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Single and Multiple Positive Solutions for Nonlinear Discrete Problems

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Abstract

In this paper, we discuss the existence of single and multiple positive solutions to the nonlinear second order discrete three-point boundary value problems. To prove the main results, we will use the well-known result of the fixed point theorems in cones.

Key words: Boundary value problems; Difference equations; Fixed point theorems; Positive solutions.

Özet

Bu çalışmada, lineer olmayan ikinci mertebe diskret üç-nokta sınır değer problemleri için tek ve birden fazla pozitif çözümlerin varlığı incelenecek. Ana sonuçları ispatlamak için konilerde sabit nokta teoremlerinin iyi bilinen sonuçlarını kullanacağız.

Anahtar Kelimeler: Sınır değer problemleri; Fark denklemleri; Sabit nokta teoremleri; Pozitif çözümler.

1. Introduction

There are many authors who studied the existence of solutions to second order two-point boundary value problems on difference equations. For some recent results, we refer the reader to [2], [3], [4], [5], [7], [9], [10]. However, to the best of the author's knowledge, there are few results for the existence of solutions to second order three-point boundary value problems on difference equations.

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In paper [1], Agarwal, Thompson and Tisdell studied existence results for solutions to second-order discrete boundary value problem

$$\Delta^2 y_{k+1} = f(x_k, y_k, \Delta y_k), \quad k = 1, 2, \dots, n-1$$

$$G(y_0, y_n, y_d) = (0, 0), \quad d \in \{1, \dots, n-1\}$$

where f and G are continuous.

In paper [11], Zhang and Medina studied existence of positive solutions for the nonlinear discrete three-point boundary value problem

$$\Delta^2 x_{k-1} + f(x_k) = 0, \quad k = 1, 2, \dots, n$$

$$x_0 = 0, \quad x_{n+1} - ax_l = b$$

where $n \in \{2, 3, \dots\}$, $l \in \{1, 2, \dots, n\}$, a and b are positive numbers and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$.

We are interested in the existence of one or two positive solutions of the following three-point boundary value problem:

$$\begin{cases} \Delta^2 y(n) + h(n)f(n, y(n+1)) = 0, & n \in [a, c] \\ \Delta y(a) = 0, \quad \alpha y(c+1) + \beta \Delta y(c+1) = \Delta y(b) \end{cases} \quad (1)$$

where $a < b < c$ are distinct integers, $\alpha > 0$, $\beta > 1$ and Δ denotes the forward difference operator defined by $\Delta y(n) = y(n+1) - y(n)$. Additionally, throughout the paper we assume $h: [a, c] \rightarrow [0, \infty)$ is continuous such that $h(n_0) > 0$ for at least one $n_0 \in [b, c]$, $f: [a, c] \times [0, \infty) \rightarrow [0, \infty)$ is continuous.

Problem (1) is equivalent to the problem

$$\begin{aligned} y(n+2) - 2y(n+1) + y(n) + h(n)f(n, y(n+1)) &= 0, \quad n \in [a, c] \\ y(a+1) &= y(a), \quad \beta y(c+2) + (\alpha - \beta)y(c+1) = y(b+1) - y(b). \end{aligned}$$

2. Preliminaries

Let $G(n, s)$ be Green's function for the boundary value problem

$$\Delta^2 y(n) = 0, \quad n \in [a, c] \subset \mathbb{Z}$$

$$\Delta y(a) = 0, \quad \alpha y(c+1) + \beta \Delta y(c+1) = \Delta y(b).$$

A direct calculation gives

$$G(n, s) = \begin{cases} s \in [a, b] \begin{cases} c + \frac{\beta-1}{\alpha} + 1 - n, & s+1 \leq n \\ c - s + \frac{\beta-1}{\alpha}, & n \leq s \end{cases} \\ s \in [b, c] \begin{cases} c + \frac{\beta}{\alpha} + 1 - n, & s+1 \leq n \\ c - s + \frac{\beta}{\alpha}, & n \leq s \end{cases} \end{cases} \quad (2)$$

To state and prove the main results of this paper, we need the following lemmas.

Lemma 1 Let $\alpha > 0$, $\beta > 1$. Then the Green's function $G(n, s)$ in (2) satisfies the following inequality

$$G(n, s) \geq \frac{n-a}{c-a+1} G(c+1, s)$$

For $(n, s) \in [a, c+1] \times [a, c]$.

