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Some inclusion results for the new Tribonacci-Lucas matrix

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ABSTRACT

The main purpose of this paper is first to establish a new regular matrix by using one of the important sequences of integer number called Tribonacci-Lucas. Also, we class this new Tribonacci-Lucas matrix with some well-known summability methods such as Riesz means, Nörlund means and Cesaro means. To do this, we show that the Tribonacci-Lucas matrix is a regular summability method and in addition to this, we give some inclusion results and finally prove that Cesaro matrix is stronger than the Tribonacci-Lucas matrix.

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1. Introduction

In 1963, Tribonacci concept was introduced by Feinberg in [1]. Later, Tribonacci and Tribonacci-Lucas numbers were investigated by Catalani in [2]. These numbers must be regarded as a generalization of the well-known Fibonacci numbers. Also, the Tribonacci-Lucas numbers are members of the following general Tribonacci recurrence

$$U_{n+1} = U_n + U_{n-1} + U_{n-2}, \quad U_0 = 0, U_1 = U_2 = 1.$$

The Tribonacci-Lucas sequence is

$$(v_n) = (3, 1, 3, 7, 11, 21, 39, 71, 131, 241, \dots)$$

and it can be easily seen from the elements of the sequence (v_n) that $v_0 = 3, v_1 = 1, v_2 = 3$ and

$$v_{n+1} = v_n + v_{n-1} + v_{n-2}.$$

The following expressions for the sums of the Tribonacci and Tribonacci-Lucas numbers can be found in [3-4]:

$$\sum_{k=1}^n U_k = \frac{U_{n+2} + U_n - 1}{2},$$

$$\sum_{k=1}^n (-1)^{k-1} U_k = \frac{(-1)^{n+1} (U_{n+1} - U_{n-1}) + 1}{2},$$

$$\sum_{k=1}^n v_k = \frac{v_{n+2} + v_n - 6}{2},$$

$$\sum_{k=1}^n (-1)^{k-1} v_k = \frac{(-1)^{n+1} (v_{n+1} - v_{n-1}) + 2}{2}.$$

The sequences of integer number defined by recurrence relations have been studied by many authors in [5-10]. In these studies, authors have given Fibonacci, Lucas, Padovan and Catalan numbers and their various properties.

Let us denote the space of all real valued sequences by w and each vector subspace of w is named sequence space. We indicate the spaces of null, convergent, bounded sequences and

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p - absolutely convergent series by c_0, c, ℓ_∞ and ℓ_p ($1 \leq p < \infty$).

Let $s = (s_n)$ be a sequence of non-negative real numbers with $s_0 > 0$ and take $S_n = \sum_{k=0}^n s_k$ for all $n \in \mathbb{N}$. Now, we give the following some well-known examples of particular summability matrices which satisfy the Toeplitz conditions.

Definition 1.1. The Riesz means according to the sequence $s = (s_n)$ is defined by the following matrix for all $n, k \in \mathbb{N}$:

$$a_{nk} = \begin{cases} \frac{s_k}{S_n}, & 0 \leq k \leq n \\ 0, & k > n \end{cases}.$$

Riesz mean (R, s) is also stated for a sequence (x_n) as follows:

$$s_n = \frac{s_1 x_1 + s_2 x_2 + \dots + s_n x_n}{S_n} \quad [11].$$

Definition 1.2. The Nörlund means according to the sequence $s = (s_n)$ is defined by the following matrix for all $n, k \in \mathbb{N}$:

$$\tilde{a}_{nk} = \begin{cases} \frac{s_{n-k+1}}{S_n}, & k \leq n \\ 0, & k > n \end{cases}.$$

Nörlund mean (N, s) is also stated for a sequence (x_n) as follows:

$$\tilde{s}_n = \frac{s_n x_1 + s_{n-1} x_2 + \dots + s_1 x_n}{S_n} \quad [11].$$

