Süleyman Demirel Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 29, Sayı 1, 220-227, 2025

Süleyman Demirel University
Journal of Natural and Applied Sciences
Volume 29, Issue 1, 220-227, 2025

DOI: 10.19113/sdufenbed.1666339

Unidefiners

Selim ÇETİN[*1]

[1]Burdur Mehmet Akif Ersoy Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, 15030, Burdur, Türkiye

(Alınış / Received: 26.03.2025, Kabul / Accepted: 12.04.2025, Online Yayınlanma / Published Online: 25.04.2025)

Keywords
Unidefiner, t −definer, t −codefiner, uninorm, identity element, monoid.

Abstract: In this paper, the concept of unidefiners is introduced as a unified aggregation framework on the interval $[0,\infty]$ that generalizes both $t-$definers and $t-$codefiners. Traditional aggregation operators, typically defined on $[0,1]$, are deemed inadequate for applications where data naturally extends beyond the unit interval. Motivated by challenges in multi-criteria decision-making and fuzzy logic, unidefiners are defined as binary operations on $[0,\infty]$ equipped with a neutral element and characterized by associativity, commutativity, and monotonicity. It is shown that when the neutral element is set to 0, a t −definer is obtained, whereas choosing ∞ yields a t −codefiner; a strong duality and transformation methods are established accordingly. The proposed framework is argued to enhance the theoretical understanding of aggregation on unbounded domains and to expand its practical applicability.

Unibelirleyiciler

Anahtar Kelimeler
Unibelirleyici,
t −belirleyici, t −eşbelirleyici, Uninorm, Birim eleman, Monoid.

Öz: Bu çalışmada, hem t −belirleyicileri hem de t −eşbelirleyicileri genelleyen, $[0,\infty]$ aralığında tanımlı birleşik bir toplulaştırma çerçevesi olarak unibelirleyici kavramı tanıtılmıştır. Geleneksel toplulaştırma işlemleri genellikle $[0,1]$ aralığında tanımlandığından, verilerin birim aralığı aşabildiği uygulamalar için yetersiz kalmaktadır. Çok ölçütlü karar verme ve bulanık mantık gibi alanlardaki zorluklardan hareketle, unibelirleyiciler; bir birim elemana sahip, birleşmeli, değişmeli ve monoton olan, $[0,\infty]$ üzerinde tanımlı ikili işlemler olarak tanımlanmıştır. Birim eleman 0 olarak alındığında bir t −belirleyici, ∞ olarak alındığında ise bir t −eşbelirleyici elde edildiği gösterilmiş; bu iki yapı arasında güçlü bir dualite ve dönüşüm yöntemleri ortaya konulmuştur. Önerilen çerçevenin, sınırsız aralıklar üzerindeki toplulaştırma işlemlerinin kuramsal kavranışını geliştirdiği ve uygulama alanını genişlettiği savunulmuştur.

1. Introduction

Aggregation operators are widely used in diverse fields, including fuzzy logic, probabilistic metric spaces, and multi-criteria decision-making. However, most existing studies focus on aggregation operators defined on the bounded interval $[0,1]$, leaving a gap in the literature for models that operate on unbounded domains. This paper introduces a novel framework (unidefiners) which unifies t −definers and t −codefiners on the interval $[0,\infty]$, thereby addressing practical issues where data and metrics naturally extend beyond the unit interval.

The motivation behind this research stems from the need for a unified aggregation framework that is both mathematically robust and versatile enough for a wide scientific audience. In many real-world applications, such as decision-making systems and clustering algorithms, the underlying data is not confined to $[0,1]$ but spans the entire non-negative real axis. By extending the well-established concept of uninorms to the $[0,\infty]$ interval via unidefiners, this work offers a more flexible approach that can capture both 'union-like' and 'intersection-like' behaviors in a single framework.

Despite the extensive body of research on uninorms and related operators on bounded intervals, there is a

*Corresponding author: seliimcetin@gmail.com

notable absence of a general theory that covers aggregation on unbounded domains. The development of unidefiners not only fills this theoretical gap but also provides new insights and tools for practical applications that require more general aggregation methods.

A uninorm is a binary operation on the unit interval $[0, 1]$ with a neutral element $e \in [0, 1]$, which is associative, commutative, and monotonic, and is known for yielding a $t-$conorm when $e = 0$ and a $t-$norm when $e = 1$ [12, 13]. This flexibility unifies both "intersection-like" and "union-like" behaviors under one framework, offering broad applicability in practical scenarios.

