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A LIFETIME REGRESSION ANALYSIS WITH UNIT LINDLEY-WEIBULL DISTRIBUTION

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Abstract: In this paper, a new lifetime distribution is introduced. Motivation is provided to obtain this distribution. The closed-form expressions of probability density and cumulative distribution functions are provided. Several distributional properties are obtained and the statistical inference are discussed on unknown parameters. The most important novelty of this study is to bring a lifetime regression analysis with the re-parameterized log-transform of the new distribution.

Key words: Unit-Lindley distribution, Weibull distribution, Monte Carlo simulation, Estimation, Lifetime regression, Confidence interval.

1. Introduction

In the last two decades, many statistical distributions have been introduced. Most of them are derived from various compounding methods. [8] introduced a Beta-normal distribution with cumulative distribution function (cdf) $R(G(x))$, where R is beta cdf, and G is normal cdf [15], [16], and [17] have introduced new distributions by using Gumbel, Fréchet and exponential cdfs for G in $R(G(x))$. All these distributions belong to Beta-G family. However, all these works don't give cdf in explicit form because of the structure of beta cdf.

In order to get an explicit cdf, [5] considers the Kumaraswamy cdf for R in [8]'s formula and they obtained different distributions by changing cdf G. The distribution family in [5] is called "Kw-G family" [7] and [19] introduced new distributions by using Kw-G family.

[2] introduced a new family of distribution by getting inspired by [8]'s formula. They consider cdf $R(W(G(x)))$, where R and G are any cdf of continuous random variables and W is a function that satisfies certain conditions. It is noted that, If $W(x) = x$ and R and G are assigned as beta and normal cdfs, respectively, then the distribution in [8] is achieved.

In this paper, we introduce a new distribution, which is a member of [2] family. Some general distributional and inferential properties of the introduced distribution are studied. Here, there are two crucial discussions on statistical inference.

The first discussion is related to the confidence intervals (CIs) for unknown parameters. In general, the CIs for unknown parameters are discussed through asymptotical normality of maximum likelihood estimates (MLEs). Here, the CIs based on asymptotical normality of MLEs are denoted

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by AN CIs. However, the limits of AN CIs sometimes turn out to be outside of the parameter space. It is an undesired outcome in practice. It should also be remembered that a large sample is needed to good approximation to the normality of MLEs when the number of the parameter is more than two. Furthermore, it is also needed a large sample to get true coverage probabilities (CPs) of AN CIs. In Subsection 3.2, uncorrected likelihood ratio (ULR) type CIs for the unknown parameters are discussed as an alternative to AN CIs. It is pointed out that the ULR type CIs have wonderful properties: The limits of ULR type CIs are always within parameter space. In the simulation given in Subsection 3.2, it is also observed that the CPs of ULR CIs are better than the CPs of AN CIs.

The second discussion is focused on the lifetime regression issue in the survival data analysis: In the lifetime regression analysis, a functional relationship between the dependent variable (lifetime or log-lifetime) and covariates are studied. A common assumption that there is a linear relationship between the location parameter and covariates in the models. These models can be used to determine the sign and magnitudes of covariates on the log-lifetimes through the location parameter. In practice, the survival data obeys to distribution, which has various types of failure rate functions. From this point of view, there is a demand for new lifetime distributions in the survival analysis.

In this study, a new lifetime distribution is introduced by using the [2]'s method. In order to obtain a new distribution with explicit cdf, W, R and G are assigned by an identity function, Unit-Lindley cdf and Weibull cdf, respectively. In Section 2, a new distribution is described with motivation and exact moments are obtained. In addition, the properties of hazard function and stochastic ordering are studied. An accepting rejecting algorithm is also provided to generate data from the new distribution. In Section 3, the several point estimators and CIs of unknown parameters are discussed through Monte Carlo simulation studies. In Section 4, a lifetime regression analysis based on introduced distribution is studied, and an extensive simulation study is performed. A practical real data set is given to illustrate the applicability of the new distribution in Section 5.

2. Unit-Lindley-Weibull distribution

In this section, we introduce a new distribution and discuss its distributional properties. Recently, Unit-Lindley (UL) distribution is introduced by [14]. If T is UL random variable, the pdf and cdf of T are given, respectively, by

$$r(t;\theta) = \left(\frac{\theta^2}{(1+\theta)(1-t)^3}\right)\exp\left(\frac{\theta t}{t-1}\right)\mathbb{I}_{(0,1)}(t)$$

and

$$R(t;\theta) = 1-\left(1-\frac{\theta t}{(1+\theta)(t-1)}\right)\exp\left(\frac{\theta t}{t-1}\right),$$

where $\theta > 0$ is a parameter and $\mathbb{I}_A(\cdot)$ is an indicator function on A. Let us also consider a Weibull random variable Y with pdf and cdf

$$g(y;\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^{\beta-1}\exp\left(-\left(\frac{y}{\alpha}\right)^{\beta}\right)\mathbb{I}_{\mathbb{R}_+}(y) \tag{2.1}$$

and

$$G(y;\alpha,\beta) = 1-\exp\left(-\left(\frac{y}{\alpha}\right)^{\beta}\right),$$

respectively. Let us assign W is an identity function and consider UL cdf and Weibull cdf for R and G in $F(x) = R(W(G(x)))$, a valid cdf is obtained by

$$F(x;\boldsymbol{\Xi}) = 1-\left(1+\frac{\theta\left(1-\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)\right)}{(1+\theta)\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)}\right)\exp\left(-\frac{\left(1-\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)\right)}{\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)}\right), \tag{2.2}$$

where $\Xi = (\alpha, \beta, \theta) \in \mathbb{R}_+^3$ is the parameter vector, α is a scale, β and θ are shape parameters. A distribution with cdf (2.2) is called unit-Lindley Weibull (ULW) and it is denoted by ULW(Ξ). Let X be the ULW(Ξ) random variable with cdf (2.2). Then, the pdf of X is given by

$$f(x;\Xi) = \frac{\beta\theta^2 x^{\beta-1}}{\alpha^\beta (1+\theta)} \exp\left\{-\theta \exp\left(\left(\frac{x}{\alpha}\right)^\beta\right) + \theta + 2\left(\frac{x}{\alpha}\right)^\beta\right\} \mathbb{I}_{\mathbb{R}_+}(x). \tag{2.3}$$

For some selected values of parameters, the pdf plots of ULW distribution are given in Figure 1. It is concluded from Figure 1, the pdf of ULW distribution can be unimodal or decreasing. It is also observed that the pdf can be skewed at right or left.

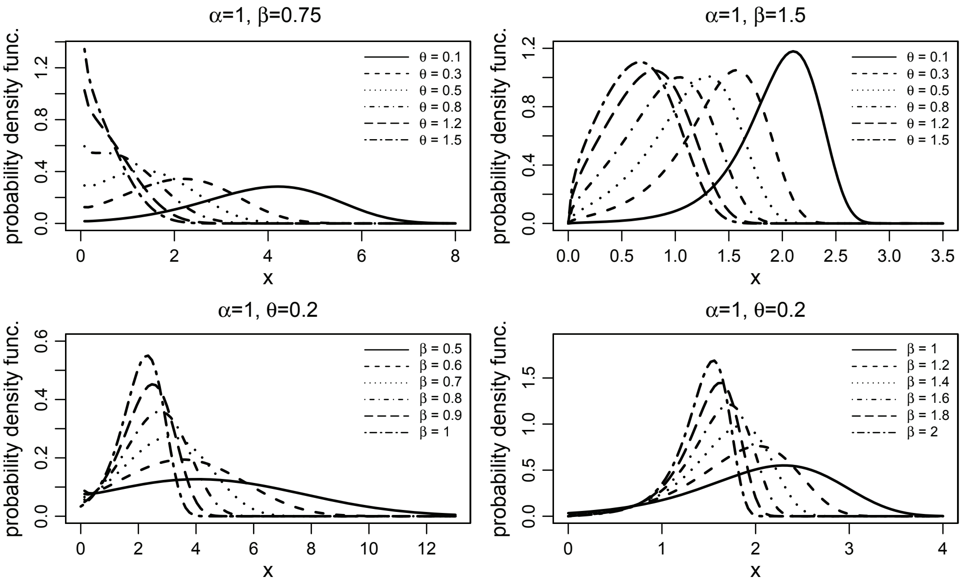

FIGURE 1. Probability density function plots for ULW distribution

2.1. Hazard function

The hazard function (hf) of ULW(Ξ) distribution can be written by

$$h(x;\Xi) = \frac{\beta x^{\beta-1}\theta^2}{\alpha^\beta \exp\left(-\left(\frac{x}{\alpha}\right)^\beta\right)\left(\exp\left(-\left(\frac{x}{\alpha}\right)^\beta\right)+\theta\right)} \mathbb{I}_{\mathbb{R}_+}(x).$$

For some selected values of parameters, the hf of ULW distribution is plotted in Figure 2. From Figure 2, it is observed that the hf of ULW distribution has increasing or bathtub shapes. In the following, we discuss these properties of hf. Let us consider the first-order derivative of hf

$$h'(x;\Xi) = \frac{\beta\theta^2 x^{\beta-2}\left(2\beta\left(\frac{x}{\alpha}\right)^\beta + \beta - 1\right)\exp\left(-\left(\frac{x}{\alpha}\right)^\beta\right) + \theta\left(\beta\left(\frac{x}{\alpha}\right)^\beta + \beta - 1\right)}{\alpha^\beta \exp\left(-\left(\frac{x}{\alpha}\right)^\beta\right)\left(\exp\left(-\left(\frac{x}{\alpha}\right)^\beta\right)+\theta\right)^2}.$$

It can be easily seen that $h'(x;\Xi) > 0$ for $\beta > 1$ and hence hf is increasing. In addition, $h'(x;\Xi) < 0$ for $x < \alpha\left(\frac{1-\beta}{2\beta}\right)^{1/\beta}$ and $h'(x;\Xi) > 0$ for $x > \alpha\left(\frac{1-\beta}{\beta}\right)^{1/\beta}$ under the condition $\beta < 1$. According to this discussion, it can be observed that hf decrease at first and increases as time progress for $\beta < 1$. Furthermore, from Figure 2, it is observed that the hf exhibits bath-tube type when $\beta < 1$.

