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Some Applications to Lebesgue Points in Variable Exponent Lebesgue Spaces

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Özet. Dağılım parametreli sistemler için optimal kontrol teorisinde kullanışlı olan, Lebesgue regüler noktalarının bazı sonuçları ispatlanmıştır.

Anahtar Kelimeler. Değişken üslü Lebesgue uzayı, regüler Lebesgue noktaları, optimal kontrol.

Abstract. Some corollaries of Lebesgue's regular points which are useful in the theory of optimal control for distributed parameter systems are proved.

Keywords. Variable exponent Lebesgue spaces, Lebesgue regular points, optimal control.

1. Introduction

In recent years, stimulated by some problems in fluid dynamics and differential equations, a great interest has arisen in spaces with variable exponents and in the extension of the results of classical Harmonic Analysis to the variable exponent setting. The mathematical applications of these problems have been carried out by the variational integrals with non-standard growth [1, 7].

Let $p: \Omega \to [1, \infty)$ be a measurable bounded function which is called the variable exponent on $\Omega \in \mathbb{R}^n$, and write $p^+ = \sup_{t\in\Omega} p(t)$ and $p^- = \inf_{t\in\Omega} p(t)$. The variable exponent Lebesgue space $L_{p(\cdot)}(\Omega)$ consists of all measurable functions $f : \Omega \to \mathbb{R}$ such that the modular

$$I(f) := \int_{\Omega} |f(t)|^{p(t)} dt$$

is finite. If $p^+ < \infty$, then

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : I\left(\frac{f}{\lambda}\right) \leq 1 \right\}$$

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defines a *Luxemburg norm* on $L_{p(\cdot)}(\Omega)$. This makes $L_{p(\cdot)}(\Omega)$ a Banach space. If p is a constant function, then the variable exponent Lebesgue space $L_{p(\cdot)}(\Omega)$ coincides with the classical Lebesgue space. The basic properties of these spaces can be found in [3, 4, 8].

Proposition 1. [3, 8] *Let p be a measurable function such that $1 < p^- \leq p(t) \leq p^+ < \infty$, and Ω be a measurable set in $\mathbb{R}^n$. Then,*

$$\|f\|_{L_{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|f\|^{p^-}_{L_{p(\cdot)}(\Omega)} \leq I(f) \leq \|f\|^{p^+}_{L_{p(\cdot)}(\Omega)},$$
$$\|f\|_{L_{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|f\|^{p^+}_{L_{p(\cdot)}(\Omega)} \leq I(f) \leq \|f\|^{p^-}_{L_{p(\cdot)}(\Omega)}.$$

Let

$$\Omega \equiv (a,b) \equiv \{t \in \mathbb{R}^n : \ a_i < t_i < b_i, \ i = 1, 2, \dots, n\}$$

be an open parallelepiped in $\mathbb{R}^n$, and

$$Q(\tau, r) \equiv \left\{t \in \mathbb{R}^n : \ \tau_i - \frac{r}{2} \leq t_i \leq \tau_i + \frac{r}{2}, \ i = 1, 2, \dots, n\right\}$$

be a closed cube with its centre at $\tau \in \Omega$ and its edges parallel to coordinate axes with their length equal to $r > 0$. The classical Lebesgue theorem on regular points accepts various generalizations and may be used in many different applications including in the calculation of variations of functionals of distributed problems in the theory of optimization. The non-trivial Lebesgue Theorem 1 on regular points in variable exponent Lebesgue spaces (see [6]) and its consequences will be used here to calculate the functional's variations. Moreover, it may be applied in various problems of optimal control theory.

2. The Main Results

Theorem 1. (i) *Assume that $\Omega \in \mathbb{R}^n$ is a bounded measurable set, $f(s,y) : \Omega \times \mathbb{R}^l \to \mathbb{R}$ is a Carathéodory function (i.e the map $y \to f(s,y)$ is continuous for a.e. $s \in \Omega$, while $s \to f(s,y)$ is measurable on Ω for all $y \in \mathbb{R}^l$), and the condition $|f(s,y)| \leq a(s) + C|y|$ holds, where $s \in \Omega$, $\mathcal{C} = const. > 0$, and $a(\cdot) \in L_{p(\cdot)}(\Omega)$, i.e. $\|a(\cdot)\|_{L_{p(\cdot)}(\Omega)} \leq A$;*