Since their introduction, uninorms have garnered significant interest, have been extensively studied, and remain a subject of active research across various domains, including characterization [16, 27], construction [4, 5, 6], decomposition [20, 25], distributivity [32], direct product operations [1], idempotency properties [24], induced order relations [2], migrativity [18], modularity [28], ordinal sum representations [11], and topological aspects [21, 22].

Furthermore, various related function classes, including $2-$uninorms [3], micanorms [29], nuninorms [10], null-uninorms [17, 31], nullnorms [9], pseudo$-$uninorms [26], $S-$uninorms [30], semi-uninorms [8], unima [14], uni$-$nullnorms [17, 19, 31], and uninorm-like parametric activation functions [7], have also been studied extensively in diverse frameworks.

However, in certain applications, one may need to work with metric or quasi$-$metric structures defined on $[0, \infty]$, for instance, $\star-$metric spaces or other generalized distance concepts.

In this context, the notion of a unidefiner comes into play: a unidefiner is a binary operation on $[0, \infty]$, equipped with a neutral element ($e \in [0, \infty]$), and is associative, commutative, and monotonic. This structure has two special cases:

$t-$definer ($e = 0$): A generalization of operations resembling addition or maximum.

$t-$codefiner ($e = \infty$): A generalization of operations resembling minimum or other behaviors at the opposite extreme.

Between these two endpoints lies a value e that leads to the so-called proper unidefiner, situated between $t-$definer and $t-$codefiner. Much like uninorms (where $e \in (0, 1)$), unidefiners (with $e \in (0, \infty)$) continue to offer a unified approach to aggregation and provide the flexibility needed in many different applications.

The Unidefiners paper presents this approach in detail, demonstrating a duality relationship between t-definer and $t-$codefiner, thus laying the theoretical groundwork for unidefiners.

Similar to t-norm/uninorm theory, it is thereby possible to design a wide array of aggregation operations on $[0, \infty]$ by selecting different identity elements.

In summary, the main contributions of this paper are: (i) the introduction of the unidefiner as a generalized aggregation operator on $[0, \infty]$, (ii) the establishment of a strong duality between $t-$definers and $t-$codefiners through this new framework, and (iii) the demonstration of how proper unidefiners can be transformed into both $t-$definers and $t-$codefiners, thereby broadening the scope of aggregation techniques applicable to various scientific fields.

The remainder of the paper is organized as follows. Section 2 provides the fundamental definitions and properties of $t-$definers and $t-$codefiners. Section 3 introduces the concept of unidefiners, details their theoretical properties, and discusses the duality relationships. Finally, Section 4 outlines potential applications of the proposed framework and offers suggestions for future research directions.

2. Material and Method

In this section, we provide some basic definitions regarding t-definers. More detail can be
found in [23, 15].
We point out that the t-definers in [23] are assumed to be continuous by definition. Here, we consider a more general case where continuity is not required.

Definition 2.1. [23] An operation $\star$ on $[0, \infty]$ that satisfies the following conditions for all
$a, b, c \in [0, \infty]$ is called a $t-$definer.

($D0$) $a \star 0 = a$

($D1$) $a \star (b \star c) = (a \star b) \star c$

($D2$) $a \star b = b \star a$

($D3$) $a \leq b \Rightarrow a \star c \leq b \star c$

In particular, if the condition

($D4$) $\star$ is continuous in its first component,

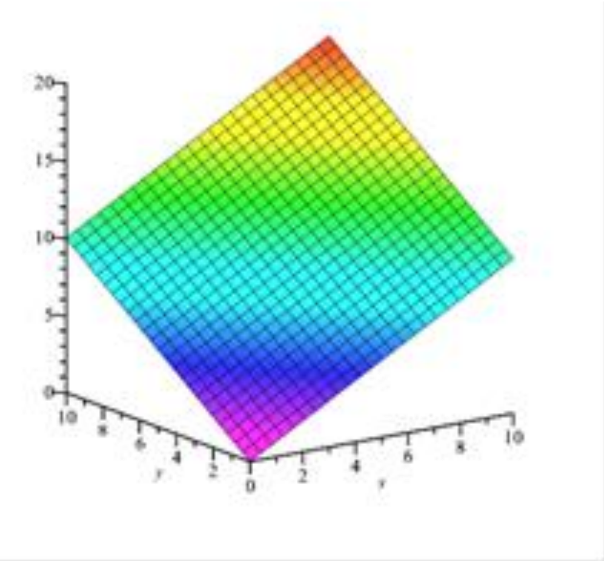
is satisfied, we call this a continuous $t-$definer.

Note that ∞ always behaves as an absorbing element for a $t-$definer, because the identity0 and monotonicity imply that for every $a \in [0, \infty]$, $\infty = 0 \star \infty \leq a \star \infty$, which gives $a \star \infty = \infty$.