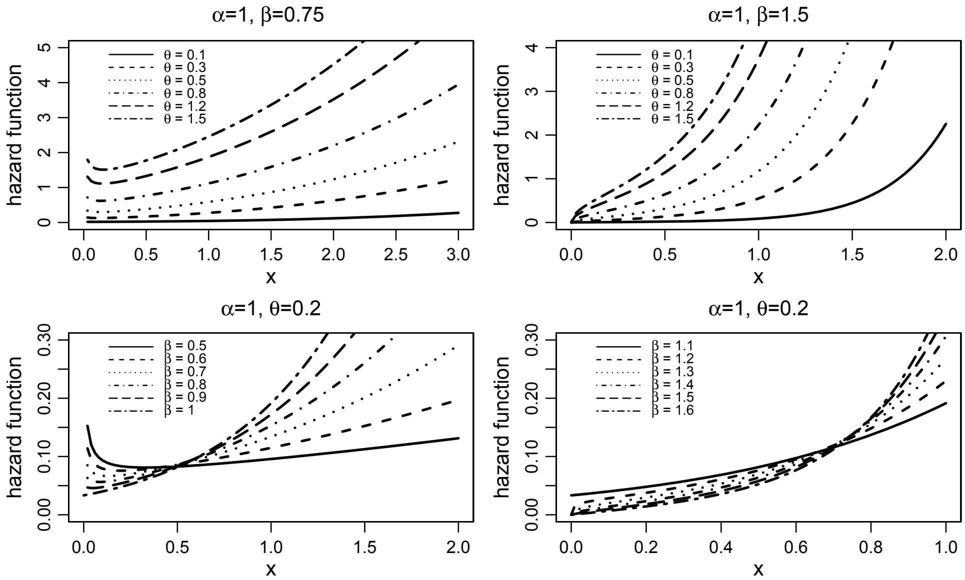

FIGURE 2. Hazard function plots for ULW distribution

2.2. Motivation for the ULW distribution

The pdf of the ULW distribution given in Eq. (2.3) can be obtained in a different ways using the method of [18]. Let Y be the Weibull random variable with pdf $g(x;\alpha,\beta)$ given in Eq. (2.1). According to [18], the pdf of the weighted random variable Y^w is defined by

$$f_w(y)=\frac{w(y;\alpha,\beta,\theta)}{E(w(Y;\alpha,\beta,\theta))}g(y;\alpha,\beta)\,\mathbb{I}_{\mathbb{R}_+}(y). \tag{2.4}$$

Let us consider $w(y;\alpha,\beta,\theta)=\exp\left\{-\theta\exp\left(\left(\frac{y}{\alpha}\right)^{\beta}\right)+\theta+3\left(\frac{y}{\alpha}\right)^{\beta}\right\}$ in Eq. (2.4) and we get

$$E(w(Y;\alpha,\beta,\theta))=\frac{\theta+1}{\theta^2}.$$

Thus, the pdf of Y^w is identical to the pdf of introduced ULW($\boldsymbol{\Xi}$) distribution with pdf (2.3).

2.3. Moments

In this subsection, exact moments of ULW($\boldsymbol{\Xi}$) distribution under a certain conditon. Let us consider the result of [10] given by

$$\int_1^\infty (\log(x))^m x^{v-1}\exp(-\mu x)dx=\frac{\partial^m}{\partial v^m}\left\{\theta^{-v}\Gamma(v,\theta)\right\}.$$

Under the condition $r/\beta\in\mathbb{Z}^+$, the r^{th} moment of a random variable X having ULW($\boldsymbol{\Xi}$) is obtained by

$$\begin{aligned}
E(X^r) &= \int_0^\infty x^r f(x)dx\\
&= \int_0^\infty x^r\frac{\theta^2\beta\alpha^{-\beta}x^{\beta-1}}{1+\theta}\exp\left(-\theta\exp\left(\left(\frac{x}{\alpha}\right)^{\beta}\right)+\theta+2\left(\frac{x}{\alpha}\right)^{\beta}\right)dx\\
&= \frac{\theta^2\alpha^r e^\theta}{1+\theta}\int_1^\infty (\log(t))^{r/\beta}t\exp(-\theta t)dx
\end{aligned}$$

$$= \frac{\theta^2 \alpha^r e^\theta}{1+\theta} \times \frac{\partial^m}{\partial v^m} \left\{ \theta^{-v} \Gamma(v,\theta) \right\} \bigg|_{v=2}, \; r \in \mathbb{N}_+$$
$$= \frac{\alpha^r e^\theta}{1+\theta} MeijerG([[1,1],[]],[[2],[0,0]],\theta),$$

where $\Gamma(v,\theta)$ is incomplete gamma function, $m = r/\beta$ and $MeijerG$ is the well-known Meijer G function which is available in Maple software. Some numerical values of first four moments are presented in Table 1.

TABLE 1. The first four moments of the ULW distribution

r	β	α	θ	$E(X^r)$
1	1	3	2	1.3613
2				2.8004
3				7.1059
4				20.6759
1	0.5	3	2	0.9334
2				2.2973
3				8.5639
4				41.3767

2.4. Stochastic ordering

For a positive continuous random variable, stochastic ordering and the other ordering are important tools for judging the comparative behavior. The following theorem shows that the ULW random variables can be ordered with respect to the likelihood ratio.

THEOREM 1. *Let $X \sim ULW(\alpha,\beta,\theta_1)$ and $Y \sim ULW(\alpha,\beta,\theta_2)$. If $\theta_1 > \theta_2$ then X is smaller than Y in the likelihood ratio order, i.e., the ratio function of the corresponding pdfs is decreasing in x.*

COROLLARY 1. *It follows from [21] that X is also smaller than Y in the hazard ratio, mean residual life and stochastic orders under the conditions given in Theorem 1.*

2.5. Data generating algorithm

In this subsection, we give an algorithm to generate data from ULW(Ξ) distribution. Since the inverse transformation method does not give an explicit formula, we propose an acceptance-rejection (AR) sampling algorithm. In this algorithm, the Weibull distribution is chosen as a proposal distribution. The AR algorithm is given as follows:

Algorithm 1.

A1. Generate data on random variable Y from Weibull distribution with pdf g given in Eq. (2.1)

A2. Generate U from standard uniform distribution (independent of Y).

A3. If
$$U < \frac{f(Y;\Xi)}{k \times g(Y;\alpha,\beta)},$$
then set $X = Y$ ("accept"); otherwise go back to A1 ("reject"), where pdf f is given as in Eq. (2.3) and
$$k = \max_{z \in \mathbb{R}_+} \frac{f(z;\Xi)}{g(z;\alpha,\beta)}.$$
The output of this algorithm suggest a random data on X from ULW(Ξ) distribution. It is noted that Algorithm 1 is used for all simulations in the paper.

3. Statistical inference on distribution parameters

In this section, we propose several estimators for estimating the unknown parameters of the ULW(Ξ) distribution. We discuss the maximum likelihood, least-squares, weighted least squares, Cramér-von Mises type, and Anderson-Darling type estimation methods. Furthermore, the two types of CIs for the parameters are discussed. Simulation studies are also performed to observe the performances of the methods discussed here.

3.1. Point estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample from the ULW(Ξ) distribution and $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denotes the corresponding order statistics. Furthermore, $x_{(i)}$ denotes the observed value of $X_{(i)}$ for $i = 1, 2, \ldots, n$. Based on this sample, the log-likelihood function is given by

$$\ell(\Xi) = n\log\left(\beta\theta^2\right) - n\log\left(\alpha\left(1+\theta\right)\right) + (\beta-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\alpha}\right) + \sum_{i=1}^{n}\log\left(\exp\left(-\theta\exp\left(\left(\frac{x_i}{\alpha}\right)^{\beta}\right)+\theta+2\left(\frac{x_i}{\alpha}\right)^{\beta}\right)\right). \tag{3.1}$$

