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On Some Inequalities for Convex and s -Convex Functions

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Abstract

In this paper, we present some integral inequalities for convex and s -convex functions and we give some remarks and propositions.

Keywords: Convex functions, Hermite-Hadamard Inequality, s -convex functions.

1. Introduction

Let f be defined from I which is a subset of $[0, \infty)$ to $\mathbb{R}$ and a differentiable mapping on the interior of the interval I (I°) such that $f' \in L[a, b]$ where a and b belongs to I with a is less than b . If $|f'|$ is bounded and its upper bound is M , then the inequality given below holds:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]. \quad (1.1)$$

This inequality is Ostrowski inequality which is well known in the literature (see [5]). One can check [6] to see some new results related to Ostrowski inequality.

The function f defined from $[a, b]$ interval which is a subset of $\mathbb{R}$ to $\mathbb{R}$, is called to be convex if the inequality given below holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. If $(-f)$ is convex then f is called as concave.

In [3], Hudzik and Maligranda defined a new class of convex functions namely s -convex in the second sense..

This class of functions are defined from R_+ to R and it is said to be s -convex in the second sense providing that the inequality

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

holds for $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ with some fixed $s \in (0, 1]$ and for all $x, y \in [0, \infty)$. K_s^2 is used to symbolise this class of functions.

For $s = 1$ it is easy to see that, classical convexity is gathered from definition of s -convexity defined on $[0, \infty)$.

Alomari has handled the next equality to prove some theorems in [1].

Lemma 1 [1] Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) . For $\lambda \in [0, 1]$, $a + \lambda \frac{b-a}{2} \leq x \leq \frac{a+b}{2}$ and the mapping $K(x, t)$ defined as.

$$K(x, t) = \begin{cases} t - \left(a + \lambda \frac{b-a}{2} \right), & t \in [a, x] \\ t - \frac{a+b}{2}, & t \in (x, a+b-x] \\ t - \left(b - \lambda \frac{b-a}{2} \right), & t \in (a+b-x, b] \end{cases}$$

we have

$$\int_a^b K(x, t) f'(t) dt = (b-a) \left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) \frac{f(x)+f(a+b-x)}{2} \right] - \int_a^b f(t) dt.$$

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Now we will give some theorems including Ostrowski type inequalities and also remarks and propositions by using previous lemma proved by Alomari .

2. Main Results

Theorem 1 Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable and absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$ and $f' \in L[a, b]$ we have

$$\begin{aligned} & \left| (b-a) \left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) \frac{f(x)+f(a+b-x)}{2} \right] - \int_a^b f(t) dt \right| \\ & \leq \left[\frac{\lambda^3(b-a)^3}{24(x-a)} + \frac{(x-a)^2}{3} - \frac{\lambda(b-a)(x-a)}{4} + \frac{(a+b-2x)^2}{8} \right] \\ & \quad \times [|f'(x)| + |f'(a+b-2x)|] \\ & \quad + \left[\frac{(2x-2a-2\lambda(b-a))^3}{24(x-a)} + \frac{(x-a)^2}{3} + \frac{(2a-2x+\lambda(b-a))(x-a)}{4} \right] \\ & \quad \times [|f'(a)| + |f'(b)|] \end{aligned}$$

for all $\lambda \in [0,1]$ and $a + \lambda \frac{b-a}{2} \leq x \leq \frac{a+b}{2}$.

Proof. From Lemma 1 and using the properties of the modulus, we have

$$\begin{aligned} |M| &= \left| (b-a) \left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) \frac{f(x)+f(a+b-x)}{2} \right] - \int_a^b f(t) dt \right| \\ &\leq \int_a^b |K(x,t)| |f'(t)| dt \\ &= \int_a^x \left| t - \left(a + \lambda \frac{b-a}{2} \right) \right| |f'(t)| dt + \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| |f'(t)| dt \\ &\quad + \int_{a+b-x}^b \left| t - \left(b - \lambda \frac{b-a}{2} \right) \right| |f'(t)| dt. \end{aligned}$$