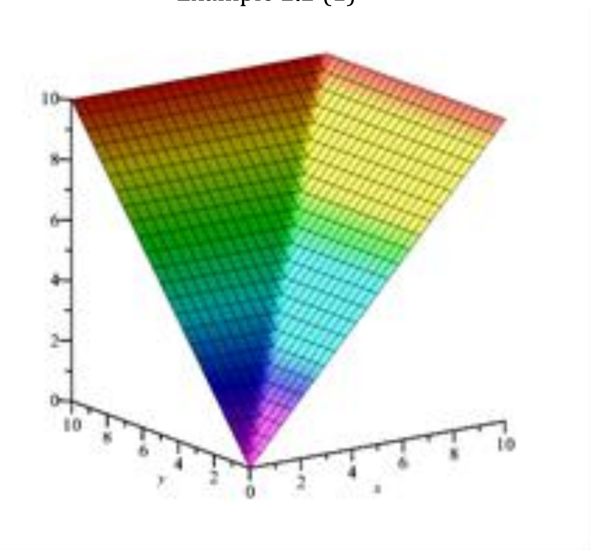
Example 2.2. The following are examples of $t-$definers on $[0, \infty]$.

1. $a \star b = a + b$
2. $a \star b = max(a, b)$
3. $a \star b = a + b + ab$
4. $a \star b = \sqrt{a^2 + b^2}$
5. $a \star b = \left(\sqrt{a} + \sqrt{b}\right)^2$

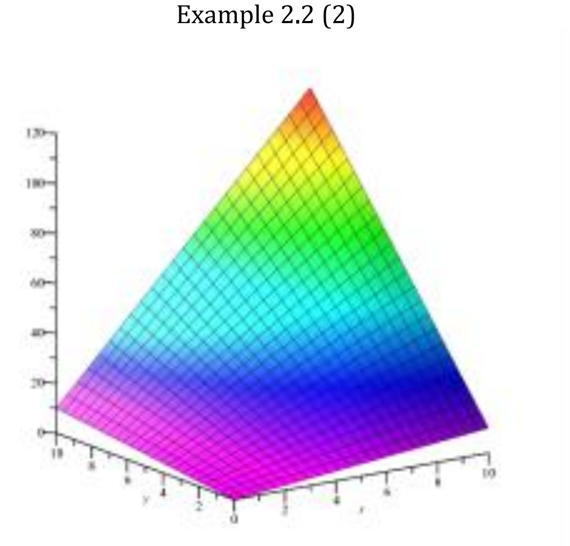
The t −definers in Example 2.2 are illustrated in Figure 1.



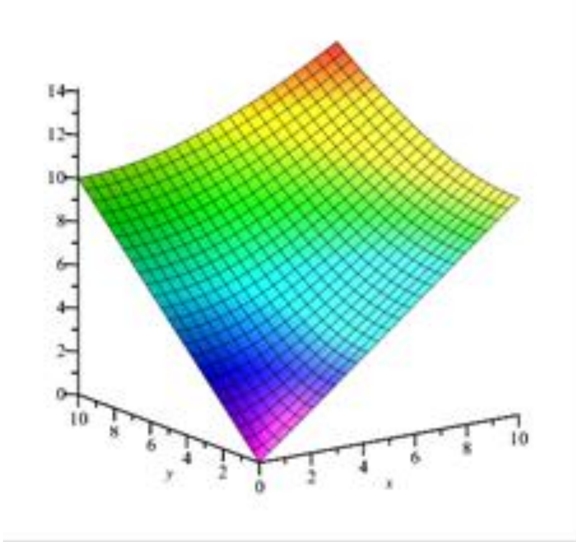
Example 2.2 (1)



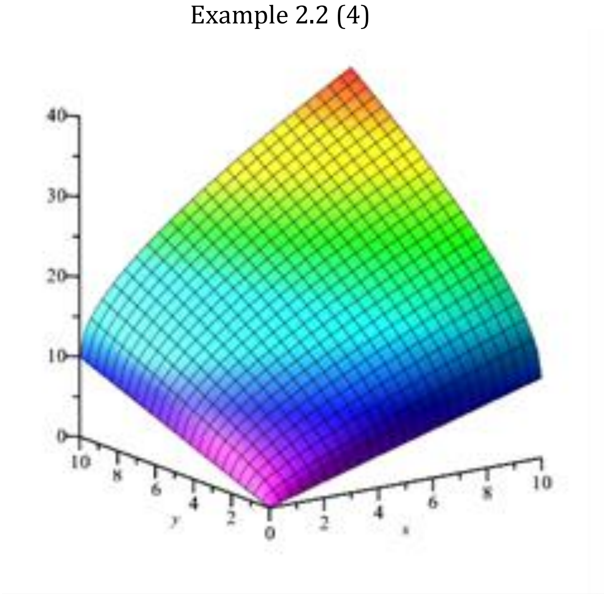
Example 2.2 (2)



Example 2.2 (3)



Example 2.2 (4)



Example 2.2 (5)

Figure 1. Some examples of t −definers

Building upon the foundational concepts introduced above, we now shift our focus to the detailed properties of unidefiners, establishing their relationship with existing aggregation operators.