Then, the MLEs of α, β and θ are given by

$$\widehat{\Xi}_1 = \arg\max_{\Xi}\left\{\ell\left(\Xi\right)\right\}, \tag{3.2}$$

where $\Xi = (\alpha, \beta, \theta)$ and $\widehat{\Xi}_1 = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\theta}\right)$. Let us define the following functions which are used to define the different type of estimators:

$$Q_{LS}\left(\Xi\right) = \sum_{i=1}^{n}\left(F\left(x_{(i)}\right) - \frac{i}{n+1}\right)^2,$$
$$Q_{WLS}\left(\Xi\right) = \sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(F\left(x_{(i)}\right) - \frac{i}{n+1}\right)^2,$$
$$Q_{CvM}\left(\Xi\right) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(x_{(i)}\right) - \frac{2i-1}{2n}\right)^2$$

and

$$Q_{AD}\left(\Xi\right) = -n - \frac{1}{n}\sum_{i=1}^{n}\left\{(2i-1)\log\left(F\left(x_{(i)}\right)\right)\right\} + \frac{1}{n}\sum_{i=1}^{n}\log\left\{1-F\left(x_{(i)}\right)\right\},$$

where $F(\cdot)$ is cdf of ULW(Ξ) distribution given in Eq. (2.2). Then, the least squares estimator (LSE), weighted least squares estimator (WLSE), Anderson Darling estimator (ADE) and the Cramér-von Mises estimator (CvME) of Ξ are given, respectively, by

$$\widehat{\Xi}_2 = \arg\min_{\Xi}\left\{Q_{LS}\left(\Xi\right)\right\}, \tag{3.3}$$
$$\widehat{\Xi}_3 = \arg\min_{\Xi}\left\{Q_{WLS}\left(\Xi\right)\right\}, \tag{3.4}$$
$$\widehat{\Xi}_4 = \arg\min_{\Xi}\left\{Q_{AD}\left(\Xi\right)\right\}, \tag{3.5}$$
$$\widehat{\Xi}_5 = \arg\min_{\Xi}\left\{Q_{CvM}\left(\Xi\right)\right\}. \tag{3.6}$$

It is noted that these estimates are discussed before in [12], [13], and [23]. All maximization and minimization problems can be solved by some numerical methods such as Nelder-Mead, BFGS, or CG. These methods can be easily conducted by **optim** function in R.

3.2. Interval estimation

In this subsection, the CIs are discussed for the parameters θ, β and α. In the statistical literature, CIs are usually constructed by using a pivotal quantity based on MLEs of parameters. However, an exact CIs can not be obtained since the MLEs are usually obtained by a numerical method to optimize the likelihood. Consequently, asymptotic CIs based on the asymptotic normality of MLEs are most popular in the all fields of statistics and it has widespread usage. It is well-known that the AN of MLEs can be stated by

$$\widehat{\boldsymbol{\Xi}}_1 \xrightarrow{d} N_3\left(\boldsymbol{\Xi}, \mathbb{I}^{-1}\left(\boldsymbol{\Xi}\right)\right),$$

where $\widehat{\boldsymbol{\Xi}}_1$ is MLE of $\boldsymbol{\Xi}$ given in Eq. (3.2) and $\mathbb{I}\left(\boldsymbol{\Xi}\right)$ is Fisher information matrix. Using this result, the $100 \times (1-\alpha)\,\%$ AN CIs of parameters α, β and θ are constructed, respectively, by

$$\begin{aligned} &\widehat{\alpha} \pm z_{1-\frac{\alpha}{2}} \times se\left(\widehat{\alpha}\right), \\ &\widehat{\beta} \pm z_{1-\frac{\alpha}{2}} \times se\left(\widehat{\beta}\right), \\ &\widehat{\theta} \pm z_{1-\frac{\alpha}{2}} \times se\left(\widehat{\theta}\right), \end{aligned}$$

where z_a, is the a^{th} quantile of standard normal distribution, $se\left(\widehat{\alpha}\right)$, $se\left(\widehat{\beta}\right)$ and $se\left(\widehat{\theta}\right)$ are the roots of the diagonal member of $\mathbb{I}^{-1}\left(\widehat{\boldsymbol{\Xi}}_1\right)$ which is a consistent estimate of $\mathbb{I}^{-1}\left(\boldsymbol{\Xi}\right)$ and the $se\left(\cdot\right)$ stands for standard error.

By the way, there is another method called ULR, which is not used in most of statistical software, but it has interesting properties. AN and ULR CIs are asymptotically equivalent [9]. The ULR CIs are transformation invariant. It is range preserving that means the CIs it produces will always be inside of the parameter space. There is no need to compute/estimate the variance of the estimates, unlike to AN. In addition, the ULR method doesn't necessarily give symmetric intervals around MLE.

Under usual regularity assumptions on the likelihood function, if the θ is true parameter, then $-2\log\left(\ell\left(\theta, \widetilde{\boldsymbol{\lambda}}\right) - \ell\left(\widehat{\boldsymbol{\Xi}}_1\right)\right)$ distributed χ^2 with degrees of freedom 1, where $\boldsymbol{\lambda} = (\alpha, \beta)$ are the nuisance parameters, ℓ is the log-likelihood function as in Eq. (3.1), $\widehat{\boldsymbol{\Xi}}_1$ is the joint MLEs of (θ, β, α) given in Eq. (3.2), $\widetilde{\boldsymbol{\lambda}} = \left(\widetilde{\alpha}, \widetilde{\beta}\right)$ is the restricted MLEs of $\boldsymbol{\lambda}$ given a fixed value of θ. Using this fact, $100 \times (1-\alpha)\,\%$ ULR CI limits (θ_L, θ_U) that satisfy

$$\ell\left(\theta, \widetilde{\boldsymbol{\lambda}}\right) = \underbrace{\ell\left(\widehat{\boldsymbol{\Xi}}_1\right) - \frac{1}{2}\chi^2_{(1)}\left(1-\alpha\right)}_{\text{LR Bound}},$$

with $\theta_L < \theta$ and $\theta_U > \theta$, where $\chi^2_{(1)}\left(a\right)$ is the a^{th} quantile of the χ^2 distribution with 1 degrees of freedom. The $100 \times (1-\alpha)\,\%$ ULR CIs can be produced in the same manner for the other parameters α and β.

3.3. Simulation study for point estimates

In the simulation study, 5000 trials are used to estimates the bias and mean squared errors (MSEs) of the MLEs, LSEs, WLSEs, ADEs and CVMEs estimates. Different sample sizes are considered in the study. Two parameter settings are considered. The results are given in the Tables 2-4.

The simulation study is performed based on the following algorithm (for one cycle):

Algorithm 2.

A1. Given true parameters, generate the data from ULW(Ξ) distribution by using AR sampling given in Algorithm 1.

A2. True parameters are used as initial values in optimization.

A3. The numerical method BFGS is used for the optimization problem given in Eq.'s (3.2), (3.3)-(3.6).

A4. If there is no solution or there is an estimate out of the parameter space, go to **A1**.

From the Tables 2-4, it is observed that the bias and MSEs of the all estimates decrease to zero as expected. The MLEs are best estimates in terms of MSEs. In general, the CVMEs have smaller bias than the others.

3.4. Simulation study for CIs

In the simulation study, 5000 trials are used to predict the CPs of the AN and ULR CIs. The nominal level is fixed at 0.95. In order to get CPs of ULR CIs, there is no need to obtain the CIs limits. It is possible that the CPs of ULR CIs can be simulated by a likelihood ratio test on the true parameter. The simulated CPs of these intervals are given in Table 5. Let us discuss the case $\Xi = (0.5, 1, 1)$. From Table 5, it is observed that the CPs of ULR reach to desired level when the sample of size greater than 100 for all parameters. However, the CPs of AN can not reach the desired level even if a large sample of size is available. In the case of $\Xi = (2.5, 1, 1)$, CIs of AN CIs reach to nominal level when the sample of size greater than 300 for parameters α and β. However, more than 800 sample of size is needed to achive nominal level for parameter θ. The CPs of ULR of CIs reach to nominal level for the all parameters when the sample of size greater than 200.

Under discussion given here, it is indicated that ULR CIs powerful tool to construct the CIs for the ULW parameters.

Table 2. Average bias and MSEs of the estimates for the true parameters $\boldsymbol{\Xi} = (0.5, 1, 1)$

		Bias			MSE		
n		θ	α	β	θ	α	β
MLEs	100	-0.1635	-0.3226	-0.1792	0.0419	0.1601	0.0520
	250	-0.1585	-0.3016	-0.1683	0.0384	0.1412	0.0443
	500	-0.1455	-0.2711	-0.1485	0.0319	0.1141	0.0349
	750	-0.1329	-0.2436	-0.1351	0.0263	0.0911	0.0279
	1000	-0.1146	-0.2080	-0.1149	0.0211	0.0711	0.0216
	1250	-0.1106	-0.1987	-0.1098	0.0195	0.0649	0.0197
	1500	-0.0988	-0.1746	-0.0975	0.0156	0.0504	0.0154
	2000	-0.0917	-0.1617	-0.0886	0.0135	0.0433	0.0130
LSEs	100	-0.0776	-0.2175	-0.1277	0.0966	0.2323	0.0741
	250	-0.0834	-0.2039	-0.1151	0.0682	0.1927	0.0611
	500	-0.0474	-0.1452	-0.0771	0.0755	0.1645	0.0511
	750	-0.0630	-0.1528	-0.0830	0.0491	0.1340	0.0421
	1000	-0.0484	-0.1277	-0.0686	0.0471	0.1237	0.0390
	1250	-0.0413	-0.1139	-0.0608	0.0465	0.1158	0.0369
	1500	-0.0372	-0.1009	-0.0541	0.0406	0.1028	0.0329
	2000	-0.0469	-0.1151	-0.0609	0.0382	0.0958	0.0304
WLSEs	100	-0.1061	-0.2618	-0.1515	0.0935	0.2098	0.0703
	250	-0.0927	-0.2271	-0.1304	0.1047	0.1828	0.0563
	500	-0.0892	-0.1994	-0.1092	0.0545	0.1403	0.0440
	750	-0.0981	-0.2003	-0.1115	0.0388	0.1154	0.0362
	1000	-0.0836	-0.1732	-0.0960	0.0354	0.1007	0.0311
	1250	-0.0599	-0.1377	-0.0757	0.0411	0.0992	0.0312
	1500	-0.0751	-0.1503	-0.0839	0.028	0.0802	0.0252
	2000	-0.0798	-0.1561	-0.0855	0.0251	0.0747	0.0230
ADEs	100	-0.1330	-0.3050	-0.1767	0.0998	0.2137	0.0684
	250	-0.1305	-0.2735	-0.1553	0.0595	0.1751	0.0557
	500	-0.0992	-0.2137	-0.1175	0.0527	0.1391	0.0432
	750	-0.1107	-0.2189	-0.1221	0.0368	0.1152	0.0358
	1000	-0.0973	-0.1923	-0.1069	0.0315	0.0991	0.0305
	1250	-0.0819	-0.1668	-0.0924	0.0324	0.0945	0.0294
	1500	-0.0806	-0.1581	-0.0883	0.0269	0.0793	0.0247
	2000	-0.0852	-0.1639	-0.0899	0.0242	0.0743	0.0229
CvMEs	100	-0.0611	-0.1960	-0.1056	0.1125	0.2445	0.0774
	250	-0.0760	-0.1940	-0.1053	0.0731	0.1984	0.0627
	500	-0.0453	-0.1412	-0.0726	0.0746	0.1654	0.0514
	750	-0.0580	-0.1462	-0.0778	0.0516	0.1376	0.0433
	1000	-0.0444	-0.1228	-0.0646	0.0493	0.1269	0.0401
	1250	-0.0385	-0.1101	-0.0577	0.0477	0.1175	0.0375
	1500	-0.0343	-0.0972	-0.0512	0.0420	0.1049	0.0336
	2000	-0.0443	-0.1119	-0.0584	0.0396	0.0977	0.0310