The transformation (R, s) is regular if $S_n \rightarrow \infty$ ($n \rightarrow \infty$) and (N, s) is regular if $s_n / S_n \rightarrow 0$ ($n \rightarrow \infty$) [12]. Also, both the Riesz and the Nörlund means are reduced to the following Cesaro mean $(C, 1)$ in the case $s_n = 1$ for all n :

$$C_{nk} = \begin{cases} \frac{1}{n}, & k \leq n \\ 0, & k > n \end{cases}.$$

Definition 1.3. Let (λ_n) be a strictly increasing sequence of positive integers. For a sequence (x_n) , C_λ -transformation is defined as follows:

$$s_n = \frac{x_1 + x_2 + \dots + x_{\lambda_n}}{\lambda_n} \quad [13].$$

Definition 1.4. The matrix $B = (B_{m,n})$ is a (M) matrix if B is triangular and

$$\left| \sum_{k=1}^n b_{m,k} x_k \right| \leq T \left| \sum_{k=1}^{n'} b_{n',k} x_k \right|$$

for some $n', n' = n'(n)$ ($0 \leq n' \leq n$), ($n = 1, 2, 3, \dots$) and for all m ($m \geq n$) [11]. Herein, n' is interdependent n and $\{x_n\}$ but it is independent of m . Also, the class (M) isn't confined to the regular matrices.

If $k < n+1$ for the matrix $(C, 1)$, then we have $\frac{1}{n+1} \sum_{m=0}^k t_m \leq \frac{1}{k+1} \sum_{m=0}^k t_m$ and so the matrix $(C, 1)$ is a (M) matrix [11].

Theorem 1.5. Let $A = (a_{m,n})$ and $B = (b_{m,n})$ be regular triangular matrices and A be a (M) matrix. Therefore, if

$$\sum_{n=1}^m \left| \frac{b_{m,n}}{a_{m,n}} - \frac{b_{m,n+1}}{a_{m,n+1}} \right| < K,$$

from which it is concluded that B is stronger than A [11].

Theorem 1.6. The matrix $A = (a_{m,n})$ is (M) matrix if it is triangular and holds the following conditions:

$$a_{m,k} = 0, 0 \leq \frac{a_{m,k}}{a_{n,k}} \leq T (0 \leq k \leq n \leq m) \quad (1)$$

and

$$\frac{a_{m,k}}{a_{n,k}} \geq \frac{a_{m,k+1}}{a_{n,k+1}} (0 \leq k \leq n \leq m) \quad (2)$$

2. Inclusion results for the Tribonacci-Lucas matrix

In this part of the paper, we are first going to introduce a new Tribonacci-Lucas matrix. Then, we give some relations and inclusion results between the matrix $V = (v_{nk})$ and some well-known summability matrices by comparing them.

Now, let us define our new Tribonacci-Lucas matrix as follows:

$$V = (v_{nk}) = \begin{cases} \frac{2v_k}{v_{n+2} + v_n - 6}, & 1 \leq k \leq n \\ 0, & k > n \end{cases} \quad (3)$$

If we write the terms of this matrix, then we have

$$V = \begin{bmatrix} \frac{2v_1}{v_3 + v_1 - 6} & 0 & 0 & 0 & \dots \\ \frac{2v_1}{v_4 + v_2 - 6} & \frac{2v_2}{v_4 + v_2 - 6} & 0 & 0 & \dots \\ \frac{2v_1}{v_5 + v_3 - 6} & \frac{2v_2}{v_5 + v_3 - 6} & \frac{2v_3}{v_5 + v_3 - 6} & 0 & \dots \\ \frac{2v_1}{v_6 + v_4 - 6} & \frac{2v_2}{v_6 + v_4 - 6} & \frac{2v_3}{v_6 + v_4 - 6} & \frac{2v_4}{v_6 + v_4 - 6} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and so,

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & \dots \\ \frac{1}{11} & \frac{3}{11} & \frac{7}{11} & 0 & \dots \\ \frac{1}{22} & \frac{3}{22} & \frac{7}{22} & \frac{11}{22} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It can be clearly seen from above that the Tribonacci-Lucas matrix is triangular.