(ii) *$y_0(s)$ and $\{y_h(s) : h \equiv \{\tau, r\} \in \Omega \times [0, \infty)\} \in L^l_{p(\cdot)}(\Omega)$, such that*

$$\int_\Omega |y_h(\cdot)|^{p(\cdot)} dt \leq B, \ \|y_h - y_0\|_{L^1_{p(\cdot)}(\Omega)} \to 0,$$

as $r \to 0$ uniformly for $\tau \in \Omega$;

(iii) $1 < p^- \le p(t) \le p^+ < \infty$ *and* $f \in L_{p(\cdot)}(\Omega)$.

Then, for almost everywhere,

$$\lim_{r\to 0} \frac{1}{r^n} \int\limits_{\Omega_r} |f(t,y_h) - f(\tau, y_0)|^{p(t)} dt = 0, \ \tau \in \Omega,$$

uniformly in y_0, y_h, where $\Omega_r \equiv \Omega_r(\tau) = \{t \in \Omega : t \in [\tau, \tau + r]\}$, i.e. τ is known as a Lebesgue point.

Proof. Let $\{r_i\}_{i=1}^{\infty}$ be countable, and dense everywhere in $[0,\infty)$. Since $f \in L_{p(\cdot)}(\Omega)$ and $p^+ < \infty$, then $|f(t,y_h) - r_i|^{p(t)} \in L^1_{\text{loc}}(\Omega_t)$ uniformly. Indeed, with fixed i, we have

$$\begin{aligned}
\int\limits_{\Omega_r} |f(t,y_h) - r_i|^{p(t)} dt &\le 2^{p^+-1} \int\limits_{\Omega_r} |f(t,y_h) - r_i|^{p(t)} dt \\
&\le 2^{p^+-1} \int\limits_{\Omega_r} \left[|r_i|^{p(t)} + 2^{p^+-1} \left(|a(t)|^{p(t)} + C^{p^+} |y_h|^{p(t)} \right) \right] dt \\
&\le 2^{p^+-1} \int\limits_{\Omega_r} \left[|r_i|^{p^+} + 2^{p^+-1} \left(|A|^{p^+} + C^{p^+} |B|^{p^+} \right) \right] dt < \infty.
\end{aligned}$$

By the Lebesgue theorem, for each $i \in \mathbb{N}$, a subset $E_i \subset \Omega_r$, $\left(E = \bigcup\limits_{i=1}^{\infty} E_i \right)$ of Lebesque measure zero is available, and with $\forall t \in \Omega_r \setminus E$, we obtain

$$\lim_{r\to 0} \frac{1}{r^n} \int\limits_{\Omega_r} |f(t,y_h) - r_i|^{p(t)} dt = |f(\tau, y_h) - r_i|^{p(\tau)}.$$

Assume that $0 < \varepsilon < 1$, $0 < \varepsilon_1 < \frac{1}{4}$, $0 < \varepsilon_2 < \frac{1}{4}$, and $\tau \in \Omega_r/E$. Choose r_i such that

$$|f(\cdot, y_h) - r_i|^{p(\cdot)} < \frac{\varepsilon_1}{2^{p^+-1}}$$

and

$$|f(\cdot, y_0) - r_i|^{p(\cdot)} < \frac{\varepsilon_2}{2^{p^+-1}}$$

with all and any $y_0, y_h \in L^l_{p(\cdot)}(\Omega)$. Thus, we get

$$\limsup_{r\to 0}\frac{1}{r^n}\int\limits_{\Omega_r}|f(t,y_h)-f(\tau,y_0)|^{p(t)}dt$$
$$\leq 2^{p^+-1}\left(\limsup_{r\to 0}\left[\frac{1}{r^n}\int\limits_{\Omega_r}|f(t,y_h)-r_i|^{p(t)}dt+\frac{1}{r^n}\int\limits_{\Omega_r}|r_i-f(\tau,y_0)|^{p(t)}dt\right]\right)$$
$$\leq 2^{p^+-1}(|f(\tau,y_h)-r_i|^{p(t)}+|f(\tau,y_0)-r_i|)\leq 2^{p^+-1}\left(\frac{\varepsilon_1}{2^{p^+-1}}+\frac{\varepsilon_2}{2^{p^+-1}}\right)<\varepsilon.$$

