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Research Article

On Graphs of Dualities of Bipartite Posets

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Abstract

In this paper we introduce some new graphs obtained from bipartite posets. We show that lower-minimal graph of a bipartite poset is isomorphic to upper-maximal graph of dual of the poset by using set representations of the posets by using set representations of the posets.

Keywords: Poset, Lower-minimal graph, Upper-maximal graph

1. Preliminaries

In this section we give some definitions we shall use in this paper. We study with finite posets and finite simple graphs.

Definition 1.1. A partial order (Simovici, Dan A. and Djeraba, Chabane, 2008) is a binary relation $\leq$ over a set P if it has:

- $a \leq a$ for all $a \in P$ (reflexivity),
- if $a \leq b$ and $b \leq a$ then $a = b$, $a, b \in P$ (antisymmetry),
- if $a \leq b$ and $b \leq c$ then $a \leq c$, $a,b ,c \in P$ (transitivity).

Definition 1.2. (Simovici, Dan A. and Djeraba, Chabane, 2008) Let $P = (X,\leq_P)$ be a poset and $x, y \in X$. If $x \leq_P y$ and $x \neq y$ then $x <_P y$.

Definition 1.3. (Simovici, Dan A. and Djeraba, Chabane, 2008) Let $P = (X,\leq_P)$ be a poset. An element $x \in X$ is called a maximal element (respectively, a minimal element) of P if there is no element $y \in X$ with $x <_P y$ in P (resp., $y <_P x$ in P). We denote the set of all maximal elements of a poset P by max(P), while min(P) denotes the set of all minimal elements of P.

Definition 1.4. (Steiner, G., and Stewart, L. K., 1987) A bipartite poset is a triple $P = (X,Y;\leq)$, where $\leq$ is a partial order on $X \cup Y$ and if $x < y$ in P, then $x \in X$ and $y \in Y$. $X = \max(P)$ and $Y = \min(P)$.

Definition 1.5. A dual poset P^d of a poset P is defined to be $x \leq y$ holds in P^d if and only if $y \leq x$ holds in P.

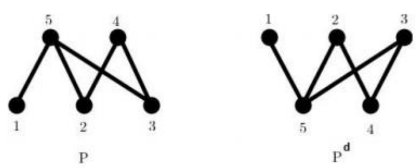

Figure 1. An example for P^d of a poset P

Definition 1.6. (Civan, Y., 2013) Let $P = (X,\leq_P)$ be a poset. For a given $x \in X$, we define $\min(x) := \{c \in \min(P) : c \leq_P x\}$.

Definition 1.7. A graph G is an ordered pair of disjoint sets (V,E), where $E \subseteq V \times V$. Set V is called the vertex or node set, while set E is the edge set of graph G. A simple graph does not contain self-loops.

Definition 1.8. (Chartrand, G., 1985) Let $G = (V, E)$ and $G_1 = (V_1, E_1)$ be graphs. G and G_1 are said to be isomorphic ($G \sim G_1$) if there exist a pair of functions $f : V \longrightarrow V_1$ and $f : E \longrightarrow E_1$ such that f associates each element in V with exactly one element in V_1 and vice versa; g associates each element in E with exactly one element in E_1 and vice versa, and for each $v \in V$, and each $e \in E$, if v is an endpoint of the edge e, then f(v) is an endpoint of the edge g(e).

Definition 1.9. (Skienna. S, 2003) Chromatic number of a graph G, $\chi(G)$ is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

Definition 1.10. (Civan, Y., 2013) Let $P = (X,\leq_P)$ be a poset. For a given $x \in X$, we define $\max(x) := \{c \in \max(P) : x \leq_P c\}$.

Definition 1.11. (Civan, Y., 2013) The upper-maximal graph $UM(P) = (X, EUM(P))$ of $P = (X,\leq)$ is defined to be the simple graph on X with $xy \in UM(P)$ if and only if $x \neq y$ and either $\max(x) \subseteq \max(y)$ or $\max(y) \subseteq \min(x)$ holds. The graph is called UM-graph.

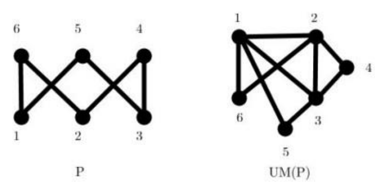

Figure 2. An example for UM-graph of a poset P

Definition 1.12. (Civan, Y., 2013) The lower-minimal graph $LM(P) = (X, ELM(P))$ of $P = (X,\leq)$ is defined to be the simple graph on X with $xy \in LM(P)$ if and only if $x \neq y$ and

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either $\min(x) \subseteq \min(y)$ or $\min(y) \subseteq \min(x)$ holds. The graph is called LM-graph.

