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Integrating Second Order ODE's: the Pseudo-Wronskian

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Abstract

We give a survey of results regarding the influence of the quantity $W(x,t) = x' - \frac{x}{t}$ in studying the linear-like solutions of the ordinary differential equation $x'' + f(t,x,x') = 0$.

Key-words: ordinary differential equation, linear-like solution, prescribed behavior, fixed point theory.

1. Introduction

Let us consider the general second order ordinary differential equation (ODE) below

$$x'' + f(t,x,x') = 0, \quad t \geq t_0 \geq 1, \qquad (1)$$

where the nonlinearity $f:[t_0,+\infty)\times IR^2 \to IR$ is assumed continuous. By a *linear-like* solution of equation (1) we mean any C^2 function x defined locally near $+\infty$ that verifies the equation throughout its entire domain of existence and can be asymptotically developed either as

$$x(t) = c \bullet t + o(t),\ x'(t) = c + o(1) \quad \text{when } t \to +\infty \qquad (2)$$

or as

$$x(t) = c_1 \bullet t + c_2 + o(1),\ x'(t) = c_1 + o(t^{-1}) \quad \text{when } t \to +\infty \qquad (3)$$

for some real constants c, c_1, c_2.

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Motivated by some questions regarding certain reaction-diffusion equations, see the references [16], [14], we are interested here in the influence that the quantity (called *pseudo-wronskian* in the sequel)

$$W(x,t) = \frac{1}{t}\begin{vmatrix} x'(t) & 1 \\ x(t) & t \end{vmatrix} = x'(t) - \frac{x(t)}{t},\ t \geq t_0,$$

has over the restrictions imposed on the nonlinearity $f(t,x,x')$ in the literature devoted to linear-like solutions of ODE's.

2. A general existence result

An existence result for linear-like solutions in a large particular case of (1) can be found in the note [28]. For general results, see [2].

Theorem 1 ([28], Theorem 1)**.** Assume that f does not depend explicitly of x' and

$$|f(t,u)| \leq h_1(t) g\left(\frac{|u|}{t}\right) + h_2(t),$$

where h_1, h_2, g are nonnegative-valued, continuous functions such that

$$\int_{t_0}^{+\infty} t^{\lambda}[h_1(t) + h_2(t)]dt < +\infty$$

for a fixed $\lambda \in [0,1]$. Then, equation (1) has a solution x which verifies (2) if $\lambda = 0$, (3) if $\lambda = 1$ and, for $\lambda \in (0,1)$, reads as

$$x(t) = c \bullet t + o(t^{1-\lambda}),\ x'(t) = c + o(t^{-\lambda}) \quad \text{when } t \to +\infty.$$

Proof (sketch of). Introduce the Banach space $(X(T,\lambda), \|*\|)$, where $X(T,\lambda)$ is the set of all real-valued continuous functions $v(t)$ defined in $[T,+\infty)$ which satisfy $\lim_{t\to+\infty} t^{\lambda} v(t) = l_{\lambda}(v) \in IR$ and $\|v\| = \sup_{t \geq T} t^{\lambda} |v(t)|$. Given c_1, c_2, introduce also the set

$$S(c_2) = \left\{ v \in X(T,\lambda) : \left|t^{\lambda} v(t) - c_2\right| \leq \int_t^{+\infty} \tau^{\lambda}[G h_1(\tau) + h_2(\tau)]d\tau, t \geq T \right\},$$

where $G = \sup\{g(u) : 0 \leq |c_1| + 2|c_2| + 1\}$.

The operator $O : S(c_2) \to S(c_2)$ with the formula

$$(Ov)(t) = t^{-\lambda}\left[c_2 - \int_t^{+\infty} \tau^{\lambda} f(\tau, u(v, c_1, c_2)(\tau)) d\tau\right],\ t \geq t_0,$$

where

$$u(v,c_1,c_2)(t)=[c_1+c_2(1-\operatorname{sgn}\lambda)]t+\lambda t\int_t^{+\infty}\frac{v(\tau)}{\tau}d\tau-(1-\lambda)\int_{t_0}^{t}v(\tau)d\tau,$$

satisfies the requirements of Schauder's fixed point theorem. It has, consequently, a fixed point in $S(c_2)$ – which is our solution.

The history of asymptotic integration of ODE's (with an emphasis on asymptotic equivalence, polynomial-like solutions, boundedness and so on) has been long and fructuous. The reader can find in the references [1], [5]-[13], [15], [17]-[27], [34]-[42] many interesting details.