TABLE 3. Average bias and MSEs of the estimates for the true parameters $\Xi = (2.5, 1, 1)$

		Bias			MSE		
n		θ	α	β	θ	α	β
MLEs	100	-0.9446	-0.3368	-0.1242	1.2437	0.1612	0.0389
	250	-0.7282	-0.2465	-0.0928	0.7688	0.0920	0.0207
	500	-0.5585	-0.1843	-0.0640	0.4591	0.0504	0.0090
	750	-0.4759	-0.1525	-0.0524	0.3332	0.0343	0.0055
	1000	-0.4052	-0.1280	-0.0429	0.2417	0.0243	0.0039
	1250	-0.3778	-0.1186	-0.0395	0.2176	0.0217	0.0033
	1500	-0.3417	-0.1068	-0.036	0.1811	0.0177	0.0028
	2000	-0.2916	-0.0899	-0.0292	0.1321	0.0126	0.0018
LSEs	100	-1.1644	-0.4158	-0.2291	1.9877	0.2767	0.0802
	250	-0.5859	-0.2103	-0.1212	1.4339	0.1713	0.0368
	500	-0.5411	-0.1856	-0.0870	0.8559	0.0970	0.0190
	750	-0.3531	-0.1216	-0.0593	0.7588	0.0770	0.0119
	1000	-0.3449	-0.1161	-0.0516	0.5621	0.0570	0.0091
	1250	-0.3094	-0.1019	-0.0458	0.489	0.0491	0.0071
	1500	-0.2766	-0.0903	-0.0406	0.4242	0.0423	0.0058
	2000	-0.2576	-0.0823	-0.0330	0.3002	0.0301	0.0038
WLSEs	100	-1.1623	-0.4183	-0.2125	1.8529	0.2536	0.0736
	250	-0.7235	-0.2451	-0.1111	0.9373	0.1120	0.0266
	500	-0.5608	-0.1853	-0.0743	0.5703	0.0630	0.0121
	750	-0.4401	-0.1420	-0.0547	0.4080	0.0423	0.0072
	1000	-0.4043	-0.1289	-0.048	0.3226	0.0329	0.0056
	1250	-0.3526	-0.1112	-0.0415	0.2766	0.0277	0.0043
	1500	-0.3257	-0.1023	-0.0381	0.2347	0.0234	0.0036
	2000	-0.2845	-0.0881	-0.0307	0.1723	0.0169	0.0023
ADEs	100	-1.1124	-0.3982	-0.1929	1.7112	0.2314	0.0642
	250	-0.7076	-0.2397	-0.1063	0.9011	0.108	0.0250
	500	-0.5514	-0.1823	-0.0721	0.5528	0.0613	0.0117
	750	-0.4369	-0.1410	-0.0538	0.4007	0.0417	0.0070
	1000	-0.4001	-0.1277	-0.0473	0.3188	0.0326	0.0056
	1250	-0.3492	-0.1101	-0.0409	0.2711	0.0273	0.0042
	1500	-0.3202	-0.1006	-0.0374	0.2302	0.0230	0.0036
	2000	-0.2802	-0.0868	-0.0301	0.1698	0.0167	0.0023
CvMEs	100	-1.085	-0.3904	-0.2027	1.8796	0.2588	0.0703
	250	-0.5378	-0.1977	-0.109	1.4186	0.1654	0.0338
	500	-0.5173	-0.1795	-0.0809	0.8367	0.0941	0.0178
	750	-0.3367	-0.1178	-0.0554	0.7528	0.0757	0.0113
	1000	-0.3328	-0.1133	-0.0488	0.5568	0.0562	0.0088
	1250	-0.2999	-0.0998	-0.0436	0.485	0.0485	0.0069
	1500	-0.2686	-0.0885	-0.0387	0.4212	0.0419	0.0056
	2000	-0.2517	-0.0810	-0.0317	0.2978	0.0298	0.0037

TABLE 4. Average bias and MSEs of the estimates for the true parameters $\Xi = (0.9, 1, 0.7)$

		Bias			MSE		
		θ	α	β	θ	α	β
MLEs	100	-0.2621	-0.3343	-0.0937	0.1162	0.1918	0.0203
	250	-0.2273	-0.2897	-0.0830	0.0949	0.1552	0.0155
	500	-0.1898	-0.2437	-0.0718	0.0748	0.1237	0.0122
	750	-0.1538	-0.1953	-0.0570	0.0542	0.0882	0.0084
	1000	-0.1395	-0.1767	-0.0509	0.0478	0.0774	0.0072
	1250	-0.1239	-0.1566	-0.0449	0.0405	0.0650	0.0059
	1500	-0.1104	-0.1393	-0.0396	0.0339	0.0541	0.0048
	2000	-0.0919	-0.1161	-0.0331	0.0270	0.0428	0.0037
LSEs	100	0.2081	0.1441	0.0232	0.3023	0.0469	0.0075
	250	0.1805	0.1407	0.0307	0.3155	0.0373	0.0047
	500	0.3649	0.1114	0.0169	0.6810	0.0269	0.0039
	750	0.2607	0.1226	0.0237	0.4100	0.0287	0.0039
	1000	0.2486	0.1264	0.0287	0.4820	0.0307	0.0048
	1250	0.4196	0.0962	0.0125	0.7386	0.0179	0.0030
	1500	0.4022	0.0967	0.0133	0.6763	0.0175	0.0030
	2000	0.3701	0.0963	0.0134	0.5429	0.0159	0.0024
WLSEs	100	0.3516	0.1031	0.0115	0.9506	0.0266	0.0062
	250	0.3277	0.1029	0.0160	0.6265	0.0201	0.0032
	500	0.3996	0.0880	0.0075	0.5159	0.0153	0.0024
	750	0.3087	0.0993	0.0140	0.3184	0.0167	0.0023
	1000	0.3954	0.0865	0.0083	0.4673	0.0131	0.0019
	1250	0.4015	0.0847	0.0075	0.4527	0.0122	0.0017
	1500	0.3509	0.0910	0.0114	0.3566	0.0138	0.0019
	2000	0.3584	0.0869	0.0091	0.3458	0.0114	0.0013
ADEs	100	0.4130	0.0916	0.0039	0.5834	0.0228	0.0055
	250	0.3729	0.0943	0.0105	0.5367	0.0174	0.0030
	500	0.4344	0.0820	0.0037	0.5408	0.0137	0.0022
	750	0.3527	0.0917	0.0094	0.3573	0.0145	0.0020
	1000	0.3881	0.0872	0.0083	0.4336	0.0136	0.0020
	1250	0.4220	0.0814	0.0052	0.4631	0.0115	0.0017
	1500	0.4142	0.0809	0.0054	0.4339	0.0110	0.0015
	2000	0.3807	0.0833	0.0068	0.3633	0.0106	0.0013
CVMEs	100	0.1796	0.1481	0.0387	0.3267	0.0494	0.0089
	250	0.1612	0.1442	0.0380	0.3270	0.0392	0.0054
	500	0.3530	0.1139	0.0210	0.6981	0.0281	0.0042
	750	0.2458	0.1253	0.0271	0.4097	0.0298	0.0042
	1000	0.2445	0.1276	0.0306	0.4962	0.0312	0.0049
	1250	0.4130	0.0978	0.0145	0.7506	0.0184	0.0031
	1500	0.3955	0.0982	0.0151	0.6842	0.0181	0.0031
	2000	0.3627	0.0977	0.0148	0.5434	0.0163	0.0025