Proof. We proceed sequentially on the branches of the Green's function $G(n, s)$ in (2).

(i) For $s \in [a, b]$ and $s+1 \leq n$, we obtain

$$G(n, s) = c + \frac{\beta-1}{\alpha} + 1 - n$$

and

$$\frac{G(n, s)}{G(c+1, s)} = \frac{c-n+1}{\frac{\beta-1}{\alpha}} + 1 > \frac{n-a}{c-a+1}.$$

(ii) Let $s \in [a, b]$ and $n \leq s$. Then

$$G(n, s) = c - s + \frac{\beta-1}{\alpha}$$

and

$$\frac{G(n, s)}{G(c+1, s)} = 1 > \frac{n-a}{c-a+1}.$$

(iii) Take $s \in [b, c]$ and $s+1 \leq n$. Then

$$G(n, s) = c + \frac{\beta}{\alpha} + 1 - n$$

and

$$\frac{G(n, s)}{G(c+1, s)} = \frac{c-n+1}{\frac{\beta}{\alpha}} + 1 \geq \frac{n-a}{c-a+1}.$$

(iv) Fix $s \in [b, c]$ and $n \leq s$. Then

$$G(n, s) = c - s + \frac{\beta}{\alpha}$$

and

$$\frac{G(n, s)}{G(c+1, s)} = 1 > \frac{n-a}{c-a+1}.$$

Lemma 2 Assume $\alpha > 0$, $\beta > 1$. Then the Green's function $G(n, s)$ in (2) satisfies

$$0 < G(n, s) \leq G(s+1, s)$$

for $(n, s) \in [a, c+1] \times [a, c]$.

Proof. Since for $s \in [a, b]$

$$G(c+1, s) = \frac{\beta-1}{\alpha} > 0$$

and for $s \in (b, c+1]$

$$G(c+1, s) = \frac{\beta}{\alpha} > 0,$$

we obtain $G(n, s) > 0$ from Lemma 1. To show that $G(n, s) \leq G(s+1, s)$, we again deal with the branches of the Green's function $G(n, s)$ in (2).

(i) For $s \in [a, b]$ and $s+1 \leq n \leq c+1$, $G(n, s)$ is decreasing in n and $G(n, s) \leq G(s+1, s)$.

(ii) Let $s \in [a, b]$ and $a \leq n \leq s$. Then it is obvious that $G(n, s) = G(s+1, s)$.

(iii) Take $s \in (b, c]$ and $s+1 \leq n \leq c+1$. Since $G(n, s)$ is decreasing in n ,

$$G(n, s) \leq G(s+1, s).$$

(iv) Fix $s \in (b, c]$ and $a \leq n \leq s$. Then it is clear that $G(n, s) = G(s+1, s)$.

Lemma 3 Suppose $\alpha > 0$, $\beta > 1$ and $s \in [a, c]$. Then the Green's function $G(n, s)$ in (2) satisfies

$$\min_{n \in [b, c+1]} G(n, s) \geq k \|G(\cdot, s)\|,$$

where

$$k = \frac{\beta-1}{\alpha(c-a) + \beta-1} \quad (3)$$

and $\|\cdot\|$ is defined by $\|x\| = \max_{n \in [a, c+1]} |x(n)|$.

Proof. Since the Green's function $G(n, s)$ in (2) is nonincreasing in n , we get

$\min_{n \in [b, c+1]} G(n, s) \geq G(c+1, s)$. Moreover, it is obvious that $\|G(\cdot, s)\| = G(s+1, s)$ for $s \in [a, c]$ by

Lemma 2. Then we have from the branches in (2) that

$$G(c+1, s) \geq k G(s+1, s).$$

Let the Banach space $\mathcal{B} = \{y : [a, c+1] \rightarrow \mathbb{R}\}$ with the norm $\|y\| = \max_{n \in [a, c+1]} |y(n)|$ and define the cone $P \subset \mathcal{B}$ by

$$P = \left\{ y \in \mathcal{B} : y(n) \geq 0, \min_{n \in [b, c+1]} y(n) \geq k \|y\| \right\} \quad (4)$$

where k is given in (3).