3. Results

In this section, the concepts of t −codefiner and unidefiner are defined, and the relevant results and their interrelations are presented.

Definition 3.1. An operation $\circledast$ on $[0,\infty]$ that satisfies the following conditions is called a t −codefiner.

$(C0)$ $a \circledast \infty = a$

$(C1)$ $a \circledast (b \circledast c) = (a \circledast b) \circledast c$

$(C2)$ $a \circledast b = b \circledast a$

$(C3)$ $a \leq b \Rightarrow a \circledast c \leq b \circledast c$

Dual to t −definers, for a t −codefiner, 0 is always an absorbing element. Indeed, since ∞ is
the identity element and by monotonicity, for every $a \in [0,\infty]$,

$$a \circledast 0 \leq \infty \circledast 0 = 0,$$
which gives $a \circledast 0 = 0$.

The following theorem establishes the duality between $t-$definers and $t-$codefiners. This result is important as it not only shows the intrinsic connection between these two operations but also provides a systematic way to generate one from the other, thereby enhancing our understanding of aggregation operators on $[0,\infty]$.

Theorem 3.2. If $\star$ is a $t-$definer on $[0,\infty]$, then the operation $\circledast$ defined by
$$a \circledast b = \frac{1}{\frac{1}{a} \star \frac{1}{b}}$$
is a $t-$codefiner on $[0,\infty]$. Conversely, if $\circledast$ is a $t-$codefiner, then the operation $\star$ defined by
$$a \star b == \frac{1}{\frac{1}{a} \circledast \frac{1}{b}}$$
is a $t-$definer.

Proof: We will show that if $\star$ is a t-definer, then the operation
$$a \circledast b = \frac{1}{\frac{1}{a} \star \frac{1}{b}}$$
is a $t-$codefiner. The transition from $t-$codefiners to $t-$definers is quite similar.
Assume that $\star$ is a $t-$definer.

(C0) $a \circledast b = \frac{1}{\frac{1}{a}\star\frac{1}{b}} = \frac{1}{\frac{1}{a}\star 0} = \frac{1}{\frac{1}{a}} = a$

(C1) $a \circledast (b \circledast c) = a \circledast \frac{1}{\frac{1}{b}\star\frac{1}{c}} = \frac{1}{\frac{1}{a}\star\frac{1}{\frac{1}{\frac{1}{b}\star\frac{1}{c}}}}$
$$= \frac{1}{(\frac{1}{a}\star\frac{1}{b})\star\frac{1}{c}} = (a \circledast b) \circledast c$$

(C2) $a \circledast b = \frac{1}{\frac{1}{a}\star\frac{1}{b}} = \frac{1}{\frac{1}{b}\star\frac{1}{a}} = b \circledast a$

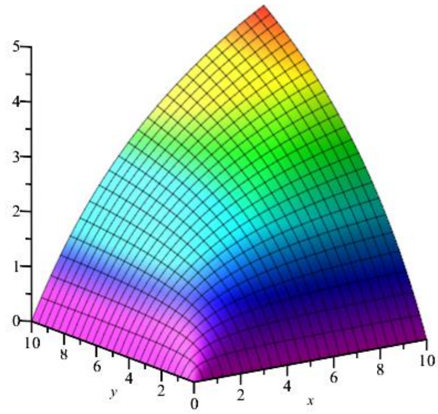
(C3) $a \leq b \Rightarrow \frac{1}{b} \leq \frac{1}{a} \Rightarrow \frac{1}{b}\star\frac{1}{c} \leq \frac{1}{a}\star\frac{1}{c}$
$$\Rightarrow \frac{1}{\frac{1}{a}\star\frac{1}{c}} \leq \frac{1}{\frac{1}{b}\star\frac{1}{c}} \Rightarrow a \circledast c \leq b \circledast c.$$

Example 3.3. The following are examples of $t-$codefiners on $[0,\infty]$, each of which is the dual of those in Example 2.2 in context of Theorem 3.2.

