tau}\left(\frac{2\rho}{\sqrt{4\rho-\tau^2}\tan\left(\frac{\sqrt{4\rho-\tau^2}}{2}(\xi+C)\right)-\tau}\right)\right\}^{\frac{1}{2n}}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-4n^2\kappa\lambda + a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \qquad (66)$$

When $\rho = 0$, $\tau \neq 0$ and $\tau^2 - 4\rho > 0$, hyperbolic function solutions are:

$$u(x,t) = \left\{ \frac{\ell - cn^2(1+n)(a-bv)\left(4\rho - \tau^2\right)}{4c^2n^4} \right.$$

$$\left. +\frac{\ell}{2c^2n^4}\left(\frac{1}{\exp\left(\tau\left(\xi+C\right)\right)-1}\right)\right\}^{\frac{1}{2n}}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-4n^2\kappa\lambda + a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \qquad (67)$$

When $\rho \neq 0$, $\tau \neq 0$ and $\tau^2 - 4\rho = 0$, rational function solutions are:

$$u(x,t) = \left\{ \frac{\ell - cn^2(1+n)(a-bv)\left(4\rho - \tau^2\right)}{4c^2n^4} \right.$$

$$\left. -\frac{\ell}{2c^2n^4}\left(\frac{\tau\left(\xi+C\right)}{2\left(\tau\left(\xi+C\right)+2\right)}\right)\right\}^{\frac{1}{2n}}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-4n^2\kappa\lambda + a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \qquad (68)$$

When $\rho = 0$, $\tau = 0$ and $\tau^2 - 4\rho = 0$, plane wave solutions are:

$$u(x,t) = \left\{ \frac{\ell - cn^2(1+n)(a-bv)\left(4\rho - \tau^2\right)}{4c^2n^4} \right.$$

$$\left. +\frac{\ell}{2c^2n^4\tau}\left(\frac{1}{\xi+C}\right)\right\}^{\frac{1}{2n}}$$

$$\times e^{i\left\{-\kappa x + \left(\frac{-4n^2\kappa\lambda + a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \qquad (69)$$

## 3. Conclusion

This paper investigated analytically the nonlinear mathematical physical model (1) by using an integration tool called the $\exp\left(-\Phi\left(\xi\right)\right)$-expansion approach. Four kinds of nonlinearities including Kerr law, power law, parabolic law and dual-power law are taken into account As a consequence, hyperbolic, trigonometric and rational function solutions are derived.

## References

[1] Zhou, Q., Liu, L., Liu, Y., Yu, H., Yao, P., Wei, C., Zhang, H., 2015. Exact optical solitons in metamaterials with cubic-quintic nonlinearity and third-order dispersion. Nonlinear Dynamics 80(3), 1365-1371.

[2] Zhou, Q., Mirzazadeh, M., Ekici, M., Sonmezoglu, A., 2016. Analytical study of solitons in non-Kerr nonlinear negative-index materials. Nonlinear Dynamics 86, 623-638.

[3] Biswas, A., Khan, K.R., Mahmood, M.F., 2014. Bright and dark solitons in optical metamaterials. Optik 125(3), 3299-3302.

[4] Xu, Y., Savescu, M., Khan, K.R., Mahmood, M.F., Biswas, A., Belic, M., 2016. Soliton propagation through nanoscale waveguides in optical metamaterials. Optics and Laser Technology 77, 177-186.

[5] Saha, M., Sarma, A.K., 2013. Modulation instability in nonlinear metamaterials induced by cubic-quintic nonlinearities and higher order dispersive effects. Optics Communications 291, 321-325.

[6] Yang, R., Zhang, Y., 2011. Exact combined solitary wave solutions in nonlinear metamaterials. Journal of the Optical Society of America B 28(1), 123-127.

[7] Yomba, E., 2005. Construction of new solutions to the fully nonlinear generalized Camassa-Holm equations by an indirect F-function method. Journal of Mathematical Physics 46, 123504-123512.

[8] He, J.H., Wu, X.H., 2006. Exp-function method for nonlinear wave equations. Chaos, Solitons and Fractals 30, 700-708.

[9] Hirota, R., 1973. Exact N-soliton of the wave equation of long waves in shallow water and in nonlinear lattices. Journal of Mathematical Physics 14, 810-814.

[10] He, J.H., 2005. Homotopy perturbation method for bifurcation of nonlinear problems. International Journal of Nonlinear Sciences and Numerical Simulation 6, 207-208.

[11] Abdou, M.A., Soliman, A.A., 2005. New applications of variational iteration method. Physica D 211, 1-8.

[12] He, J.H., 2004. Variational principles for some nonlinear partial differential equations with variable coefficients. Chaos, Solitons and Fractals 19, 847-851.

[13] Abassy, T.A., El-Tawil, M.A., Saleh, H.K., 2004. The solution of KdV and mKdV equations using Adomian

Pade approximation. International Journal of Nonlinear Sciences and Numerical Simulation 5, 327-340.

[14] Antonova, M., Biswas, A., 2009. Adiabatic parameter dynamics of perturbed solitary waves. Communications in Nonlinear Science and Numerical Simulation 14, 734-748.

[15] Wang, M.L., 1995. Solitary wave solutions for variant Boussinesq equations. Physics Letters A 199, 169-172.

[16] Ablowitz, M.J., Clarkson, P.A., 1991. Solitons: Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge.

[17] Liu, S.K., Fu, Z.T., Liu, S.D., Zhao, Q., 2001. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Physics Letters A 289, 69-74.

[18] Tascan, F., Bekir, A., 2009. Analytic solutions of the (2+1)-dimensional nonlinear evolution equations using the sine-cosine method. Applied Mathematics and Computation 215, 3134-3139.

[19] Ozis, T., Aslan, I., 2009. Symbolic computation and exact and explicit solutions of some nonlinear evolution equation in mathematical physics. Communications in Theoretical Physics 51, 577-580.

[20] Manafian, J., Lakestani, M., Bekir, A., 2016. Study of the analytical treatment of the (2+1)-dimensional Zoomeron, the Duffing and the SRLW equations via a new analytical approach. International Journal of Applied and Computational Mathematics 2(2), 243-268.

[21] Khan, K., Akbar, M.A., 2013. Application of $\exp(-\Phi(\xi))$-expansion method to find the exact solutions of modified Benjamin-Bona-Mahony equation. World Applied Sciences Journal 24(10), 1373-1377.

[22] Roshid, H.O., Kabir, M.R., Bhowmik, R.C., Datta, B.K., 2014. Investigation of solitary wave solutions for Vakhnenko-Parkes equation via exp-function and $\exp(-\Phi(\xi))$-expansion method. SpringerPlus 3:692.

[23] Kaplan, M., Bekir, A., 2016. A novel analytical method for time-fractional differential equations. Optik 127, 8209-8214.