The proof of Theorem 1 is complete. □

In the distributed parameters systems and optimal control theories we face the following problem [2, 5]:

Let $\Omega \in \mathbb{R}^n$ be a bounded measurable set and

$$\{f[m,\tau](\cdot)\in L_{p(\cdot)}(\Omega):\ m\in\mathbb{N},\ \tau\in\Omega\}$$

be a family of functions satisfying the following condition:

$$\|f[m,\tau](\cdot)\|_{L_{p(\cdot)}(\Omega)}\to 0,\ \text{as } m\to+\infty,\ \text{uniformly in } \tau\in\Omega.$$

Moreover, assume that

$$F_p[m]\,(\tau)=\frac{1}{r_m^n}\int\limits_{\Omega[m,\tau]}\left|f[m,\tau](t)\right|^{p(t)}dt,$$

where $\Omega[m,\tau]\equiv\{t\in\Omega : t\in[\tau,\tau+r_m]\}$, $r_m\to 0$ as $m\to+\infty$.

Then, the question is for which subsequence $\{m_k\}\subset\mathbb{N}$ and set $\Omega_0\subset\Omega$, $\text{meas}(\Omega_0)=\text{meas}(\Omega)$ of Lebesque measure zero

$$F_p[m_k](\tau)\to 0\,\forall\tau\in\Omega_0 \text{ as } k\to+\infty \tag{1}$$

will be valid.

Let's explain why we refer to a subsequence $\{m_k\}$ rather than a whole sequence $\{m\}$. We consider the simple case, i.e. when subintegral functions do not relate to τ and a family of functions $\{f_m(\cdot)\in L_{p(\cdot)}(\Omega):\ m\in\mathbb{N}\}$ such that $\|f_m(\cdot)\|_{L_{p(\cdot)}(\Omega)}\to 0$ as $m\to+\infty$, and

$$F_p^*[m](\tau)=\frac{1}{r_m^n}\int\limits_{\Omega[m,\tau]}|f_m(t)|^{p(t)}dt.$$

We will show that, in general, convergence $F_p^*[m](\tau) \to 0$ for almost all $\tau \in \Omega$ as $k \to +\infty$ may not be true.

Remark 1. Let $n = 1$, $\Omega = [0, 1]$, and define the sets

$$Q_{k,j} = \left[\frac{j-1}{k}, \frac{j}{k}\right), \quad P_{k,j} = \left[\frac{j-1}{k}, \frac{j}{k} - \gamma_{k,j}\right),$$

where $\gamma_{k,j} = \frac{1}{j2^k}$ for $j = 1, \ldots, k$, $k = 2, 3, \ldots$. Further, let's assume

$$P_k = \bigcup_{j=1}^{\infty} P_{k,j},\ k = 2, 3, \ldots \text{ and } Q_0 = \bigcap_{k=2}^{\infty} P_k.$$

Then it is clear that $\text{meas}(\Omega_0) > 0$. Let $\varphi_{k,j}$ be the characteristic function of sets $Q_{k,j}$, and

$$f_1(t) = \varphi_{2,1}, \quad f_2(t) = \varphi_{2,2}, \quad f_3(t) = \varphi_{3,1} \quad f_4(t) = \varphi_{3,2}, \quad f_5(t) = \varphi_{3,3}, \ldots.$$

Similarly, let constitute the following sequence

$$r_1 = \gamma_{2,1}, \quad r_2 = \gamma_{2,2}, \quad r_3 = \gamma_{3,1}, \quad r_4 = \gamma_{3,2}, \quad r_5 = \gamma_{3,3}, \ldots.$$

It is obvious that $f_m(\cdot) \subset L_{p(\cdot)}[0, 1]$ and $\|f_m(\cdot)\|_{L_{p(\cdot)}[0,1]} \to 0,\ r_m \to 0$ as $m \to +\infty$, while

$$F_p^*[m](\tau) = \frac{1}{r_m^n} \int\limits_{\Omega[m,\tau]} |f_m(t)|^{p(t)} dt \nrightarrow 0, \forall \tau \in \Omega_0 \text{ as } m \to +\infty.$$