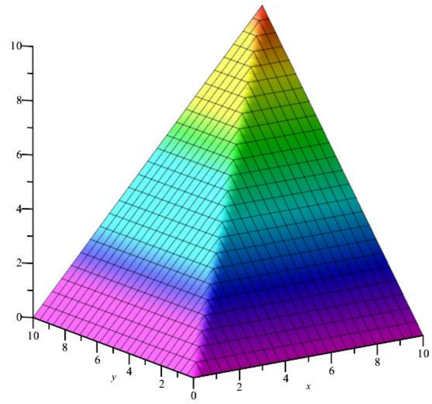
For brevity, the formulas are presented in a simple form; however, if either a or b is ∞, it is assumed to act as a neutral element, which is also observed in the limit case. As a result to simplify the expressions, instead of writing the formulas as piecewise functions for the sake of ∞, we adopt the convention that whenever a $t-$codefiner is given, ∞ functions as a neutral element.

1. $a \circledast b = \frac{ab}{a+b}$
2. $a \circledast b = min(a,b)$
3. $a \circledast b = \frac{ab}{a+b+1}$
4. $a \circledast b = \frac{ab}{\sqrt{a^2+b^2}}$
5. $a \circledast b = \frac{ab}{(\sqrt{a}+\sqrt{b})^2}$

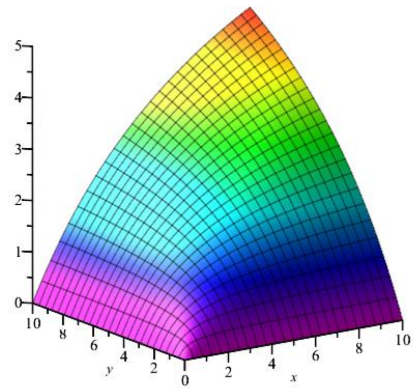
The $t-$codefiners in Example 3.2 are illustrated in Figure2.



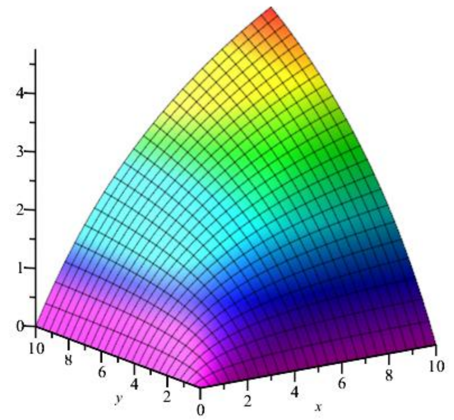
Example 3.2 (1)



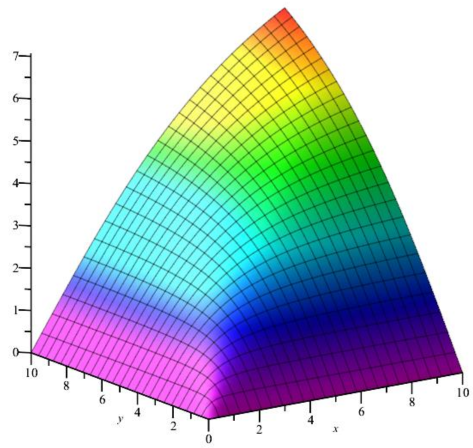
Example 3.2 (2)



Example 3.2 (3)



Example 2.2 (4)



Example 2.2 (5)

Figure 2. Some examples of t −codefiners

We note that the construction given in Theorem 3.2 is not the only nontrivial way to obtain t −codefiners from t −definers. For any $k \in (0,\infty)$, the operation

$$a \circledast b = \frac{k}{\frac{k}{a} \star \frac{k}{b}}$$

also serves the same purpose.

For instance, taking $k = 2$ and $\star$ defined by $a \star b = a + b + ab$, we obtain

$$a \circledast b = \frac{2}{\frac{2}{a}\star\frac{2}{b}} = \frac{2}{\frac{2}{a}+\frac{2}{b}+\frac{2}{a}\frac{2}{b}} = \frac{2}{\frac{2a+2b+4}{ab}} = \frac{ab}{a+b+2},$$

which differs from the one given in Example 3.3 (3). Of course, as before, we do not directly apply the formula when ∞ is involved but instead use the fact that it acts as a neutral element.

Now, we introduce the concept of unidefiner, which unifies t-definers and t −codefiners under a common framework.

Definition 3.4 An operation $*$ on $[0,\infty]$ that satisfies the following conditions is called a *unidefiner*.

($U0$) There exists an $e \in [0,\infty]$, such that $a \star e = a$ for all $a \in [0,\infty]$.

($U1$) $a * (b * c) = (a * b) * c$ for all $a, b, c \in [0,\infty]$.

($U2$) $a * b = b * a$ for all $a, b \in [0,\infty]$.