Now, let us define the following real valued sequence $y = (y_n)$ which is named V – transform of a sequence $x = (x_n)$ for all $n \in \mathbb{N}$:

$$y_n = V(x_n) = \frac{2}{v_{n+2} + v_n - 6} \sum_{k=1}^n v_k \quad (4)$$

First, we are going to give the definition of V – convergence in defiance of F – convergence in [14].

Definition 2.1. If $(V(x_n - l)) \rightarrow 0$ for $n \in \mathbb{N}$ and $l \in \mathbb{R}$, then a real valued sequence $x = (x_n)$ is named V – convergent to l .

Theorem 2.2. The Tribonacci-Lucas matrix $V = (v_{nk})$ is a regular summability method $\Leftrightarrow v_{n+2} + v_n - 6 \rightarrow \infty$ as $n \rightarrow \infty$.

Proof. Let $V = (v_{nk})$ be a regular summability method. Then,

$$\lim_{n \rightarrow \infty} v_{nk} = \lim_{n \rightarrow \infty} \frac{2v_k}{v_{n+2} + v_n - 6} = 0 \text{ from Silverman-Toeplitz theorem}$$

in [8]. Thus, $v_{n+2} + v_n - 6 \rightarrow \infty$, $n \rightarrow \infty$. Now contrarily, assume that $v_{n+2} + v_n - 6 \rightarrow \infty$ as $n \rightarrow \infty$. Therefore,

$$\sum_{k=1}^{\infty} \frac{2v_k}{v_{n+2} + v_n - 6} = \sum_{k=1}^n \frac{2v_k}{v_{n+2} + v_n - 6} = 1 \quad \text{and} \quad \text{also,}$$

$$\lim_{n \rightarrow \infty} v_{nk} = \lim_{n \rightarrow \infty} \frac{2v_k}{v_{n+2} + v_n - 6} = 0$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} v_{nk} = \lim_{n \rightarrow \infty} \sum_{k=1}^n v_{nk} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2v_k}{v_{n+2} + v_n - 6} = 1.$$

In that case, the Tribonacci-Lucas matrix V is a regular summability method.

Theorem 2.3. The Tribonacci-Lucas matrix $V = (v_{nk})$ is a (M) matrix.

Proof. Since the inequalities

$$0 \leq \frac{2v_k}{v_{m+2} + v_m - 6} \cdot \frac{v_{n+2} + v_n - 6}{2v_k} = \frac{v_{n+2} + v_n - 6}{v_{m+2} + v_m - 6} \leq \frac{v_{n+2}}{v_{m+2}} \leq 1$$

and

$$\begin{aligned} \frac{\frac{2v_{k+1}}{v_{m+2} + v_m - 6}}{\frac{2v_{k+1}}{v_{n+2} + v_n - 6}} &= \frac{v_{n+2} + v_n - 6}{v_{m+2} + v_m - 6} \leq \frac{v_{n+2} + v_n - 6}{v_{m+2} + v_m - 6} \cdot \frac{2v_k}{2v_k} \\ &= \frac{2v_k}{v_{m+2} + v_m - 6} \cdot \frac{v_{n+2} + v_n - 6}{2v_k} \end{aligned}$$

hold, the Tribonacci-Lucas matrix V is (M) matrix.

Definition 2.4. Let $x = (x_n)$ and $y = (y_n)$ be two real valued sequences. Then, if there are two positive real numbers t and T such that the inequality $t.x_n \leq y_n \leq T.x_n$ holds for all $n \in \mathbb{N}$, they are named equivalent.

Now, let us give the relation between V and (R, s) :

Theorem 2.5. Let $V = (v_{nk})$ be a Tribonacci-Lucas matrix and $x = (x_n)$ be a real valued sequence. Then, $x_n \rightarrow l(V) \Leftrightarrow x_n \rightarrow l(R, s)$ for any sequence (s_n) such that (s_n) and (v_n) are equivalent for all $n \in \mathbb{N}$.