TABLE 5. The CPs of AN and ULR CIs

		AN			ULR		
Ξ	n	α	β	θ	α	β	θ
$(0.5, 1, 1)$	100	0.8772	0.8136	0.8790	0.9566	0.9564	0.9558
	200	0.8254	0.8310	0.8830	0.9470	0.9476	0.9480
	300	0.8130	0.8210	0.8750	0.9380	0.9390	0.9382
	400	0.8268	0.8372	0.8862	0.9408	0.9402	0.9406
	500	0.8370	0.8462	0.8812	0.9434	0.9412	0.9456
	600	0.8420	0.8500	0.8834	0.9400	0.9402	0.9394
	700	0.8548	0.8612	0.8806	0.9412	0.9422	0.9424
	800	0.8450	0.8526	0.8746	0.9382	0.9362	0.9364
	900	0.8604	0.8664	0.8800	0.9414	0.9426	0.9430
	1000	0.8672	0.8712	0.8830	0.9352	0.9358	0.9344
$(2.5, 1, 1)$	100	0.9176	0.9164	0.8858	0.9602	0.9534	0.9630
	200	0.9458	0.9386	0.9156	0.9500	0.9496	0.9500
	300	0.9520	0.9450	0.9294	0.9516	0.9492	0.9492
	400	0.9584	0.9548	0.9342	0.9508	0.9536	0.9528
	500	0.9546	0.9552	0.9328	0.9520	0.9552	0.9506
	600	0.9604	0.9562	0.9410	0.9562	0.9544	0.9562
	700	0.9586	0.9602	0.9378	0.9552	0.9540	0.9538
	800	0.9626	0.9542	0.9444	0.9524	0.9522	0.9524
	900	0.9618	0.9570	0.9470	0.9520	0.9534	0.9540
	1000	0.9608	0.9588	0.9440	0.9544	0.9552	0.9532

4. ULW regression analysis

The regression models are used in different ways in survival analysis. Sometimes mean or quantiles of underlying distribution are assumed as a linear function of covariates(predictors). When the mean or quantiles have not explicit form, the location parameter is assumed as a linear function of covariates by using a suitable link function. The log-location-scale regression models are studied by several authors such as [1] and [24]. In this section, we describe the use of log-location-scale ULW regression methodology.

Let X be a ULW($\boldsymbol{\Xi}$) random variable. Let us consider are-parameterization by $\beta = 1/\sigma$ and $\alpha = \exp(\mu)$ and then, the log-lifetime $Y = \log(X)$ is a random variable with the pdf

$$h(y;\boldsymbol{\tau}) = \frac{\theta^2 \exp\left(\frac{y-\mu}{\sigma}\right)\exp\left\{-\theta\exp\left(\exp\left(\frac{y-\mu}{\sigma}\right)\right)+\theta+2\exp\left(\frac{y-\mu}{\sigma}\right)\right\}}{(1+\theta)\,\sigma}\mathbb{I}_{\mathbb{R}}(y),$$

where $\boldsymbol{\tau} = (\mu,\sigma,\theta)$, μ and σ are location and scale parameters, respectively. The cdf of Y is also given by

$$H(y;\boldsymbol{\tau}) = 1-\left(1+\frac{\theta\left(1-\exp\left(-\exp\left(\frac{y-\mu}{\sigma}\right)\right)\right)}{(1+\theta)\exp\left(-\exp\left(\frac{y-\mu}{\sigma}\right)\right)}\right)\exp\left(-\frac{\left(1-\exp\left(-\exp\left(\frac{y-\mu}{\sigma}\right)\right)\right)}{\exp\left(-\exp\left(\frac{y-\mu}{\sigma}\right)\right)}\right). \tag{4.1}$$

It is noted that the random variable Y with cdf (4.1) is denoted LULW(μ,σ,θ), where LULW stands for log-ULW distribution. Let us consider the regression model

$$\mathbf{Y} = \boldsymbol{\mu} + \sigma\boldsymbol{\varepsilon}, \tag{4.2}$$

where $\mathbf{Y} = (Y_1,\ldots,Y_n)^{\mathrm{T}}$ and $Y_1,Y_2,\ldots,Y_n$ are independent LULW random variables with parameters $\left(\mu_i = \boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\beta},\sigma,\theta\right)$, respectively. Furthermore, $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_p)^{\mathrm{T}}$, $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_n)^{\mathrm{T}}$, $\boldsymbol{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_n)^{\mathrm{T}}$, $\mu_i = \boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\beta}$ and $\boldsymbol{Z}_i = (Z_{i1},\ldots,Z_{ip})^{\mathrm{T}}$ $(= (1,Z_{i1},\ldots,Z_{ip})^{\mathrm{T}}$ when a intercept is included in a model) are ith values of covariates for $i = 1,2,\ldots,n$. In addition, $\varepsilon_i = (Y_i-\mu_i)/\sigma$ for $i = 1,2,\ldots,n$ is a random error distributed LULW with parameters $(\mu = 0, \sigma = 1, \theta)$.

Let us discuss the MLEs of parameters $\boldsymbol{\eta} = (\boldsymbol{\beta},\sigma,\theta)$ in the model (4.2) under Type-I right censoring. Suppose that the log-lifetimes $Y_i\,(i = 1,2,\ldots,n)$ are Type-I right censored (at $\log(c_i)$) from LULW(μ_i,σ,θ), where c_i is censoring time for lifetime X_i. Let us define

$$T_i = \min\{Y_i,\log(c_i)\},\; i = 1,2,\ldots,n.$$

Hence, the log-likelihood function based on the Type-I right censored sample $T_1,T_2,\ldots,T_n$ is written by

$$\ell(\boldsymbol{\eta}) = \sum_{i=1}^{n}\left\{\omega_i\log\left(h\left(t_i;\left(\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\beta},\sigma,\theta\right)\right)\right)+(1-\omega_i)\log\left(1-H\left(t_i;\left(\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\beta},\sigma,\theta\right)\right)\right)\right\}, \tag{4.3}$$

where

$$\omega_i = \begin{cases} 0\,, & T_i > \log(c_i) \\ 1\,, & T_i \le \log(c_i) \end{cases}$$

is an indicator function and t_i denotes the observed value of T_i, $i = 1,2,\ldots,n$.

The MLE of $\boldsymbol{\eta}$ can be obtained by maximizing the log-likelihood (4.3). Some numerical methods such as Nelder-Mead and BFGS can be used for a maximization problem.

4.1. Simulation study for MLEs of regression parameters

In this subsection, the bias and MSEs of MLEs are discussed for lifetime regression model parameters through a Monte Carlo simulation with 2000 trials. All simulations are run for the following model

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 Z_{i1} + \beta_2 Z_{i2} + \beta_3 Z_{i3} + \sigma\varepsilon_i \\ &= \mathbf{Z}_i^{\mathrm{T}} \boldsymbol{\beta} + \sigma\varepsilon_i, \;\; i = 1, 2, \ldots, n \end{aligned}$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2, \beta_3)^{\mathrm{T}}$, $\mathbf{Z}_i^{\mathrm{T}} = (1, Z_{i1}, Z_{i2}, Z_{i3})$ and $\varepsilon_i \sim$LUWL$(\mu = 0, \sigma = 1, \theta)$, $i = 1, 2, \ldots, n$.

In the simulation, we consider $\boldsymbol{\beta} = (-.1, -.1, -.1, -.1)$, $\theta = 1, 1.5$ and 2. The true parameter $\boldsymbol{\beta} = (.1, .1, .1, .1)$ is also considered in the simulation, but no different patterns are observed for the other one. The covariates $\boldsymbol{Z}_i$, $(i = 1, 2, \ldots, n)$ are generated in two cases: In the first case, four levels (there are 4 categories: 1,2,3,4) are considered for Z_{i1}, Z_{i2} and Z_{i3}. The other case, (Z_{i1}, Z_{i2}, Z_{i3}) are generated from multivariate normal distribution. In addition, two correlation matrix for covariates (Z_{i1}, Z_{i2}, Z_{i3}) are considered by

$$\boldsymbol{\rho}_1 = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\rho}_2 = \begin{pmatrix} 1 & 0.5 & 0.5 \\ & 1 & 0.5 \\ & & 1 \end{pmatrix}.$$

Hence, it can be observed the multicollinearity effect on the URL regression analysis. The simulation study is performed based on the following algorithm for one cycle:

Algorithm 3

A1. For a fixed n, generate the covariates with a given correlation matrix ($\boldsymbol{\rho}_1$ or $\boldsymbol{\rho}_2$) and equal marginal probabilities of four levels. R function **ordsample** in the package **GenOrd** is used in our study. Otherwise, the covariates are generated from a multivariate normal distribution with mean **0** and given correlation matrix (ρ_1 or ρ_2). R function **mvrnorm** in the package **MASS** is used in our study.

A2. Compute $\mu_i = \boldsymbol{Z}_i^{\mathrm{T}} \boldsymbol{\beta}$, $i = 1, 2, \ldots, n$.

A3. Set the true parameters $\alpha_i = \exp(\mu_i)$, β and θ.

A4. For $i = 1, 2, \ldots, n$, generate the dependent variable X_i from ULW$(\boldsymbol{\Xi}_i)$ distribution with $\boldsymbol{\Xi}_i = (\alpha_i, \beta, \theta)$ using the AR sampling given in Algorithm 1 and set $Y_i = \log(X_i) \sim$LULW(μ_i, σ, θ).

A5. The numerical methods such as Nelder-Mead, BFGS and CG are used to maximize the log-likelihood given in Eq. (4.3) and the true parameters given **A3** are used as initial values.