($U3$) $a \le b \Rightarrow a \star c \le b \star c$ for all $a, b, c \in [0,\infty]$.

In this case, a unidefiner $*$ is an operation on the set $[0,\infty]$ that is unital, associative, commutative, and monotonic; in other words, $([0,\infty],*)$ is an ordered commutative monoid. A t −definer is simply a unidefiner whose identity element is 0, whereas a t −codefiner is a unidefiner whose identity element is ∞. A unidefiner that is neither a t −definer nor a t −codefiner is called a proper unidefiner.

Example 3.5. The operation $*$ on $[0,\infty]$ defined by

$$a * b = \begin{cases} 0, & a, b < e \\ \max(a,b), & a, b \ge e \\ \min(a,b), & \text{otherwise} \end{cases}$$

is an example of a unidefiner, with an arbitrary unit $e \in (0,\infty)$, which is not a t −definer, nor a t −codefiner.

Example 3.6. The operation $*$ given by

$$a * b = \begin{cases} \min(a,b), & a, b \le e \\ \infty & a, b > e \\ \max(a,b), & \text{otherwise} \end{cases}$$

is a unidefiner with the unit e, where $e \in (0,\infty)$.

Theorem 3.7. If $*$ is a unidefiner, then $0 * \infty$ is an absorbing element.

Proof: Let e be the identity element of $*$. If $a \le e$, then from $a * 0 \le e * 0 = 0$, we get $a * 0 = 0$. Thus, we obtain $a * (0 * \infty) = (a * 0) * \infty = 0 * \infty$.
On the other hand, if $a > e$, then $a * \infty \ge e * \infty = \infty$, which gives $a * \infty = \infty$. Using commutativity and associativity, we again obtain $a * (0 * \infty) = (a * \infty) * 0 = 0 * \infty$.

We can further refine our understanding of $0 * \infty$ beyond just being an absorbing element:

Theorem 3.8. For a unidefiner $*$, we have either $0 * \infty = 0$ or $0 * \infty = \infty$.

Proof: First, if e is the identity element of $*$, then $0 * \infty \ne e$, since Theorem 3.5 states that $0 * \infty$ is an absorbing element and thus cannot be equal to the identity.

If $0 * \infty < e$, then by Theorem 3.5 and monotonicity, we have
$0 * \infty = (0 * \infty) * 0 \leq e * 0 = 0$,
which implies $0 * \infty = 0$.

Conversely, if $0 * \infty > e$, then
$0 * \infty = (0 * \infty) * \infty \geq e * \infty = \infty$,
which implies $0 * \infty = \infty$.

The result above shows that unidefiners can be classified into two distinct types. If a unidefiner satisfies $0 * \infty = 0$, then it resembles t −codefiners due to 0 being an absorbing element, and such a unidefiner is called disjunctive. If instead, $0 * \infty = \infty$, the unidefiner is said to be conjunctive.

Example 3.9. For the unidefiner given in Example 3.5., $0 * \infty = 0$, hence it is a disjunctive unidefiner. On the other hand, $0 * \infty = \infty$ for the unidefiner in Example 3.6, and therefore it is conjunctive.

From any proper unidefiner, one can obtain both a t −definer and a t −codefiner, as described in the following theorems.

Theorem 3.10. Let $*$ be a proper unidefiner with identity element e. Then, the operation $\star$ defined by
$$a \star b = ((a + e) * (b + e)) - e$$
is a t −definer.

Proof: First, we ensure that this operation is closed on $[0, \infty]$. Since $e \leq a + e$ and $e \leq b + e$, by monotonicity we obtain
$$a \star b = \big((a + e) * (b + e)\big) - e \geq (e * e) - e = 0.$$
Thus, we have $a \star b \geq 0$, ensuring that the operation remains within $[0, \infty]$.

(D0) $a \star 0 = \big((a + e) * (0 + e)\big) - e$
$= \big((a + e) * e\big) - e = (a + e) - e = a.$

(D1)
$$\begin{aligned} a \star (b \star c) &= a \star \Big(\big((b + e) * (c + e)\big) - e\Big) \\ &= \Big((a + e) * \Big(\big((b + e) * (c + e)\big) - e\Big)\Big) - e \\ &= \Big((a + e) * \big((b + e) * (c + e)\big)\Big) - e \\ &= \Big(\big((a + e) * (b + e)\big) * (c + e)\Big) - e \\ &= (a \star b) \star c \end{aligned}$$

(D2)
$$\begin{aligned} a \star b &= \big((a + e) * (b + e)\big) - e \\ &= \big((b + e) * (a + e)\big) - e \\ &= b \star a \end{aligned}$$

(D3) If $a \leq b \Longrightarrow$ then $a + e \leq b + e$, and this easily yields $a \star c \leq b \star c$ for all $c \in [0, \infty]$.