Indeed, if we take $\tau \in \Omega_0$, then $\tau \in P_k$ for $k = 2, 3, \ldots$, and there exists $j = j(k)$ such that $\tau \in P_{k,j}$. However, by this construction, if $r_m = \gamma_{k,j}$, then $\Omega[m, \tau] = [\tau, \tau + \gamma_{k,j}] \subset Q_{k,j}$, and thus $f_m(t) = \varphi_{k,j}(t) = 1$ for $t \in \Omega[m, \tau]$, and eventually $F_p^*[m](\tau) = 1$. This means the set $F_p^*[m](\tau)$ contains a subset $F_p^*[m_s](\tau) = 1$. $\square$

We will require the following definitions to obtain the main results.

Definition 1. A system M of measurable sets containing a point $\xi \in \mathbb{R}^n$ will have a compaction to the point ξ, if sets are as small in diameter as desired among system's sets and regular compaction, and a cube $Q(\xi, h) \supset e$ for all and any $e \in M$ such that $h^n \le L\,\text{meas}(e)$, where L is a constant which is invariable with e.

Definition 2. A system $\{\Omega[m, \tau] : m \in \mathbb{N}, \tau \in \Omega\}$ of measurable sets will have a regular compaction on a set Ω, provided

(i) $\tau \in \Omega[m, \tau] \quad \forall m \in \mathbb{N}$,

(ii) $\sup_{\tau\in\Omega} \operatorname{diam} \Omega[m,\tau] \to 0$ with $m \to +\infty$.

Remark 2. It is obvious that the subsystem deriving from the sets system which have regular compaction on Ω with constant $\tau \in \Omega$ will have compaction to the point τ. In addition, if the condition of regularity holds we can speak about a uniform and regular compaction sets system $\Omega[m,\tau] \equiv \{t \in \Omega : t \in [\tau, \tau + r_m]\}, \quad r_m \to 0$. It is obvious that a parallelepiped system will be uniformly and regularly compactable. It is also clear that the said class will extend further.

Theorem 2. *Assume that $f(t, y) : \ \Omega \times \mathbb{R}^l \to \mathbb{R}$ is a Carathéodory function and satisfies the following conditions:*

(i) *Sets system $\{G[m,\tau] : \ m \in \mathbb{N}, \ \tau \in \Omega\}$ is uniformly and regularly compactable on Ω, $\Omega[m,\tau] \equiv G[m,\tau] \cap \Omega$,*
(ii) $1 < p^-$, $p^+ < \infty$, *$\{|y[m,h](\cdot)| : \ m \in \mathbb{N}, \ h \in G\} \subset L^l_{p(\cdot)}(\Omega)$, and* $\exists\{b_m(\cdot) \subset L_{p(\cdot)}(\Omega)\}$ *such that $\|b_m(\cdot)\|_{L_{p(\cdot)}(\Omega)} \to 0$ as $m \to +\infty$, and* $|y[m,h](\cdot)| \le b_m(\cdot)$ $\forall m \in \mathbb{N}$, *$h \in G$ for almost all $\tau \in \Omega$,*
(iii) $\forall y \in L^l_{p(\cdot)}(\Omega)$ *we will get $f(\cdot, y(\cdot)) \in L_{p(\cdot)}(\Omega)$, where*

$$\|f(\cdot, y(\cdot))\|_{L_{p(\cdot)}(\Omega)} \to 0, \quad \|y(\cdot)\|_{L^l_{p(\cdot)}(\Omega)} \to 0.$$

Remark 3. To get the condition (ii), it is sufficient to show the following inequality holds true

$$|y[m,h](t)| \le b_m(t) N \left(\|\chi_{\Omega[m,\tau]} \Psi(t)\|_{L_{p(\cdot)}(\Omega)} \right), \ \forall m \in \mathbb{N}$$

for almost all $\tau \in \Omega$, where $b_m(\cdot) \in L_{p(\cdot)}(\Omega)$ and $\Psi(\cdot) \in L_{p(\cdot)}(\Omega)$ are some constant functions and $N(\cdot) : \ \mathbb{R}_+ \to \mathbb{R}$ is a nondecreasing function such that $N(\beta) \to +0$ with $\beta \to +0$.