(1) is equivalent to the nonlinear integral equation

$$y(n) = \sum_{s=a}^c G(n, s) h(s) f(s, y(s+1)). \quad (5)$$

We can define the operator $A : P \rightarrow \mathcal{B}$ by

$$Ay(n) = \sum_{s=a}^c G(n, s) h(s) f(s, y(s+1)), \quad (6)$$

where $y \in P$. If $y \in P$, then by Lemma 3 we have

$$\begin{aligned} \min_{n \in [b, c+1]} Ay(n) &= \sum_{s=a}^c \min_{n \in [b, c+1]} G(n, s) h(s) f(s, y(s+1)) \\ &= k \sum_{s=a}^c \max_{n \in [a, c+1]} |G(n, s)| h(s) f(s, y(s+1)) \\ &= k \|Ay\|. \end{aligned}$$

Thus $Ay \in P$ and therefore $AP \subset P$. In addition, $A: P \rightarrow P$ is completely continuous by a standard application of the Arzela-Ascoli Theorem.

We will apply the following well-known result of the fixed point theorems to prove the existence of one or two positive solutions to the (1).

Lemma 4 [6],[8] Let P be a cone in a Banach space \mathcal{B} , and let D be an open, bounded subset of \mathcal{B} with $D_p := D \cap P \neq \emptyset$ and $\bar{D}_p \neq P$. Assume that $A: \bar{D}_p \rightarrow P$ is a compact map such that $y \neq Ay$ for $y \in D_p$. The following results hold.

- (i) If $\|Ay\| \leq \|y\|$ for $y \in D_p$, then $i_p(A, D_p) = 1$.
- (ii) If there exists an $q \in P \setminus \{0\}$ such that $y \neq Ay + \lambda q$ for all $y \in D_p$ and all $\lambda > 0$, then $i_p(A, D_p) = 0$.
- (iii) Let U be open in P such that $\bar{U}_p \subset D_p$. If $i_p(A, D_p) = 1$ and $i_p(A, U_p) = 0$, then A has a fixed point in $D_p \setminus \bar{U}_p$. The same result holds if $i_p(A, D_p) = 0$ and $i_p(A, U_p) = 1$.

3. Single and Multiple Positive Solutions

For the cone P given in (4) and any positive real number r , define the convex set

$$P_r := \{y \in P : \|y\| < r\}$$

and the set

$$\Omega_r := \left\{ y \in P : \min_{n \in [b, c+1]} y(n) < kr \right\}.$$

The following results are proved in [3].

Lemma 5 The set Ω_r has the following properties.

- (i) Ω_r is open relative to P .
- (ii) $P_{kr} \subset \Omega_r \subset P_r$.
- (iii) $y \in \partial\Omega_r$ if and only if $\min_{n \in [b, c+1]} y(n) = kr$.
- (iv) If $y \in \partial\Omega_r$, then $kr \leq y(n) \leq r$ for $n \in [b, c+1]$.

For convenience, we introduce the following notations. Let

$$\begin{aligned} f_{kr}^r &:= \min \left\{ \min_{n \in [b, c+1]} \frac{f(n, y)}{r} : y \in [kr, r] \right\} \\ f_0^r &:= \max \left\{ \max_{n \in [a, c+1]} \frac{f(n, y)}{r} : y \in [0, r] \right\} \\ f^a &:= \limsup_{y \rightarrow a} \max_{n \in [a, c+1]} \frac{f(n, y)}{y} \\ f_a &:= \liminf_{y \rightarrow a} \min_{n \in [b, c+1]} \frac{f(n, y)}{y} \quad (a := 0^+, \infty). \end{aligned}$$

In the next two lemmas, we give conditions on f guaranteeing that $i_p(A, P_r) = 1$ or $i_p(A, \Omega_r) = 0$.

Lemma 6 Let

$$L := \sum_{s=a}^c G(s+1, s)h(s). \quad (7)$$

If the conditions

$$f_0^r \leq \frac{1}{L} \text{ and } y \neq Ay \text{ for } y \in \partial P_r,$$

hold, then $i_p(A, P_r) = 1$.