Example 3.11. Consider the unidefiner in Example 3.5. Then, in the notation of Theorem 3.10, we have
$$\begin{aligned} a \star b &= \big((a + e) * (b + e)\big) - e \\ &= \max(a + e, b + e) - e \\ &= \max(a, b). \end{aligned}$$
as $a + e, b + e \geq e$. This is the t −definer in Example 2.2 (2).

Example 3.12. For the unidefiner given in Example 3.6, we consider two cases. If both a and b are nonzero, then $a + e, b + e > e$ and
$$a \star b = \big((a + e) * (b + e)\big) - e = \infty - e = \infty.$$
If either a or b is zero, say $b = 0$, then
$$\begin{aligned} a \star 0 &= \big((a + e) * e\big) - e \\ &= \max(a + e, e) - e = a \end{aligned}$$
for $a \neq 0$, and
$$0 \star 0 = (e * e) - e = \min(e, e) - e = 0$$
for $a = 0$. This gives the t −definer
$$a \star b = \begin{cases} b, & a = 0 \\ a & b = 0 \\ \infty, & \text{otherwise} \end{cases}$$
which we call drastic t −definer, in analogy with the drastic t −norm.

Theorem 3.13. If $*$ is a proper unidefiner with identity e, then $a \star b = \frac{1}{\left(\frac{1+ae}{a}\right) * \left(\frac{1+be}{b}\right) - e}$ defines a t −codefiner.

Proof: By Theorem 3.7, $a \star b = ((a + e) * (b + e)) - e$ gives a t −definer and by Theorem 3.2 $a \circledast b = \frac{1}{\frac{1}{a} \star \frac{1}{b}}$ gives a t −codefiner, and
$$\begin{aligned} a \circledast b &= \frac{1}{\frac{1}{a} \star \frac{1}{b}} = \frac{1}{\left(\left(\frac{1}{a} + e\right) * \left(\frac{1}{b} + e\right)\right) - e} \\ &= \frac{1}{\left(\left(\frac{1+ae}{a}\right) * \left(\frac{1+be}{b}\right)\right) - e} \end{aligned}$$

The reduction of the expression in this final result, reveals a compact form that underscores the inherent symmetry in the operation, making it more accessible for further theoretical exploration and practical application.

Example 3.14. For the unidefiner in Example 3.5, the t −codefiner described in Theorem 3.13 is obtained as
$$\begin{aligned} a \circledast b &= \frac{1}{\left(\left(\frac{1}{a} + e\right) * \left(\frac{1}{b} + e\right)\right) - e} \\ &= \frac{1}{\max\left(\frac{1}{a} + e, \frac{1}{b} + e\right) - e} = \frac{1}{\max\left(\frac{1}{a}, \frac{1}{b}\right)} \\ &= \min(a, b), \end{aligned}$$
the t −codefiner in Example 3.3 (2).

4. Discussion and Conclusion

In this paper, we introduced unidefiners, a novel framework that unifies t −definers and t −codefiners on the interval $[0, \infty]$. By establishing a strong duality between these operators, we bridge the gap between classical uninorm theory and aggregation methods applicable to unbounded domains.

Selecting appropriate identity elements enables unidefiners to act as either disjunctive or conjunctive operators. This versatility makes them suitable for applications in multi-criteria decision-making, clustering, and fuzzy logic.

Future research should explore additional properties such as idempotency, continuity, and local behavior, along with practical implementations. Overall, unidefiners represent both a theoretical advancement and a promising tool for developing new aggregation techniques.

Declaration of Ethical Code

In this study, we undertake that all the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

References

[1] Aşıcı E., Mesiar R. 2021. On the direct product of uninorms on bounded lattices. Kybernetika 57(6), 989–1004

[2] Aşıcı E., Mesiar R. 2024. Some investigations on the U-partial order induced by uninorms. Aequationes mathematicae 1-14.

[3] Cao M., Du WS. 2023. On residual implications derived from 2-uninorms. International Journal of Approximate Reasoning 159, 108926.

[4] Çaylı GD. 2023. An alternative construction of uninorms on bounded lattices. International Journal of General Systems 52(5), 574–596.

[5] Çaylı GD. 2024. Constructing uninorms on bounded lattices through closure and interior operators. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 32(1), 109–129.

[6] Çaylı GD., Ertuğrul U., Karaçal F. 2023. Some further construction methods for uninorms on bounded lattices. International Journal of General Systems 52(4), 414–442.