Proof. Assume that $x_n \rightarrow l(V)$. In this case, we get

$$\lim_{n \rightarrow \infty} \frac{2}{v_{n+2} + v_n - 6} \sum_{k=1}^n v_k (x_k - l) = 0.$$

From here, under the supposition on (s_n) , the following inequality holds:

$$\begin{aligned} \frac{1}{S_n} \sum_{k=1}^n s_k (x_k - l) &\leq \frac{1}{S_n} \sum_{k=1}^n T.v_k (x_k - l) \\ &\leq \frac{T}{t} \frac{2}{v_{n+2} + v_n - 6} \sum_{k=1}^n v_k (x_k - l). \end{aligned} \quad (5)$$

Also, by using the similar technique, we find

$$\frac{t}{T} \frac{2}{v_{n+2} + v_n - 6} \sum_{k=1}^n v_k (x_k - l) \leq \frac{1}{S_n} \sum_{k=1}^n s_k (x_k - l). \quad (6)$$

Resulting from $x_k \rightarrow l(V)$, the inequalities (5) and (6) give us

$$\text{that } \lim_{n \rightarrow \infty} \frac{1}{S_n} \sum_{k=1}^n s_k (x_k - l) = 0. \text{ The sufficient condition of this}$$

theorem can be easily shown by use of the same method. So, the proof is completed.

Theorem 2.6. Let $B = (b_{nk})$ be a regular matrix and suppose

that $\sum_{k=1}^n |b_{nk} - v_{nk}| \rightarrow 0$ as $n \rightarrow \infty$. Then for any bounded sequence, $x_n \rightarrow l(B) \Leftrightarrow x_n \rightarrow l(V)$.

Proof. For any n and bounded sequence $x = (x_n)$, we have

$$\begin{aligned} |(Bx)_n - (Vx)_n| &= \left| \sum_{k=1}^n b_{nk} x_k - \sum_{k=1}^n v_{nk} x_k \right| \\ &\leq \sum_{k=1}^n |b_{nk} - v_{nk}| |x_k| \leq \|x\| \sum_{k=1}^n |b_{nk} - v_{nk}|. \end{aligned}$$

Hence, if $x_n \rightarrow l(B)$ and $x_n \rightarrow l(V)$, then we get

$$|(Vx)_n - l| \leq |(Vx)_n - (Bx)_n| + |(Bx)_n - l| \rightarrow 0, n \rightarrow \infty \quad (7)$$

and in a similar way

$$|(Bx)_n - l| \leq |(Bx)_n - (Vx)_n| + |(Vx)_n - l| \rightarrow 0, n \rightarrow \infty. \quad (8)$$

Consequently, the inequalities (7) and (8) complete the proof.

Now, we are going to establish the following associate matrix

$$\tilde{V} = (\tilde{v}_{nk}) :$$

$$\tilde{v}_{nk} = \begin{cases} \frac{2v_{n-k}}{v_{n+2} + v_n - 6}, & k \leq n \\ 0, & k > n \end{cases} \quad (9)$$

We can state the associate matrix $\tilde{V} = (\tilde{v}_{nk})$ as a Nörlund type Tribonacci-Lucas matrix when the matrix $\tilde{V} = (\tilde{v}_{nk})$ can be written as a Riesz type Tribonacci-Lucas matrix. Accordingly, we first give the following lemma which will be used in the next theorems.

Lemma 2.7. The series $\sum_{n=1}^{\infty} s_n x^{n-1}$ and $\sum_{n=1}^{\infty} S_n x^{n-1}$ are convergent for all x where $|x| < 1$, if (N, s_n) is a regular Nörlund matrix.

This lemma is also suitable for the matrix $\tilde{V} = (\tilde{v}_{nk})$ just because the matrix $\tilde{V} = (\tilde{v}_{nk})$ is a Nörlund type matrix. In the continuation of this study, we can use V_n in place of \tilde{V}_n having regard to the definition of $\tilde{V} = (\tilde{v}_{nk})$.