3. A study of $W(x,t)$

The simplest result concerning the pseudo-wronskian regards its *set of zeros*: if x is any C^2 function such that $x''(t)\le 0$ for every t then $W(x,t)$ either has no zero or its set of zeros is an interval (possibly degenerate). This is a consequence of the obvious identity

$$tx''(t)=\frac{d}{dt}[tW(x,t)].$$

Another immediate result reads as follows: assume that the linear ODE

$$x''+a(t)x=0,\quad t\ge t_0,$$

with continuous coefficient $a(t)$ has a solution $x(t)$ that satisfies (2) for some $c>0$. This happens if, say, $\int_{t_0}^{+\infty}t|a(t)|dt<+\infty$. Then,

$$W(y,t)\approx-\frac{1}{c\cdot t}<0\qquad \text{as } t\to+\infty,$$

where $y(t)=x(t)\int_t^{+\infty}\frac{ds}{[x(s)]^2}$, see [3], p. 360. Since $y(t)\approx 1/c$ when $t\to+\infty$, we notice that, *regardless of the sign of* x'', *linear homogenous ODE's of second order have always bounded solutions with eventually negative pseudo-wronskian.*

The presence of $W(x,t)$ in the formula of $f(t,x,x')$ from equation (1) yields a *consistent enlargement of the class of functions* h – see the hypotheses of Theorem 1.

Theorem 2 ([29], Theorem 6). Assume that there exist the continuous functions $h(t)$, $g(s)$ such that $g(s)>0$ for all $s>0$ and $xg(s)\le g(x^{1-\alpha}s)$, where $x\ge t_0$ and $s\ge 0$, for a certain $\alpha\in(0,1)$. Suppose further that

$$\left|f(t,x,x')\right| \le h(t)g\left(\left|x'-\frac{x}{t}\right|\right)$$

and, for some ε , $\delta > 0$,

$$\int_{t_0}^{+\infty} \frac{h(s)}{s^{\alpha}} ds \le \int_{\varepsilon+\delta t_0^{1-\alpha}}^{+\infty} \frac{du}{g(u)} \quad (< +\infty) .$$

Then, all the solutions x of equation (1) such that $\left|W(x,t_0)\right| \le \delta$ are defined throughout $[t_0,+\infty)$ and satisfy (2).

Sufficient conditions for the *integrability of* $W(x,t)$ are given in the next result.

Theorem 3 ([3], Theorem 6). Assume that $f(t,x,x') = f(t,x)$ in equation (1) and $\left|f(t,x)\right| \le F\left(t,\left|x\right|/t\right)$, where the comparison function F is continuous and monotone nondecreasing with respect to the second variable.

(i) Suppose that there exists $\lambda \in (0,1)$ and $c \ne 0$ such that

$$\int_{t_0}^{+\infty} t \ln\left(\frac{t}{t_0}\right) F\left(t,\frac{2}{t_0}(1+\lambda)\left|c\right|\right) dt < \lambda\left|c\right| .$$

Then, equation (1) has a solution x defined in $[t_0,+\infty)$ that can be developed as $x(t) = c(x)t + o(1)$ when $t \to +\infty$ for $c(x) \in IR$, $\operatorname{sgn} x(t) = \operatorname{sgn} c$ for all $t \ge t_0$ and

$$x(t) - \int_{t_0}^{t} \frac{x(s)}{s} ds = c + o(1) \quad \text{when } t \to +\infty .$$

(ii) Suppose that there exist $a \in IR$ and $c > 0$ such that

$$\int_{t_0}^{+\infty} t\left[1+\ln\left(\frac{t}{t_0}\right)\right] F\left(t,\left|a\right|+\frac{c}{t_0}\right) dt < c .$$

Then, equation (1) has a solution x defined in $[t_0,+\infty)$ that can be developed as $x(t) = c \cdot t + o(1)$ when $t \to +\infty$ and with $W(x,*) \in L^1((t_0,+\infty), IR)$.

Let us discuss now the effects that a perturbation of equation (1) might have on $W(x,t)$: *non-null limits* and *oscillations*.

Theorem 4 ([29], Theorem 12). Fix $u_0 \in IR$ and consider the ODE below

$$x'' + f(t,x,x') = p(t) , \; t \ge t_0 , \quad (4)$$

with continuous f, p, such that

$$\left|f(t,x,x')\right| \le h(t)\left|x' - \frac{x}{t}\right|, \quad \int_{t_0}^{+\infty} s\bullet h(s)ds, \quad \lim_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t} p(s)ds = a \in IR - \{0\}.$$

Then, equation (4) has a solution x defined in $[t_0,+\infty)$ such that $x(t_0)=u_0$ and $\lim_{t\to+\infty} W(x,t) = a$ – which means that $x(t) \approx a\bullet t\ln t$ when $t \to +\infty$.