Proof. [Remark 3] Let's point out that for all $m \in \mathbb{N}$ the set $\Omega[m,\tau]$ is contained in some closed ball of radius $\operatorname{diam}\Omega[m,\tau]$. Therefore, its measure does not exceed this ball's measure, i.e. $\operatorname{diam}\Omega[m,\tau] \le C(n)[\operatorname{diam}\Omega[m,\tau]]^n$, $C(n) = const. > 0$. □

Proof. [Theorem 2] This can be obtained immediately from the proof of Theorem 3. □

Lemma 1. *Assume $S(\Omega)$ is a space of measurable functions for almost everywhere on Ω, $l \in \mathbb{N}$, $c(\cdot)$, $d(\cdot) \in S^l(\Omega)$-measurable on Ω, l-vector functions, $c(t) \le d(t)$ for almost all $t \in \Omega$ such that*

$c, d \in \mathbb{R}^l : \ c \le d \Longleftrightarrow c_j \le d_j, \ j = 1, 2, \ldots, l, \ [c, d] = [c_1, d_1] \times [c_2, d_2] \times \ldots \times [c_l, d_l],$

and $f(t,y):\ \Omega\times\mathbb{R}^l\to\mathbb{R}$ *is measurable in* $t\in\Omega$ *and continuous in* $y\in\mathbb{R}^l$. *Then the function* $\varphi(t)\equiv\max\limits_{y\in[c(t),d(t)]} f(t,y)$ *will be measurable on* Ω *and there exists* $\theta(\cdot)\in T[c,d]\equiv\{y\in S^l(\Omega):\ y(t)\in[c(t),d(t)]\}$ *such that* $f(t,\theta(\cdot))=\varphi(t)$ *for almost all* $t\in\Omega$.

Proof. The proof can be obtained immediately from [5, Assertion 1.2, p. 326 and Theorem 1.4, p. 327]. ☐

Theorem 3. *Assume* $1<p^-\le p(t)\le p^+<\infty$ *and* $f(t,y):\Omega\times\mathbb{R}^l\to\mathbb{R}$ *is a Carathéodory function and satisfies the following conditions:*

(i) $\forall y\in L^l_{p(\cdot)}(\Omega)$ *we get* $f(\cdot,y(\cdot))\in L_{p(\cdot)}(\Omega)$,
(ii) $b_m(\cdot)\in L_{p(\cdot)}(\Omega),\ m\in\mathbb{N}$.

Then

$$\|b_m(\cdot)\|_{L_{p(\cdot)}(\Omega)}\to 0\Longleftrightarrow\int\limits_\Omega |b_m(t)|^{p(t)}dt\to 0\quad \textit{with}\quad \|y(\cdot)\|_{L^l_{p(\cdot)}(\Omega)}\to 0.$$

Moreover, for sufficiently large $m\in\mathbb{N}$, *we have*

$$\varphi(t)\equiv\max\{f(t,y):\ y\in\mathbb{R}^l,\ |y|\le b_m(t)\},\tag{2}$$

where $\varphi_m(\cdot)\in L_{p(\cdot)}(\Omega)$ *and* $\|\varphi_m(\cdot)\|_{L_{p(\cdot)}(\Omega)}\to 0$ *as* $m\to+\infty$.

Proof. In view of Lemma 1, for each $m\in\mathbb{N}$ the function $\varphi_m(t)$ is measurable on Ω. Let $\varepsilon>0$. By the condition (i), there exists a number $\delta(\varepsilon)>0$ such that

$$\int\limits_\Omega |f(t,y)|^{p(t)}dt\Longleftrightarrow\|f(t,y)\|_{L_{p(\cdot)}(\Omega)}<\varepsilon,\ \exists y\in L^l_{p(\cdot)}(\Omega),$$

$$\int\limits_\Omega |y(t)|^{p(t)}dt\Longleftrightarrow\|y(\cdot)\|_{L^l_{p(\cdot)}(\Omega)}<\delta(\varepsilon).$$

Since $\|b_m(\cdot)\|_{L_{p(\cdot)}(\Omega)}\to 0$ as $m\to+\infty$, we can choose a number $m^*=m^*(\varepsilon)\in\mathbb{N}$ sufficiently large such that

$$l\|b_m(\cdot)\|_{L_{p(\cdot)}(\Omega)}<\delta(\varepsilon)\quad\text{with}\quad m\ge m^*.$$