A6. If there is no solution or estimate out of parameter space, or negative standard error, go to **A4**.

Using Algorithm 3, a simulation study is performed with 2000 trials for a sample of size $n = 100, 200, \ldots, 1000$ and the nominal level is fixed at 0.95. Figures 3-6 are produced by settings given at the beginning of this subsection.

From Figures 3-6, the discrete or continuous covariates discussed above, does not affect the properties of estimates. If the multicollinearity level increase, the MSEs of $\widehat{\beta}_1, \widehat{\beta}_2$ and $\widehat{\beta}_3$ increase. It is an interesting observation from Figures 3-4 that, MSEs of $\widehat{\beta}_0, \widehat{\sigma}$ and $\widehat{\theta}$ are not affected by the degree of multicollinearity within covariates $\boldsymbol{Z}_1, \boldsymbol{Z}_2, \boldsymbol{Z}_3$. Although the $\widehat{\beta}_1, \widehat{\beta}_2$ and $\widehat{\beta}_3$ has a negligible bias for even if a small sample of size, the estimates $\widehat{\beta}_0, \widehat{\sigma}$ and $\widehat{\theta}$ are asymptotically unbiased. The CPs of AN CIs for β_1, β_2 and β_3 are almost equal to the nominal level for all sample size and multicollinearity cases. Furthermore, the CPs of AN CIs for θ are greater than nominal level for small sample size but it reduces to the nominal level when the sample size increases. The CPs of AN CIs for β_0 and σ are less than nominal level for small sample size, but it climbs to the nominal level when sample size increases. The mean lengths of CIs for all parameters decrease to zero when the sample size increases. The mean lengths of AN CIs for $\beta_1, \beta_2, \beta_3$ in the case multicollinearity are wider than being uncorrelated covariates. Being multicollinearity does not affect negatively on the mean lengths of AN CIs for β_0, σ and θ.

In Figures 5-6, the behaviors of estimates and CIs are also discussed according to increment in true parameter θ. When the true parameter θ is 1, bias of $\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3, \widehat{\sigma}$ and $\widehat{\theta}$ are negligible for moderate sample size. If $\theta < 1\,(>1)$ the bias of $\widehat{\beta}_0$ are positive (negative), but it reduces (increases) to zero when the sample size increases. If $\theta < 1\,(>1)$ the bias of $\widehat{\sigma}$ are negative (positive) but it increases (reduce) to zero when the sample size increases. When the θ increases, MSEs of $\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3$ and $\widehat{\theta}$ increase. For small sample size, if the θ increases, the MSEs of $\widehat{\beta}_0$ and $\widehat{\sigma}$ increase. For large sample size, if the θ increases, the MSEs of $\widehat{\beta}_0$ and $\widehat{\sigma}$ decrease. The CPs of AN CIs for β_1, β_2 and β_3 are almost equal to nominal level for $\theta = 0.5, 1$ and 2. The CPs of AN CIs for β_0 are less (greater) than nominal level when $\theta < 1\,(>1)$. If $\theta = 1$, the CPs of AN CIs for β_0 tends to nominal level for $n \geq 300$. When the θ increases, CPs of AN CIs of σ are closing to nominal level, but mean lengths of AN CIs of $\beta_1, \beta_2, \beta_3$, σ and θ increase.

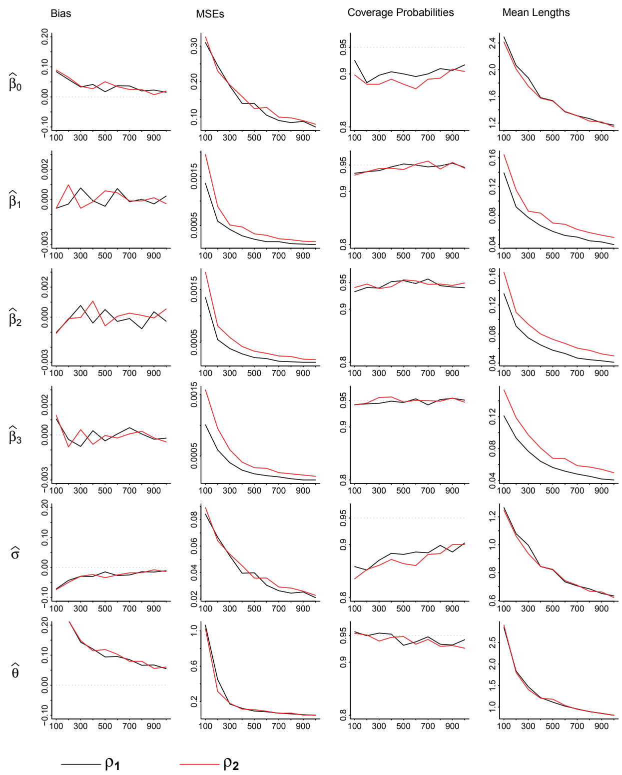

FIGURE 3. Average bias and MSEs for MLEs, CPs and mean lengths for AN CIs of ULW regression model parameters when multivariate normal covariates and $\theta = 1$

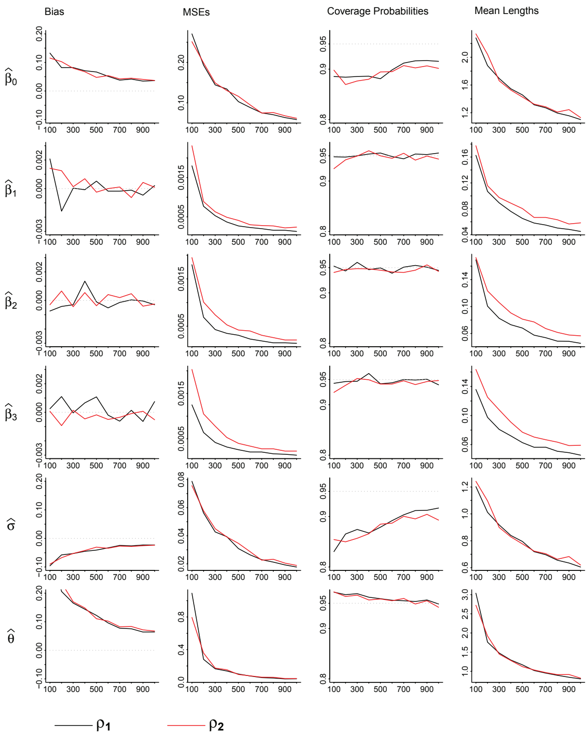

FIGURE 4. Average bias and MSEs for MLEs, CPs and mean lengths for AN CIs of ULW regression model parameters when ordinal covariates and $\theta = 1$

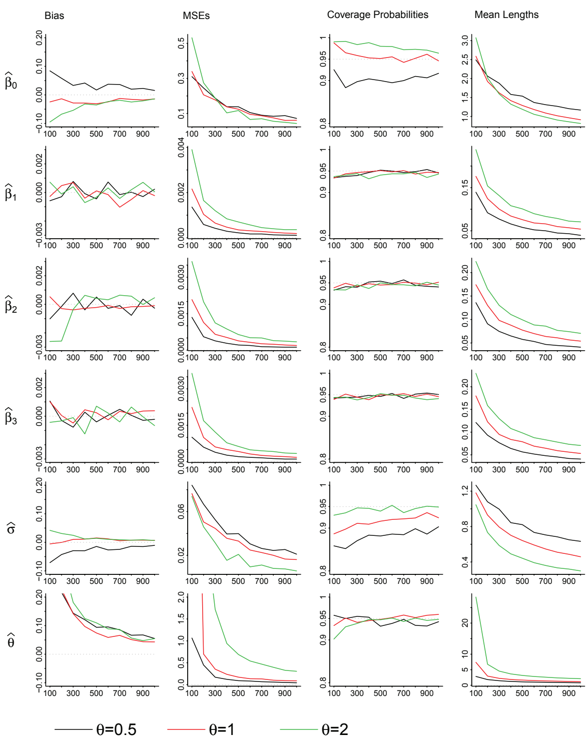

FIGURE 5. Average bias and MSEs for MLEs, CPs and mean lengths for AN CIs of ULW regression model parameters when multivariate normal covariates with correlation matrix $\boldsymbol{\rho}_1$

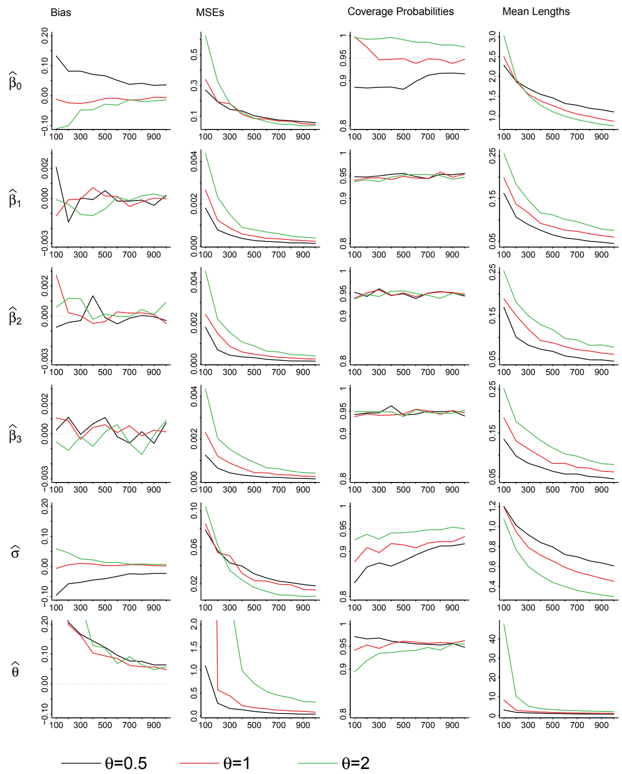

FIGURE 6. Avarage bias and MSEs for MLEs, CPs and mean lengths for AN CIs of ULW regression model parameters when ordinal covariates with correlation matrix $\boldsymbol{\rho}_1$

5. Real Data Analysis

In this section, the real data application of ULW distribution is given. The distribution fitting to total milk production data is studied.