Remark 2.8. Due to the fact that the series $v(x) = \sum_{n=1}^{\infty} v_n x^{n-1}$

and $V(x) = \sum_{n=1}^{\infty} V_n x^{n-1}$ are convergent for all $|x| < 1$, the series below are also convergent:

$$q(x) = \frac{s(x)}{v(x)} = \frac{S(x)}{V(x)}, \quad q(x) = \sum_{n=1}^{\infty} q_n x^{n-1},$$

$$r(x) = \frac{v(x)}{s(x)} = \frac{V(x)}{S(x)}, \quad r(x) = \sum_{n=1}^{\infty} r_n x^{n-1}.$$

Theorem 2.9. $(N, s_n) \subseteq (\tilde{V})$ if and only if there is $T > 0$ such that for every n $|q_1|S_n + |q_2|S_{n-1} + \dots + |q_n|S_1 \leq T.V_n$ and

$$\lim_{n \rightarrow \infty} \frac{q_n}{V_n} = 0.$$

Proof. Let (k_n) and (h_n) be the (N, s) -transformation and (\tilde{V}) -transformation of a real valued sequence (p_n) . Then, we get

$$\begin{aligned} \sum_{n=1}^{\infty} V_n h_n x^{n-1} &= \sum_{n=1}^{\infty} V_n \frac{(v_n p_1 + v_{n-1} p_2 + \dots + v_1 p_n)}{V_n} x^{n-1} \\ &= (v_1 p_1) x^0 + (v_2 p_1 + v_1 p_2) x^1 + (v_3 p_1 + v_2 p_2 + v_1 p_3) x^2 + \dots \\ &\quad + (v_n p_1 + v_{n-1} p_2 + \dots + v_1 p_n) x^{n-1} + \dots \\ &= p_1 (v_1 x^0 + v_2 x^1 + v_3 x^2 + \dots) + p_2 (v_1 x + v_2 x^2 + v_3 x^3 + \dots) \\ &\quad + p_3 (v_1 x^2 + v_2 x^3 + v_3 x^4 + \dots) + \dots + p_n (v_1 x^{n-1} + \dots) \\ &= p_1 x^0 (v_1 x^0 + v_2 x^1 + v_3 x^2 + \dots + v_n x^{n-1}) + p_2 x^1 (v_1 x^0 + v_2 x^1 + v_3 x^2 + \dots + v_{n-1} x^{n-2}) \\ &\quad + \dots + p_n x^{n-1} (v_1 x^0 + \dots) \\ &= \left(\sum_{n=1}^{\infty} p_n x^{n-1} \right) \left(\sum_{n=1}^{\infty} v_n x^{n-1} \right) = p(x) v(x). \end{aligned} \quad (10)$$

In a similar way, we also have

$$\sum_{n=1}^{\infty} S_n k_n x^{n-1} = p(x) s(x). \quad (11)$$

Now, from the hypothesis, we know that $v(x) = q(x) s(x)$ and $v(x) p(x) = q(x) s(x) p(x)$.

If we consider (17), (18) and the Cauchy product of series, then

we find $\sum_{n=1}^{\infty} V_n h_n x^{n-1} = \sum_{n=1}^{\infty} \sum_{m=1}^n q_{n-m+1} S_m k_m x^{n-1}$ and so for all $n \in \mathbb{N}$,

$$V_n h_n = q_n S_1 k_1 + q_{n-1} S_2 k_2 + \dots + q_1 S_n k_n. \quad \text{Thus, } h_n = \sum_{m=1}^n b_{nm} k_m \quad \text{and}$$

$$b_{nm} = \begin{cases} \frac{q_{n-m+1} S_m}{V_n}, & m \leq n \\ 0, & m > n \end{cases}. \quad \text{The matrix } b_{nm} \text{ is regular, in truth}$$

$$\lim_{n \rightarrow \infty} b_{nm} = \lim_{n \rightarrow \infty} \frac{q_{n-m+1} S_m}{V_n} = \lim_{n \rightarrow \infty} \frac{q_{n-m+1} S_m}{V_{n-m+1}} = 0,$$

$$\sum_{m=1}^{\infty} |b_{nm}| = \frac{|q_1| S_n + \dots + |q_n| S_1}{V_n} \leq T,$$

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n b_{nm} = \frac{q_1 S_n + \dots + q_n S_1}{V_n} = \frac{V_n}{V_n} = 1.$$

Therefore, the proof of sufficient condition is completed. The proof of necessary condition can be done by taking advantage of the specifications in the expression of theorem.