[7] Csiszar O, Pusztahazi LS., Denes-Fazakas L., Gashler MS., Kreinovich V, Csisz´ar G. 2023. Uninorm-like parametric activation functions for human-understandable neural models. Knowledge-Based Systems 260, 110095.

[8] Dan Y. 2023. A unified way to studies of t-seminorms, t-semiconorms and semi-uninorms on a complete lattice in terms of behaviour operations. International Journal of Approximate Reasoning 156, 61–76.

[9] Dan Y., Hu BQ., De Baets B. 2022. Nullnorms on bounded lattices constructed by means of closure and interior operators. Fuzzy Sets and Systems 439, 142–156.

[10] De Campos Souza PV., Lughofer E. 2022. An advanced interpretable fuzzy neural network model based on uni-nullneuron constructed from n-uninorms. Fuzzy Sets and Systems 426, 1–26.

[11] Dvorak A., Holcapek M., Paseka J. 2022. On ordinal sums of partially ordered monoids: A unified approach to ordinal sum constructions of t-norms, t-conorms and uninorms. Fuzzy Sets and Systems 446, 4–25.

[12] Fodor J., De Baets B. 2007. Uninorm basics. In Fuzzy Logic: A Spectrum of Theoretical & Practical Issues 49–64. Berlin, Heidelberg: Springer Berlin Heidelberg, 15s.

[13] Fodor JC., Yager RR., Rybalov A. 1997. Structure of uninorms. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 5(4), 411–427.

[14] Gürdal U., Oğur O., Çetin S. Unima on [0,1]. Manuscript submitted for publication.

[15] He SY., Xie LH., Yan PF. 2022. On ⋆-metric spaces. Filomat 36(18), 6173–6185.

[16] Hlinena D., Kalina M. 2011. Characterization of uninorms on bounded lattices and pre-order they induce. International Journal of Computational Intelligence Systems 14(1), 148–158.

[17] İnce MA., Karaçal F. 2023. Determination of the smallest-greatest uni-nullnorms and null-uninorms on an arbitrary bounded lattice L. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 31(1), 103–119.

[18] Jiang DX., Liu HW. 2024. Migrativity of uninorms not internal on the boundary over continuous t-(co)norms. Iranian Journal of Fuzzy Systems 21(3), 103–121.

[19]] Jocic D., Stajner-Papuga I. 2023. Distributivity of a uni-nullnorm with continuous and Archimedean underlying T-norms and T-conorms over an arbitrary uninorm. Mathematica Slovaca 73(6), 1527–1544.

[20] Karaçal F., Ertuğrul U., Arpacı S., Kesicioğlu MN. 2023. Congruence relations and direct decomposition of uninorms on bounded lattices. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 31(6), 1033–1059.

[21] Karaçal F., Ertuğrul U., Kesicioğlu M. 2021. Generating methods for principal topologies on bounded lattices. Kybernetika 57(4), 714-736.

[22] Karaçal F., Köroğlu T. 2022. A principal topology obtained from uninorms. Kybernetika 58(6), 863–882.

[23] Khatami SMA., Mirzavaziri M. 2020. Yet another generalization of the notion of a metric space. arXiv preprint. arXiv:2009.00943

[24] Mesiarova-Zemankova A., Mesiar R., Su Y., Wang ZD. 2024. Idempotent uninorms on bounded lattices with at most single point incomparable with the neutral element: Part I. International Journal of General Systems 1-34.

[25] Ouyang Y., Zhang HP., De Baets B. 2024. Decomposition and construction of uninorms on the unit interval. Fuzzy Sets and Systems 493–494, 109083.

[26] Wang SM. 2019. The logic of pseudo-uninorms and their residua, Symmetry 11(3), 368-380.

[27] Wen H., Wu X., Çaylı GD. 2023. Characterizing some types of uninorms on bounded lattices. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 31(4), 533–549.

[28] Xie A., Zhang JQ. 2024. On modularity property for uninorms with continuous underlying functions. Iranian Journal of Fuzzy Systems 21(2), 105–116.

[29] Yang E. 2021. Micanorm aggregation operators: basic logico-algebraic properties. Soft Computing 25, 13167–13180.

[30] Yang B., Lı W., Liu YH., Xu J. 2023. The distributivity of extended semi-t-operators over extended S-uninorms on fuzzy truth values. Soft Computing 28(4), 2823–2841.

[31] Zhang HP., Ouyang Y., De Baets B. 2021. Constructions of uni-nullnorms and null-uninorms on a bounded lattice. Fuzzy Sets and Systems 403, 78–87.

[32] Zong WW., Su Y., Liu HW. 2024. Conditionally distributive uninorms locally internal on the boundary. Semigroup Forum 1–9.