The ULW distribution is now fitted to the data about the total milk production in the first birth of 107 cows from SINDI race. The data is taken from [3] and the data given as follow:

0.4365, 0.4260, 0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058 ,0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768 ,0.5350, 0.4151 ,0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371 ,0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912 ,0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529 ,0.4530, 0.3891, 0.4752, 0.3134, 0.3175 ,0.1167, 0.6750, 0.5113,0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945 ,0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413,0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517 ,0.2681, 0.4049, 0.5553, 0.5878 ,0.4741 ,0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, 0.2747.

It should be pointed out that this data is also analyzed in [6] and [20]. For the comparison, beta, Weibull (W), the Lindley Weibull (LW), unit-gamma (UG), unit-logistic (ULOG), UL distributions are considered. It is noted that LW, UG, ULOG and UL are introduced by [4], [11], [14], and [22] respectively. The pdfs of these distributions are given by

$$\begin{aligned}
f_{ULW}(x) &= 1-\left\{1+\frac{p_1\left(1-\exp\left(-\left(\frac{x}{p_3}\right)^{p_2}\right)\right)}{(1+p_1)\exp\left(-\left(\frac{x}{p_3}\right)^{p_2}\right)}\right\}\exp\left\{-\frac{p_1\left(1-\exp\left(-\left(\frac{x}{p_3}\right)^{p_2}\right)\right)}{\exp\left(-\left(\frac{x}{p_3}\right)^{p_2}\right)}\right\}\mathbb{I}_{\mathbb{R}_+}(x)\\
f_W(x) &= \frac{p_1}{p_2}-\left(\frac{x}{p_1}\right)^{p_2-1}\exp\left(-\left(\frac{x}{p_1}\right)^{p_2}\right)\mathbb{I}_{\mathbb{R}_+}(x)\\
f_{LW}(x) &= \frac{x^{p_2-1}p_3^2p_2p_1^2\exp\left(-(xp_1)^{p_2}\right)}{1+p_3}\\
&\quad\times\left(1-\log\left(\exp\left(-(xp_1)^{p_2}\right)\right)\right)\left(\exp\left(-(xp_1)^{p_2}\right)\right)^{p_3-1}\mathbb{I}_{\mathbb{R}_+}(x)\\
f_{Beta}(x) &= \frac{1}{\beta(p_1,p_2)}x^{p_1-1}(1-x)^{p_2-1}\mathbb{I}_{(0,1)}(x)\\
f_{UG}(x) &= \frac{p_2^{p_1}x^{p_2-1}(-\log(x))^{p_1-1}}{\Gamma(p_1)}\mathbb{I}_{(0,1)}(x)\\
f_{ULOG}(x) &= \frac{p_2\exp(p_1)x^{p_2-1}(1-x)^{p_2-1}}{\left(x^{p_2}\exp(p_1)+(1-x)^{p_2}\right)^2}\mathbb{I}_{(0,1)}(x)\\
f_{UL}(x) &= \frac{p_1^2\exp\left(-\frac{xp_1}{1-x}\right)}{(1+p_1)(1-x)^3}\mathbb{I}_{(0,1)}(x)
\end{aligned}$$

The total time on test (TTT) plot is used to determine the hazard behavior of the data. TTT plot for the total milk production data is given in Figure 7 and it indicates that the total milk production comes from a distribution with the increasing failure rate. Therefore, the ULW distribution is a candidate for modeling this data (see, Section 2.1 and Figure 2).

In this section, seven distributions are fitted to the total milk production data with the likelihood principle. The MLEs of distribution parameters are obtained by numerical methods that try to maximize the log-likelihood. In most cases, we observe that the different initial values give different estimates, and one can not conclude which one is treated as a MLE. Therefore, an algorithm is used to get the almost correct MLEs of parameters given in Table 6. An algorithm is given as follows:

Algorithm 4.

A1. 1000 (it can be increased by optionally) initial values are uniformly generated from a subset of parameter space.

A2. Using initial values generated in Step **A1**, the numerical methods Nelder-Mead, BFGS, and CG are used to maximize the log-likelihood.

A3. The likelihoods for all estimates in Step **A2** are ordered from large to small.

A4. The estimates with the largest likelihoods are treated as MLEs of parameters.

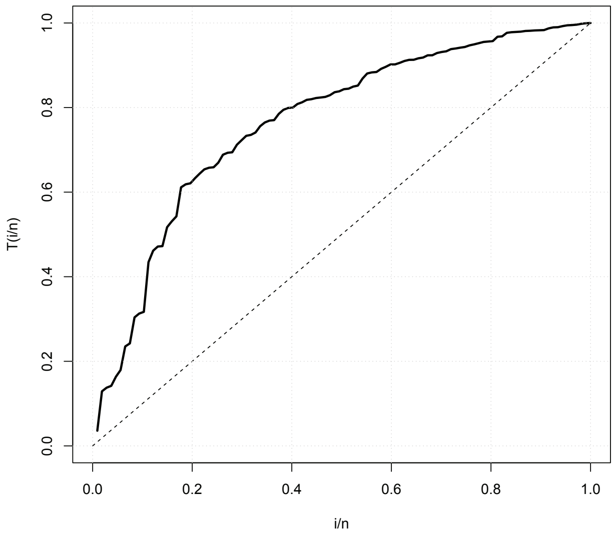

FIGURE 7. TTT plot for the total milk production data

The MLEs of parameters and related standard errors for ULW, W, LW, beta, UG, ULOG and UL distributions are given in Table 6. In this table, some comparison criteria are presented. The log-likelihood ℓ, -2ℓ, AIC, Bayesian information criterion (BIC), corrected Akaike' s information criterion (CAIC), Hannan–Quinn information criterion (HQIC), Kolmogorov-Smirnov statistic (KS), Anderson-Darling statistic(AD), Cramér von Mises statistic(CvM) and related p-values(KS p-value, AD p-value and CvM p-value), the MLE $\widehat{p}_i$ $(i=1,2,3)$ of parameter p_i with standard error $se(\widehat{p}_i)$ and AN intervals (LB_{p_i}, UB_{p_i}) are calculated and they are presented in Table 6. It is noted that some lower limit of AN CI are below the lower bound of parameter space. It can be corrected with lower bound of parameter space. In the Table 6, initial parameters, and the numerical methods are given to get MLEs for all models in the analysis. From the Table 6, the ULW distribution has the smallest values of -2ℓ, AIC, BIC, CAIC, HQIC, KS, AD and CvM. Furthermore, goodness of fit tests KS, AD and CvM confirm the ULW model validity (p values>0.05). From these results, it is concluded that the ULW distribution is better than the others in terms of all criteria. Figure 9 presents the overlapping of the fitted ULW cdf on the empirical cdf. From Figure 9, it is observed that fitted cdf of ULW distribution exhibits better than the others.

Using discussion in Subsection 3.2, 95% ULR CIs for θ, α and β are calculated by (0.0560,1.7914), (0.1515,0.7142) and (0.5473,2.1491), respectively. Figure 8 represents the 95% ULR CI of parameter θ. A logarithmic scale is used for x-axis to improve the quality of graphical view.