Definition 2.10. Let $\beta = (\beta_n)$ be a strictly increasing sequence of positive integers. Let us define the V_{β} -transformation of a sequence $x = (x_n)$ as follows:

$$z_n = \frac{2}{v_{\beta(n)+2} + v_{\beta(n)} - 6} \sum_{k=1}^{\beta(n)} v_k x_k.$$

Little o notation, also called Landau's symbol is usually used in mathematics. Informally, $f(t) = o(g(t))$ is supposed to mean that f grows much slower than g and it is insignificant in comparison. Formally, we write $f(t) = o(g(t))$ if and only if for every $T > 0$ there exists a real number N such that for all $t > N$ we get $|f(t)| < T|g(t)|$ and if $g(t) \neq 0$, this is equivalent to $\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = 0$.

Theorem 2.11. Let $\beta = \{\beta(n)\}$ and $\gamma = \{\gamma(n)\}$ be a strictly increasing sequences of natural number. Then, V_β is equivalent to V_γ on ℓ_∞ if $\lim_{n \rightarrow \infty} \frac{v_{\beta(n)+2} + v_{\beta(n)} - 6}{v_{\gamma(n)+2} + v_{\gamma(n)} - 6} = 1$.

Proof. Let $x = x(n)$ be a bounded sequence and $T(n) = \max\{\beta(n), \gamma(n)\}$, $t(n) = \min\{\beta(n), \gamma(n)\}$. Then, we have for any n

$$\begin{aligned} |(V_\beta x)_n - (V_\gamma x)_n| &= \left| \frac{2}{v_{\beta(n)+2} + v_{\beta(n)} - 6} \sum_{k=1}^{\beta(n)} v_k x_k - \frac{2}{v_{\gamma(n)+2} + v_{\gamma(n)} - 6} \sum_{k=1}^{\gamma(n)} v_k x_k \right| \\ &= \left| \frac{2}{v_{T(n)+2} + v_{T(n)} - 6} \sum_{k=1}^{T(n)} v_k x_k - \frac{2}{v_{t(n)+2} + v_{t(n)} - 6} \sum_{k=1}^{t(n)} v_k x_k \right| \\ &= \left| \frac{2}{v_{T(n)+2} + v_{T(n)} - 6} \sum_{k=1}^{t(n)} v_k x_k + \frac{2}{v_{T(n)+2} + v_{T(n)} - 6} \sum_{k=t(n)+1}^{T(n)} v_k x_k - \frac{2}{v_{t(n)+2} + v_{t(n)} - 6} \sum_{k=1}^{t(n)} v_k x_k \right| \\ &= \left| \sum_{k=1}^{t(n)} v_k x_k \left(\frac{2}{v_{T(n)+2} + v_{T(n)} - 6} - \frac{2}{v_{t(n)+2} + v_{t(n)} - 6} \right) + \frac{2}{v_{T(n)+2} + v_{T(n)} - 6} \sum_{k=t(n)+1}^{T(n)} v_k x_k \right| \\ &\leq \|x\|_\infty \left(\sum_{k=1}^{t(n)} v_k \left| \frac{2(v_{t(n)+2} + v_{t(n)} - v_{T(n)+2} - v_{T(n)})}{(v_{T(n)+2} + v_{T(n)} - 6)(v_{t(n)+2} + v_{t(n)} - 6)} \right| + \sum_{k=t(n)+1}^{T(n)} v_k \left| \frac{2}{v_{T(n)+2} + v_{T(n)} - 6} \right| \right) \\ &\leq \|x\|_\infty \left(\frac{(v_{t(n)+2} + v_{t(n)} - v_{T(n)+2} - v_{T(n)})(v_{t(n)+2} + v_{t(n)} - 6)}{(v_{T(n)+2} + v_{T(n)} - 6)(v_{t(n)+2} + v_{t(n)} - 6)} + \frac{v_{T(n)+2} + v_{T(n)} - v_{t(n)+2} - v_{t(n)}}{v_{T(n)+2} + v_{T(n)} - 6} \right) \\ &\leq 2\|x\|_\infty \left(\frac{(v_{T(n)+2} + v_{T(n)} - 6) - (v_{t(n)+2} + v_{t(n)} - 6)}{v_{T(n)+2} + v_{T(n)} - 6} \right) \\ &\leq 2\|x\|_\infty \left(1 - \frac{v_{t(n)+2} + v_{t(n)} - 6}{v_{T(n)+2} + v_{T(n)} - 6} \right) = o(1) \end{aligned}$$