Proof. If $y \in \partial P_r$, then using Lemma 2, we have

$$\begin{aligned} Ay(n) &= \sum_{s=a}^c G(n, s)h(s)f(s, y(s+1)) \\ &\leq \|f(\cdot, y)\| \sum_{s=a}^c G(s+1, s)h(s) \\ &\leq r = \|y\| \end{aligned}$$

It follows that $\|Ay\| \leq \|y\|$ for $y \in \partial P_r$. By Lemma 4(i), we get $i_p(A, P_r) = 1$.

Lemma 7 Let

$$N := \left(\sum_{s=b}^c G(c+1, s)h(s) \right)^{-1} \quad (8)$$

If the conditions

$$f_{kr}^r \geq Nk \text{ and } y \neq Ay \text{ for } y \in \partial \Omega_r,$$

hold, then $i_p(A, \Omega_r) = 0$.

Proof. Let $q(n) \equiv 1$ for $n \in [a, c+1]$, then $q \in \partial P_1$. Assume there exist $y_0 \in \partial \Omega_r$ and $\lambda_0 > 0$ such that $y_0 = Ay_0 + \lambda_0 q$. Then for $n \in [b, c+1]$ we have

$$y_0(n) = Ay_0(n) + \lambda_0 q(n)$$

$$\begin{aligned}
 &\geq \sum_{s=b}^c G(n, s)h(s)f(s, y_0(s+1)) + \lambda_0 \\
 &\geq Nkr \sum_{s=b}^c G(c+1, s)h(s) + \lambda_0 \\
 &= kr + \lambda_0
 \end{aligned}$$

But this implies that $kr \geq kr + \lambda_0$, a contradiction. Hence, for $y_0 \in \partial\Omega_r$ and $\lambda_0 > 0$, so by Lemma 4(ii), we get $i_p(A, \Omega_r) = 0$.

Theorem 8 Let k , L , and N be as in (3), (7), and (8), respectively. Suppose that one of the following conditions holds.

(C1) There exist constants $c_1, c_2, c_3 \in \mathbb{R}$ with $0 < c_1 < kc_2$ and $c_2 < c_3$ such that

$$f_0^{c_1}, f_0^{c_3} \leq \frac{1}{L}, f_{kc_2}^{c_2} \geq Nk, \text{ and } y \neq Ay \text{ for } y \in \partial\Omega_{c_2}.$$

(C2) There exist constants $c_1, c_2, c_3 \in \mathbb{R}$ with $0 < c_1 < c_2 < kc_3$ such that

$$f_{kc_1}^{c_1}, f_{kc_3}^{c_3} \geq Nk, f_0^{c_2} \leq \frac{1}{L}, \text{ and } y \neq Ay \text{ for } y \in \partial P_{c_2}.$$

Then (1) has two positive solutions. Additionally, if in (C1) the condition $f_0^{c_1} \leq \frac{1}{L}$ is replaced

by $f_0^{c_1} < \frac{1}{L}$, then (1) has a third positive solution in P_{c_1} .

Proof. Assume that (C1) holds. We show that either A has a fixed point in ∂P_{c_1} or in $\Omega_{c_2} \setminus \bar{P}_{c_1}$. If $y \neq Ay$ for $y \in \partial P_{c_1}$, then by Lemma 6, we have $i_p(A, P_{c_1}) = 1$. Since $f_{kc_2}^{c_2} \geq Nk$ and $y \neq Ay$ for $y \in \partial\Omega_{c_2}$, from Lemma 7 we get $i_p(A, \Omega_{c_2}) = 0$. By Lemma 5(ii) and $c_1 < kc_2$, we have $\bar{P}_{c_1} \subset \bar{P}_{kc_2} \subset \Omega_{c_2}$. From Lemma 4(iii), A has a fixed point in $\Omega_{c_2} \setminus \bar{P}_{c_1}$. If $y \neq Ay$ for $y \in \partial P_{c_3}$, then from Lemma 6 $i_p(A, P_{c_3}) = 1$. By Lemma 5(ii) and Lemma 4(iii), A has a fixed point in $P_{c_3} \setminus \bar{\Omega}_{c_2}$. The proof is similar when (C2) holds and we omit it here.