The proof of this theorem relies on an application of the Leray-Schauder alternative in the function space $X(t_0,-1)$ for the integral operator

$$(Tv)(t) = \int_{t_0}^{t} sb(s)ds + \int_{t}^{+\infty} g\left(s, v(s), \int_{t_0}^{s}\frac{v(\tau)}{\tau^2}d\tau\right)ds, \quad v \in X(t_0,-1),$$

where $g(t,v,w) = tf\left(t, t\left(\frac{u_0}{t_0}+w\right), \frac{u_0}{t_0}+w+\frac{v}{t}\right)$.

Theorem 5 ([30], Theorem 1, Remark 3). Assume that $f(t,x,x') = f(t,x)$ in equation (4) and $\left|f(t,x)\right| \le F(t,|x|)$ for a continuous and monotone nondecreasing in the second variable comparison function F. Suppose also that

$$\int_{t}^{+\infty} sF\left(s, |P(s)| + \sup_{\tau\ge s}\{q(\tau)\}\right)ds \le q(t), \quad t \ge t_0,$$

for a certain positive-valued, bounded (possibly decaying to 0 as $t \to +\infty$), continuous function $q(t)$. Here, $P''(t) = p(t)$ for all $t \ge t_0$. Assume further that

$$\limsup_{t\to+\infty}\left[t\frac{W(P,t)}{q(t)}\right] > 1 \quad \text{and} \quad \liminf_{t\to+\infty}\left[t\frac{W(P,t)}{q(t)}\right] < -1.$$

Then, equation (4) has a solution x defined in $[t_0,+\infty)$ such that

$$x(t) = P(t) + o(1) \quad \text{as } t \to +\infty$$

and $W(x,*)$ oscillates – this means that there exist the sequences $\{t_n : n \ge 1\}$ and $\{t_n{}^0 : n \ge 1\}$, increasing and unbounded from above, with the property that $W(x,t_n) < W(x,t_n{}^0) = 0 < W(x,t_{n+1})$ for every $n \ge 1$.

The proof of this result is based on a Kummer-like decomposition of the equation (4), see [2], pp. 47-48. We have the identities

$$a(t)y(t) = \int_{t}^{+\infty} a(s)\left[f(s,x(s)) + q(s)a(s)\int_{s}^{+\infty}\frac{y(\tau)}{a(\tau)}d\tau\right]ds \quad (5)$$

and

$$x(t) = P(t) - a(t)\int_t^{+\infty} \frac{y(s)}{a(s)} ds\,, \quad (6)$$

where $a(t)$ is a positive solution of the linear homogenous ODE below

$$z'' + q(t)z = 0\,, \ \ t \ge t_0\,, \quad (7)$$

such that (the coefficient $q(t)$ being continuous)

$$\int_{t_0}^{+\infty} \frac{ds}{[a(s)]^2} < +\infty\,, \ \lim_{t\to+\infty} \frac{a'(t)}{a(t)} = 0 \ \text{ and } \ \int_{t_0}^{+\infty} |q(s)| \left\{ a(s) \int_s^{+\infty} \frac{d\tau}{[a(\tau)]^2} \right\} ds < +\infty\,.$$

Then, we have the asymptotic developments

$$x(t) = P(t) + o\left(a(t) \int_t^{+\infty} \frac{ds}{[a(s)]^2} \right) \quad (8)$$

and

$$W(x,t) = W(P,t) - W(a,t)\int_t^{+\infty} \frac{y(s)}{a(s)} ds + y(t)$$

$$= W(P,t) + o\left(a(t) \int_t^{+\infty} \frac{ds}{[a(s)]^2} \right) \quad \text{when } t \to +\infty\,.$$

The latter estimate follows from

$$\lim_{t\to+\infty} \left\{ [a(t)]^2 \int_t^{+\infty} \frac{ds}{[a(s)]^2} \right\}^{-1} = \lim_{t\to+\infty} \frac{a'(t)}{a(t)} = 0$$

and $\quad y(t)\left\{ a(t) \int_t^{+\infty} \frac{ds}{[a(s)]^2} \right\}^{-1} = a(t)y(t)\left\{ [a(t)]^2 \int_t^{+\infty} \frac{ds}{[a(s)]^2} \right\}^{-1} = o(1) \quad$ when $t \to +\infty$.