By Lemma 1, we have

$$\exists\,\theta_m(\cdot)\in S^l(\Omega):\ |\theta_m(t)|\le b_m(t),$$

for almost all $t \in \Omega,\ \varphi_m(t) = f(t, \theta_m(t))$. Thus, if we consider

$$\|\theta_m(\cdot)\|_{L^l_{p(\cdot)}(\Omega)} = \|\,|\theta_m|\,\|_{L_{p(\cdot)}(\Omega)} \le l\|b_m(\cdot)\|_{L_{p(\cdot)}(\Omega)},$$

we will get $\varphi_m(\cdot) \in L_{p(\cdot)}(\Omega),\ \|\varphi_m(\cdot)\|_{L_{p(\cdot)}(\Omega)} \le \varepsilon, \forall m \ge m^*(\varepsilon)$. This implies that (2) is fulfilled. Therefore, the proof of Theorem 3 is complete. □

Theorem 4. *Assume conditions* (i), (ii)*, and* (iii) *of Theorem 2 hold with* $p^+ < \infty$. *Then there exists a subsequence* $m_k \to +\infty$ *as* $k \to +\infty$*, and a set* $\Omega_0 \subset \Omega$ *with* $\text{meas}(\Omega_0) = \text{meas}(\Omega)$, *such that* $\forall \tau \in \Omega_0$ *and* $\forall h \in G$

$$\frac{1}{\text{meas}(G[m_k, \tau])} \int\limits_{\Omega[m_k,\tau]} |f(t, y[m_k, h](t))|^{p(t)} dt \to 0 \textit{ as } k \to +\infty.$$

Proof. We may assume that $f(t, y) \ge 0$ for almost all $t \in \Omega$, $\exists y \in \mathbb{R}^l$. Set

$$\varphi_m \equiv \max_{y \in [-b(t), b(t)]_l} f(t, y).$$

Then by (2), we get

$$f(t, y[m, h](t)) \le \varphi_m(t) \quad \forall h \in G, \text{ for almost all } t \in \Omega. \tag{3}$$

By Theorem 3, $\varphi_m(\cdot) \in L_{p(\cdot)}(\Omega)$ for all sufficiently large $m \in \mathbb{N}$ and

$$\|\varphi_m(\cdot)\|_{L_{p(\cdot)}(\Omega)} \to 0 \text{ as } m \to +\infty.$$

Thanks to the Theorem on the Convergence Regulator, a subsequence $\varepsilon_k \searrow +0$ as $k \to +\infty$ and a function $\omega(\cdot) \in L_{p(\cdot)}(\Omega)$ can be found so that

$$|\varphi_m(t)| \le \varepsilon_k\, \omega(t) \text{ for almost all } t \in \Omega. \tag{4}$$

Let's choose any closed parallelepiped $[\Omega] \supset \Omega$ including all sets $G[m, \tau]$, $\forall m \in \mathbb{N}$, $\forall \tau \in \Omega$ along with a set Ω and extend function $\omega(\cdot)$ to this parallelepiped with zero. Then, obviously, $\omega(\cdot) \in L_{p(\cdot)}(\Omega)$, and directly from (3) and (4) and by contractiveness of system $\{G[m,\tau],\ m \in \mathbb{N},\ \tau \in \Omega\}$ to the point $\tau \in \Omega$, it follows that

$$\frac{1}{\operatorname{meas}(G[m_k,\tau])} \int\limits_{\Omega[m_k,\tau]} |f(t, y[m_k, h](t))|^{p(t)} dt$$

$$= \frac{\varepsilon_k}{\operatorname{meas}(G[m_k,\tau])} \int\limits_{\Omega[m_k,\tau]} |\omega(t)|^{p(t)} dt$$

$$= \frac{\varepsilon_k}{\operatorname{meas}(G[m_k,\tau])} \int\limits_{G[m_k,\tau]} |\omega(t)|^{p(t)} dt \to 0 \cdot |\omega(\tau)|^{p(\tau)} = 0,$$

for almost all $\tau \in \Omega$. Hence, the proof of Theorem 4 is complete. □

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