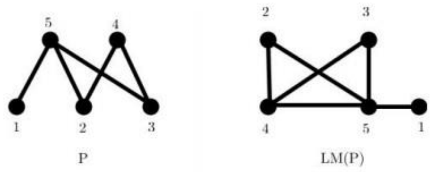

Figure 3. An example for LM-graph of a poset P

2. Set Representations of Graphs of Bipartite Posets

We want to obtain lower-minimal graph and upper-maximal graph of dual poset of a bipartite poset by using representations in Definition 1.3 and Definition 1.4 in order to analyze graph theotretical relations between the graphs.

Definition 2.1 Let $P = (X,Y ; \leq)$ be a bipartite poset. Set terms are elements of P under interpretation of [[ ]] such that $[[y]]= \{x1, x2, x3, ..., xn\}$ where $y \in Y$, $x1, x2, x3, ..., xn \in X$ and $y < x1, y < x2,…,y < xn$.

Definition 2.2 Let $P = (X,Y ;\leq)$ be a bipartite poset. Upper set terms are elements of P under interpretation of [[ ]] such that $[[y]]^U= \{x1, x2, x3, ..., xn\}$ where $y \in Y$, $x1, x2, x3, ..., xn \in X$ and $y>x1, y>x2,…,y>xn$.

3. Proofs

Proposition 2.1. LM-graph of every bipartite poset is representable by set terms of the poset as Definition 2.1.

Proof. Let $P = (X,Y ;\leq)$ be a bipartite poset. One can obtain $[[y]]= \{x1, x2, x3, ..., xn\}$ for all $y \in Y$ and , $x1, x2, x3, ..., xn \in X$ by taking Y=min(P), X=max(P) and $y<x1$, $y<x2,…,y<xn$. Under the circumtances, ELM(XUY) is obtained by taking $xiy \in$ ELM(XUY) and $\min(xi) \subseteq \min(y)$ for all $y \in Y$ and $1 \leq i < n$. On the other hand, it is true that $xixj \in$ ELM(XUY) since $\min(xi) \subseteq \min(xj)$ or $\min(xj) \subseteq \min(xi)$ for $y<xi$, $y<xj$ for $1\leq i ,j \leq n$. Therefore, the lower-minimal graph is LM(P)=(XUY, ELM(XUY)).

Lemma 2.2. Let $P = (X,Y ;\leq)$ be a bipartite poset with min(P)=X, max(P)=Y and [[xi]] are set terms of P where $1\leq i \leq n$ and $yj \in Y$ such that $1\leq j \leq m$. Then all [[yj]] which hold the condition '' if $yj \in [[xi]]$ then $xi \in [[yj]]$ '' are set terms of P^d for all $xi \in X$, $1\leq i \leq n$ and for all $yj \in Y$, $1\leq j \leq m$.

Proof. Let $P = (X,Y ;\leq)$ be a bipartite poset with min(P)=X, max(P)=Y and [[xi]] are set terms of P where $1\leq i \leq n$ and $yj \in Y$ such that $1\leq j \leq m$. It is obvious that $P^d=(X, Y, \leq)$ is a bipartite poset with max(P)=X and min(P)=Y. İf $xi < yj$ in P than $xi > yj$ in P^d from Definition 2.1. Therefore, every [[yj]] is a set term in P^d for all $yj \in P^d$.

Theorem 2.3. If $P = (X,Y ;\leq)$ is a bipartite poset with min(P)=X, max(P)=Y and [[xi]] are set terms of P where $1\leq i \leq n$ then $[[xi]]^U$ are upper set terms of P^d where $1\leq i \leq n$.

Proof. Let $P = (X,Y ; \leq)$ be a bipartite poset with min(P)=X, max(P)=Y and [[xi]] are set terms of P where $1\leq i \leq n$. Then there exist $xi < y1$, $x < y2$, …, $xi < yj$, $1\leq j \leq m$ in P. $xi > y1$, $xi > y2$, …, $xi > yj$ in P^d from Definition 1.5. We conclude $[[xi]]^U$ are upper set terms for P^d where $1\leq i \leq n$ from Definition 2.2.

Corollary 2.4. If P is bipartite poset then $LM(P) \sim UM(P^d)$ and $UM(P) \sim LM(P^d)$.

Proof. It is easy to see from Definition 1.8 and Theorem 2.3.

Corollary 2.5. If P is bipartite poset then $\chi(LM(P)) = \chi(UM(P^d))$ and $\chi(LM(P^d)) = \chi(UM(P))$.

Proof. It is easy to see from Corollary 2.4.

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