In the fundamental particular case when $\int_{t_0}^{+\infty} t|q(t)|dt < +\infty$, all the solutions $a(t)$ of equation (7) verify the formulas (2) and thus (8) reads as $x(t) = P(t) + o(1)$ when $t \to +\infty$ – which is the formula obtained at Theorem 5.

Open problem. Other conditions regarding the coefficient $q(t)$ that will lead to the existence of such a solution $a(t)$ of equation (7) are still unknown. According to the fundamental paper by Hartman [18], for the equation (7) to be nonoscillatory it is necessary that either

$$\lim_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(\tau)d\tau ds=-\infty \text{ or } \lim_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(\tau)d\tau ds\in IR.$$

Further, a necessary and sufficient condition to have $\lim_{t\to+\infty}[a'(t)/a(t)]=0$ is that

$$\lim_{u\to+\infty}\left(\sup_{v\geq 0}\frac{1}{1+v}\left|\int_{u}^{u+v}q(s)ds\right|\right)=0.$$

Another topic regarding the pseudo-wronskian is concerned with the *multiplicity of solutions* to a problem attached to equation (1).

Theorem 6 ([31], Theorem) Consider the problem

$$x''=\frac{1}{t}g(tx'-x),\quad t\geq t_0\geq 1,$$

$$t_0x'(t_0)-x(t_0)=c>0,$$

where the nonlinearity $g:IR\to IR$ is assumed continuous, with $g(c)=g(3c)=0$ and $g(\alpha)>0$ for every real $\alpha\neq c,3c$ and

$$\int_{c+}^{2c}\frac{du}{g(u)}<+\infty,\qquad \int_{2c}^{(3c)-}\frac{du}{g(u)}=+\infty.$$

Then, the above problem has infinitely many linear-like solutions that verify formulas (3).

The proof relies on the fact that the problem admits the next one-parametric family of solutions

$$x_T(t)=t\left[\frac{u_0}{t_0}+\int_{t_0}^{t}\frac{y_T(s)}{s^2}ds\right],\quad x(t_0)=u_0\in IR,\qquad T>0, \tag{9}$$

where

$$y_T(t)=c,\ t\in[t_0,t_0+T]\quad\text{and}\quad y_T(t)=G^{-1}(t-t_0-T),\ t\geq t_0+T$$

for the function $G:[c,3c)\to[0,+\infty)$ given by the formulas

$$G(c)=0\quad\text{and}\quad G(x)=\int_{c+}^{x}\frac{du}{g(u)},\ x\in(c,3c).$$

We obtain the asymptotic development $x_T(t)=a_Tt+b_T+o(1)$ when $t\to+\infty$, where

$$a_T=\frac{u_0}{t_0}+\int_{t_0}^{+\infty}\frac{y_T(s)}{s^2}ds\quad\text{and}\quad b_T=-3c.$$

We also remark that

$$\frac{da_T}{dT} = -\frac{c(t_0+T-1)}{(t_0+T)^2} - \int_{t_0+T}^{+\infty} \frac{g(y_T(s))}{s^2} ds < 0, \ \ T > 0,$$

which means that the solutions from (9) are not only different for each other but they have also *different slopes* for their oblique asymptotes $X_T = a_T t + b_T$.

Open problem. It is still unknown how to build examples of initial value problems with an infinity of solutions verifying (3) when both a_T and b_T vary with T.

Under appropriate conditions, *we can prescribe the zero(s) and size* of the pseudo-wronskian.

Theorem 7 ([32], Proposition 1). Assume that $a, b > 0$, the coefficient $q(t)$ of equation (7) is nonnegative-valued, with eventually isolated zeros, and

$$(a+b)\int_{t_0}^{+\infty} s \bullet q(s) ds \le b.$$

Then, equation (7) has a solution x which verifies (2) for $c = a$ and also satisfies the relations $W(x, t_0) = 0$ and

$$b - a \le x'(t) \le \left(1 - \frac{1}{t}\int_{t_0}^{t} s^2 q(s) ds\right)\frac{x(t)}{t} < \frac{x(t)}{t} \le a + b, \ \ t > t_0.$$

The proof is based on an application of the Banach contraction principle to the integral operator

$$(Tx)(t) = t\left[a + \int_t^{+\infty} \frac{1}{s^2}\int_{t_0}^{s} \tau q(\tau) x(\tau) d\tau ds\right], \ t \ge t_0.$$

We can also produce the *oscillation of the pseudo-wronskian* under several conditions.