Corollary 9 If there exist a constant $c > 0$ such that one of the following conditions holds:

$$(H1) \quad 0 \leq f^0, f^\infty < \frac{1}{L}, f_{kc}^c \geq Nk, \text{ and } y \neq Ay \text{ for } y \in \partial\Omega_c.$$

$$(H2) \quad N < f_0, f_\infty \leq \infty, f_0^c \leq \frac{1}{L}, \text{ and } y \neq Ay \text{ for } y \in \partial P_c.$$

Then (1) has two positive solutions.

Proof. We show that (H1) implies (C1). It is easy to verify that $0 \leq f^0 < \frac{1}{L}$ implies that there

exists $c_1 \in (0, kc)$ such that $f_0^{c_1} < \frac{1}{L}$. Let $m \in \left(f^\infty, \frac{1}{L}\right)$. Then there exists $c_2 > c$ such that

$$\max_{n \in [a, c+1]} f(n, y) \leq ky \text{ for } y \in [c_2, \infty) \text{ because } 0 \leq f^\infty < \frac{1}{L}. \text{ Let}$$

$$n = \max \left\{ \max_{n \in [a, c+1]} f(n, y) : 0 \leq y \leq c_2 \right\} \text{ and } c_4 > \max \left\{ \frac{n}{\frac{1}{L} - m}, \frac{1}{L} \right\}.$$

Then we have

$$\max_{n \in [a, c+1]} f(n, y) \leq my + n \leq mc_4 + n < \frac{1}{L} c_4 \text{ for } y \in [0, c_4].$$

This implies that $f_0^{c_4} < \frac{1}{L}$ and (C1) holds. Similarly, (H2) implies (C2).

As a special case of Theorem 8 and Corollary 9, we have the following two results.

Theorem 10 Assume that one of the following conditions holds.

(C3) There exist constants $c_1, c_2 \in \mathbb{R}$ with $0 < c_1 < kc_2$ such that

$$f_0^{c_1} \leq \frac{1}{L} \text{ and } f_{kc_2}^{c_2} \geq Nk.$$

(C4) There exist constants $c_1, c_2 \in \mathbb{R}$ with $0 < c_1 < c_2$ such that

$$f_{kc_1}^{c_1} \geq Nk \text{ and } f_0^{c_2} \leq \frac{1}{L}.$$

Then (1) has a positive solution.

Corollary 11 Assume that one of the following conditions holds:

$$(H3) \quad 0 \leq f^0 < \frac{1}{L} \text{ and } N < f_\infty \leq \infty.$$

$$(H4) \quad 0 \leq f^\infty < \frac{1}{L} \text{ and } N < f_0 \leq \infty.$$

Then (1) has a positive solution.

References

1. R. P. Agarwal, H. B. Thompson and C. C. Tisdell, Three point boundary value problems for second-order discrete equations, *Comput. Math. Appl.*, 45 (2003), 1429-1435.
2. D. R. Anderson, I. Rachunkova and C. C. Tisdell, Solvability of discrete Neumann boundary value problems, *J. Math. Anal. Appl.*, 331 (2007), 736-741.
3. D. Bai and Y. Xu, Nontrivial solutions of boundary value problems of second-order difference equations, *J. Math. Anal. Appl.*, 326 (2007), 297-302.
4. A. Cabada, A. Iannizzotto and S. Tersion, Multiple solutions for discrete boundary value problems, *J. Math. Anal. Appl.*, 356 (2009), 293-307.

5. X. Cai and J. Yu, Existence theorems for second-order discrete boundary value problems, *J. Math. Anal. Appl.*, 320 (2006), 649-661.
6. D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.
7. X. He and X. Wu, Existence and multiplicity of solutions for nonlinear second-order difference boundary value problems, *Comput. Math. Appl.*, 57 (2009), 1-8.
8. K.Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, *J. London Math. Soc.*, 63 (2001), 690-704.
9. H. Liang and P. Weng, Existence and multiple solutions for a second-order difference boundary value problem via critical point theory, *J. Math. Anal. Appl.*, 326 (2007), 511-520.
10. R. Ma and H. Ma, Existence of sign-changing periodic solutions of second-order difference equations, *Appl. Math. Comput.*, 203 (2008), 463-470.
11. G. Zhang, and R. Medina, Three-point boundary value problems for difference equations, *Comput. Math Appl.*, 48, (2004) 1791-1799.