since $\lim_{n \rightarrow \infty} \frac{v_{\beta(n)+2} + v_{\beta(n)} - 6}{v_{\gamma(n)+2} + v_{\gamma(n)} - 6} = 1$ and $\lim_{n \rightarrow \infty} \frac{v_{t(n)+2} + v_{t(n)} - 6}{v_{T(n)+2} + v_{T(n)} - 6} = 1$.

Therefore, if x is V_β -summable to L , then we obtain

$$0 \leq |(V_\gamma x)_n - L| \leq |(V_\gamma x)_n - (V_\beta x)_n| + |(V_\beta x)_n - L| = o(1) + o(1) = o(1)$$

and in a similar way, if x is V_γ -summable to L , then we obtain

$$0 \leq |(V_\beta x)_n - L| \leq |(V_\beta x)_n - (V_\gamma x)_n| + |(V_\gamma x)_n - L| = o(1) + o(1) = o(1).$$

Theorem 2.12. Cesaro matrix C_{nk} is stronger than Tribonacci-Lucas matrix $V = (v_{nk})$.

Proof. From Theorem 1.5, if we take $B = V$ (Tribonacci-Lucas matrix) and $A = C$ (Cesaro matrix), then we find

$$\begin{aligned} \sum_{k=1}^n \left| \frac{2nv_k}{v_{n+2} + v_n - 6} - \frac{2nv_{k+1}}{v_{n+2} + v_n - 6} \right| &= \sum_{k=1}^n \frac{2n}{v_{n+2} + v_n - 6} (v_k - v_{k+1}) \\ &= \frac{2n}{v_{n+2} + v_n - 6} \sum_{k=1}^n (v_k - v_{k+1}). \end{aligned}$$

Since the inequality $2n \leq v_{n+2} + v_n - 6$ holds for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \frac{2n}{v_{n+2} + v_n - 6} \sum_{k=1}^n (v_k - v_{k+1}) &\leq \frac{v_{n+2} + v_n - 6}{v_{n+2} + v_n - 6} \sum_{k=1}^n (v_k - v_{k+1}) \\ &\leq v_1 - v_2 + v_2 - v_3 + v_3 - v_4 + \dots + v_n - v_{n+1} \leq 1 - v_{n+1} < 1. \end{aligned}$$

Consequently, we obtain $V \subset C$ and the proof is completed.

In general, the converse of this theorem is not true. Indeed, for the sequence $x_n = \frac{(-1)^n}{n}$, $(C_n x) = \frac{1}{n} \sum_{k=1}^n \frac{(-1)^k}{k}$ is convergent but the V -transformation of (x_n) , that is

$$(V_n x) = \frac{2}{v_{n+2} + v_n - 6} \sum_{k=1}^n v_k \frac{(-1)^k}{k} \text{ is not convergent.}$$

3. Summary and Conclusions

In our study, a new regular matrix was first defined by using the well-known sequence of integer number called Tribonacci-Lucas. Then, we compared the Tribonacci-Lucas matrix with the other summability matrices such as Nörlund mean, Riesz mean and Cesaro mean and also investigated the relation between these matrices. Since the matrix $V = (v_{nk})$ is regular, the sequence $(V_n x)$ is convergent for a sequence (x_n) . So, for the matrix $V = (v_{nk})$, both statistical convergence and the studies with regular matrices can be investigated.

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