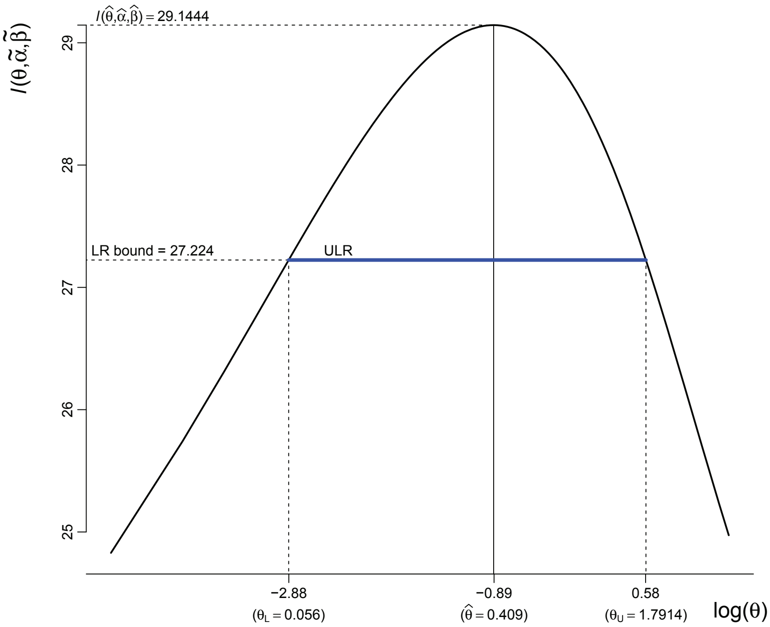

FIGURE 8. Confidence limits for parameter θ based on ULR

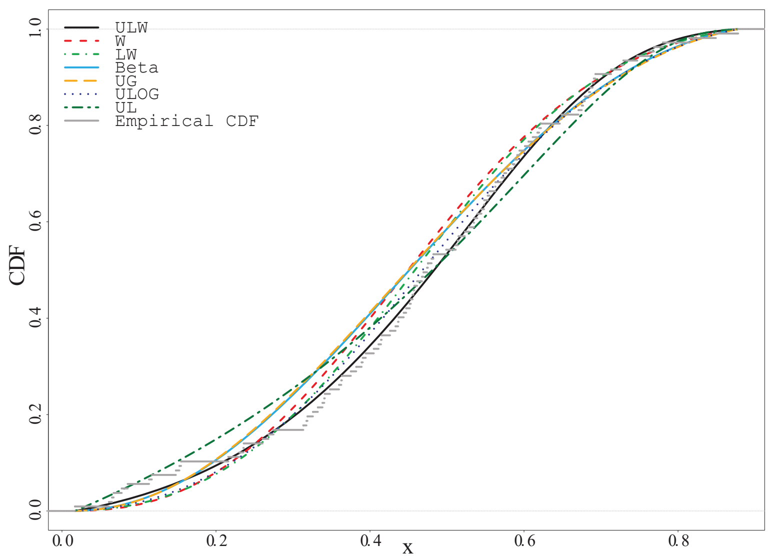

FIGURE 9. Fitted and empirical cdf plots for the total milk production data

TABLE 6. Data analysis results for the total milk production data

	ULW	W	LW	Beta	UG	ULOG	UL
ℓ	29.1444	21.3475	23.6708	23.7772	23.0467	24.8400	25.3805
-2ℓ	-58.2888	-42.6950	-47.3417	-47.5545	-46.0934	-49.6800	-50.7609
AIC	-52.2888	-38.6950	-41.3417	-43.5545	-42.0934	-45.6800	-48.7609
BIC	-44.2703	-33.3494	-33.3232	-38.2088	-36.7477	-40.3343	-46.0881
CAIC	-52.0557	-38.5796	-41.1087	-43.4391	-41.9780	-45.5646	-48.7229
HQIC	-49.0382	-36.5280	-38.0911	-41.3874	-39.9263	-43.5129	-47.6774
KS	0.0459	0.0832	0.0653	0.0910	0.0939	0.0571	0.1096
AD	0.2332	1.4841	1.0403	1.3853	1.4997	0.8646	1.3116
CVM	0.0292	0.1895	0.1104	0.2282	0.2450	0.0771	0.2286
KS p value	0.9778	0.4487	0.7518	0.3384	0.3021	0.8767	0.1532
AD p value	0.9785	0.1804	0.3366	0.2064	0.1766	0.4364	0.2286
CVM p value	0.9792	0.2891	0.5371	0.2190	0.1949	0.7095	0.2184
$\widehat{p}_1$	0.4091	0.5236	3.0722	2.4125	2.6767	0.2073	1.2001
$\widehat{p}_2$	1.1486	2.6012	2.2558	2.8297	2.9774	1.9104	
$\widehat{p}_3$	0.3454		0.5823				
LB_{p_1}	-0.2404	0.4839	0.8143	1.7961	1.9994	-0.1166	1.0258
LB_{p_2}	0.3768	2.1899	1.7933	2.0958	2.1489	1.6022	
LB_{p_3}	-0.0211		-0.1751				
UB_{p_1}	1.0586	0.5633	5.3301	3.0289	3.3541	0.5311	1.3743
UB_{p_2}	1.9205	3.0124	2.7182	3.5635	3.8060	2.2185	
UB_{p_3}	0.7119		1.3397				
$SE_{\widehat{p}_1}$	0.3314	0.0202	1.1520	0.3145	0.3456	0.1652	0.0889
$SE_{\widehat{p}_2}$	0.3938	0.2098	0.2359	0.3744	0.4228	0.1572	
$SE_{\widehat{p}_3}$	0.1870		0.3864				
Numerical Method	BFGS	BFGS	BFGS	CG	CG	BFGS	CG
Inital value for $\widehat{p}_1$	46.5515	91.4019	16.6926	10.4744	91.4145	61.8622	45.8126
Inital value for $\widehat{p}_2$	55.8020	24.2207	13.9139	16.9355	95.3848	47.9454	
Inital value for $\widehat{p}_3$	4.5296		22.7243				

6. Conclusions

In this paper, a new lifetime regression analysis with a newly introduced distribution is provided. The simulation study given in Subsection 4.1 indicates that proposed regression analysis can be used without any doubt.

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References

[1] Altun, E., Yousof, H.M. and Hamedani, G.G. (2018). A new log-location regression model with influence diagnostics and residual analysis. *International Journal of Applied Mathematics and Statistics*, 33(3), 417-449.

[2] Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71, 63-79.

[3] Brito, R.S. (2009). *Estudo de expansoes assintoticas, avaliacao numerica de momentos das distribuicoes beta generalizadas, aplicacoes em modelos de regressao e analise discriminante.* [Master's thesis, Universidade Federal Rural de Pernambuco].

[4] Cordeiro, G.M., Afify, A.Z., Yousof, H.M., Çakmakyapan, S. and Özel, G. (2018). The Lindley Weibull distribution: properties and applications. *Anais da Academia Brasileira de Ciências*, 90, 2579-2598.

[5] Cordeiro, G.M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81, 883-898.

[6] Cordeiro, G.M. and dos Santos Brito, R. (2012). The beta power distribution. *Brazilian Journal of Probability and Statistics*, 26, 88-112.

[7] Cordeiro, G.M., Ortega, E.M. and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute*, 347, 1399-1429.

[8] Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*, 31, 497-512.

[9] Fraser, D.A.S. (1976). *Probability and Statistics: Theory and Applications.* Duxbury Press, North Scituate, Mass.

[10] Gradshteyn, I.S. and Ryzhik, I.M. (2014). *Table of integrals, Series, and Products.* Eighth Edition, Academic Press.

[11] Grassia, A. (1977). On a family of distributions with argument between 0 and 1 obtained by transformation of the gamma and derived compound distributions. *Australian Journal of Statistics*, 19, 108-114.

[12] Korkmaz, M.C. (2020). A new heavy-tailed distribution defined on the bounded interval: the logit slash distribution and its application. *Journal of Applied Statistics*, 47(12), 2097-2119.

[13] Korkmaz, M.C., Altun, E., Yousof, H.M. and Hamedani, G.G. (2019). The odd power Lindley generator of probability distributions: properties, characterizations and regression modeling. *International Journal of Statistics and Probability*, 8, 70-89.

[14] Mazucheli, J., Menezes, A.F.B. and Chakraborty, S. (2019). On the one parameter unit-Lindley distribution and its associated regression model for proportion data. *Journal of Applied Statistics*, 46, 700-714.

[15] Nadarajah, S. and Gupta, A.K. (2004). The beta Fréchet distribution. *Far East Journal of Theoretical Statistics*, 14, 15-24.

[16] Nadarajah, S. and Kotz, S. (2004). The beta Gumbel distribution. *Mathematical Problems in Engineering*, 4, 323-332.

[17] Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution. *Reliability Engineering and System Safety*, 91, 689-697.

[18] Patil, G.P. and Rao, C.R. (1978). Weighted distributions and size biased sampling with applications to wildlife populations and human families. *Biometrics*, 34, 179-189.

[19] Pascoa, M.A., Ortega, E.M. and Cordeiro, G.M. (2011). The Kumaraswamy generalized gamma distribution with application in survival analysis. *Statistical Methodology*, 8, 411-433.

[20] Saraçoğlu, B. and Tanış, C. (2018). A new statistical distribution: cubic rank transmuted Kumaraswamy distribution and its properties. *Journal of the National Science Foundation of Sri Lanka*, 46, 505-518.

[21] Shaked, M. and Shanthikumar, J.G. (1994). *Stochastic Orders and Their Applications.* Academic Press, London.

[22] Tadikamalla, P.R., Johnson, N. L. (1982). Systems of frequency curves generated by transformations of logistic variables. *Biometrika*, 69, 461-465.

[23] Tanış, C. and Saraçoğlu, B. (2019). Comparisons of six different estimation methods for log-Kumaraswamy distribution. *Thermal Science*, 23, 344-344.

[24] Yousof, H.M., Altun, E., Rasekhi, M., Alizadeh, M., Hamedan, G.G. and Ali, M.M. (2019). A new lifetime model with regression models, characterizations and applications. *Communications in Statistics - Simulation and Computation*, 48, 264-286.

Appendix P*roof of Theorem 1*

For any $x > 0$, the ratio of the densities is given by

$$g(x) = \frac{\theta_1^2 (1+\theta_2) \exp\left(-\theta_1 \exp\left(\left(\frac{x}{\alpha}\right)^\beta\right) + \theta_1 + 2\left(\frac{x}{\alpha}\right)^\beta\right)}{\theta_2^2 (1+\theta_1) \exp\left(-\theta_2 \exp\left(\left(\frac{x}{\alpha}\right)^\beta\right) + \theta_2 + 2\left(\frac{x}{\alpha}\right)^\beta\right)}.$$

Consider the derivative of $\log(g(x))$ in x

$$\frac{d\log(g(x))}{dx} = \frac{(\theta_2 - \theta_1)\,\beta \left(\frac{x}{\alpha}\right)^\beta \exp\left(\left(\frac{x}{\alpha}\right)^\beta\right)}{x} < 0$$

for $\theta_1 > \theta_2$ and hence proof is completed.