Theorem 8 ([33], Theorem 7, Remark 1). Fix $p \in (0,1)$, $c \ne 0$ and assume that the coefficient $q(t)$ of equation (7) verifies the following conditions

$$\int_{t_0}^{+\infty} \left\{ t\left[\int_t^{+\infty} s^2 |q(s)| ds\right]^{-1}\right\}^{1-p} t^2 |q(t)| dt < +\infty$$

and

$$L_+ = \limsup_{t \to +\infty} \frac{t\int_t^{+\infty} s^2 q(s) ds}{\int_t^{+\infty} s^2 |q(s)| ds} > 0 > L_- = \liminf_{t \to +\infty} \frac{t\int_t^{+\infty} s^2 q(s) ds}{\int_t^{+\infty} s^2 |q(s)| ds}.$$

Then, the equation (7) has a solution x that satisfies (3) for $c_1 = c$ and $c_2 = 0$, $W(x,*)$ oscillates and also $W(x,*) \in L^p((t_0,+\infty), IR)$.

The proof consists of an application of the Banach contraction principle to the integral operator – recall the decomposition (5), (6) and take $z'' = 0$, $t \geq t_0$, as auxiliary equation –

$$(Ty)(t) = \frac{1}{t}\int_t^{+\infty} sq(s)x(s)ds, \quad x(t) = c - s\int_t^{+\infty} \frac{y(s)}{s}ds, \quad t \geq t_0.$$

To give an example of coefficient $q(t)$ which obeys the restrictions from Theorem 8, set $\alpha > \frac{2-p}{p}$ and introduce the sequence $\{a_k = k^{-\alpha} - (k+1)^{-\alpha} : k \geq 1\}$. Consider also the function $Q : [9,+\infty) \to IR$ with the formula

$$Q(t) = \begin{cases} a_k(t-9k), t \in [9k, 9k+1], \\ a_k(9k+2-t), t \in [9k+1, 9k+3], \\ a_k(t-9k-4), t \in [9k+3, 9k+4], \\ a_k(9k+4-t), t \in [9k+4, 9k+5], \\ a_k(t-9k-6), t \in [9k+5, 9k+7], \\ a_k(9k+8-t), t \in [9k+7, 9k+8], \\ 0, t \in [9k+8, 9(k+1)], \end{cases} \quad k \geq 1.$$

Then, we can take $q(t) = t^{-2}Q(t)$ for all $t \geq 9$. We have also $L_+ = -L_- = \frac{9\alpha}{4}$.

Open problem. What can be said about the case when $c = 0$?

Let us close this study with an analysis of *the size of the pseudo-wronksian for nonlinear differential equations.*

Theorem 9 ([4], Theorem 8, Corollary 3). Set $t_0, \lambda \geq 1$, $a, b \geq 0$, $c \in (0,1]$ and $\varepsilon \in (0,1)$. Assume that the continuous function $q : [t_0,+\infty) \to [0,+\infty)$ verifies the conditions

$$\lambda(a+\varepsilon)^{\lambda-1} I_c < c, \quad \frac{b}{t_0} + (a+\varepsilon)^{\lambda} \frac{I_c}{ct_0^{\,c}} < \varepsilon, \quad I_c = \int_{t_0}^{+\infty} t^{\lambda+c} q(t)dt.$$

Then, the Emden-Fowler like equation

$$x''+q(t)x^{\lambda}=0, \quad t\geq t_0,$$

admits a solution $x:[t_0,+\infty)\to[b,+\infty)$ with the asymptotic profile given by $x(t)=a\bullet t+O(t^{1-c})$ when $t\to+\infty$ such that

$$a^{\lambda}\bullet\frac{1}{t}\int_{t_0}^{t}s^{\lambda+1}q(s)ds\leq\frac{x(t)-b}{t}-x'(t)\leq(a+\varepsilon)^{\lambda}\bullet\frac{1}{t^c}\int_{t_0}^{t}s^{\lambda+c}q(s)ds, \quad t\geq t_0.$$

In particular, $W(x,t)=O(t^{-c})$ as $t\to+\infty$.

The proof relies on an application of the Banach contraction principle to the integral operator

$$(Ty)(t)=-\frac{1}{t}\int_{t_0}^{t}sq(s)[x(s)]^{\lambda}ds, \quad x(t)=at+b-s\int_{t}^{+\infty}\frac{y(s)}{s}ds, \quad t\geq t_0.$$

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