Since $|f'|$ is convex on $[a, b] = [a, x] \cup (x, a+b-x] \cup (a+b-x, b]$, we have

$$|f'(t)| \leq \frac{t-a}{x-a} |f'(x)| + \frac{x-t}{x-a} |f'(a)|, \quad t \in [a, x];$$

$$|f'(t)| \leq \frac{t-x}{a+b-2x} |f'(a+b-x)| + \frac{a+b-x-t}{a+b-2x} |f'(x)|, \quad t \in (x, a+b-x];$$

and

$$|f'(t)| \leq \frac{t-a-b+x}{x-a} |f'(b)| + \frac{b-t}{x-a} |f'(a+b-x)|, \quad t \in (a+b-x, b];$$

which follows that

$$\begin{aligned}
|M| &\leq \int_a^x \left| t - \left(a + \lambda \frac{b-a}{2} \right) \right| \left[\frac{t-a}{x-a} |f'(x)| + \frac{x-t}{x-a} |f'(a)| \right] dt \\
&+ \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| \left[\frac{t-x}{a+b-2x} |f'(a+b-x)| + \frac{a+b-x-t}{a+b-2x} |f'(x)| \right] dt \\
&+ \int_{a+b-x}^b \left| t - \left(b - \lambda \frac{b-a}{2} \right) \right| \left[\frac{t-a-b+x}{x-a} |f'(b)| + \frac{b-t}{x-a} |f'(a+b-x)| \right] dt \\
&= \int_a^{a+\lambda \frac{b-a}{2}} \left(a + \lambda \frac{b-a}{2} - t \right) \left[\frac{t-a}{x-a} |f'(x)| + \frac{x-t}{x-a} |f'(a)| \right] dt \\
&+ \int_{a+\lambda \frac{b-a}{2}}^x \left(t - a - \lambda \frac{b-a}{2} \right) \left[\frac{t-a}{x-a} |f'(x)| + \frac{x-t}{x-a} |f'(a)| \right] dt \\
&+ \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) \left[\frac{t-x}{a+b-2x} |f'(a+b-x)| + \frac{a+b-x-t}{a+b-2x} |f'(x)| \right] dt \\
&+ \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2} \right) \left[\frac{t-x}{a+b-2x} |f'(a+b-x)| + \frac{a+b-x-t}{a+b-2x} |f'(x)| \right] dt \\
&+ \int_{a+b-x}^{b-\lambda \frac{b-a}{2}} \left(b - \lambda \frac{b-a}{2} - t \right) \left[\frac{t-a-b+x}{x-a} |f'(b)| + \frac{b-t}{x-a} |f'(a+b-x)| \right] dt \\
&+ \int_{b-\lambda \frac{b-a}{2}}^b \left(t - b + \lambda \frac{b-a}{2} \right) \left[\frac{t-a-b+x}{x-a} |f'(b)| + \frac{b-t}{x-a} |f'(a+b-x)| \right] dt \\
&= \frac{1}{24} \left[3(a+b-2x)^2 + 8(a-x)^2 - 6(a-b)(a-x)\lambda + \frac{(a-b)^3 \lambda^3}{a-x} \right] \\
&\times (|f'(x)| + |f'(a+b-x)|) \\
&+ \frac{1}{24} \left[4(a-x)^2 - 6(a-b)(a-x)\lambda + 6(a-b)^2 \lambda^2 - \frac{(a-b)^3 \lambda^3}{a-x} \right] \\
&\times (|f'(a)| + |f'(b)|)
\end{aligned}$$

where we use the facts

$$\begin{aligned}
\int_a^{a+\lambda \frac{b-a}{2}} \left(a + \lambda \frac{b-a}{2} - t \right) (t-a) dt &= \int_{b-\lambda \frac{b-a}{2}}^b \left(t-b + \lambda \frac{b-a}{2} \right) (b-t) dt \\
&= \frac{1}{48} \lambda^3 (b-a)^3,
\end{aligned}$$

$$\begin{aligned}
\int_{a+\lambda \frac{b-a}{2}}^x \left(t - a - \lambda \frac{b-a}{2} \right) (x-t) dt &= \int_{a+b-x}^{b-\lambda \frac{b-a}{2}} \left(b - \lambda \frac{b-a}{2} - t \right) (t-a-b+x) dt \\
&= \frac{1}{48} (2x-2a+a\lambda-b\lambda)^3,
\end{aligned}$$

$$\begin{aligned}
\int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) (t-x) dt &= \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2} \right) (a+b-x-t) dt \\
&= \frac{1}{48} (a+b-2x)^3,
\end{aligned}$$

$$\begin{aligned}
\int_{\frac{a+b}{2}}^{a+b-x} \left(\frac{a+b}{2} - t \right) (a+b-x-t) dt &= \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2} \right) (t-x) dt \\
&= \frac{5}{48} (a+b-2x)^3,
\end{aligned}$$

$$\int_{a+b-x}^{b-\lambda \frac{b-a}{2}} \left(b - \lambda \frac{b-a}{2} - t \right) (b-t) dt = \int_{a+\lambda \frac{b-a}{2}}^x \left(t - a - \lambda \frac{b-a}{2} \right) (t-a) dt$$

$$= -\frac{1}{48}(2x - 2a + a\lambda - b\lambda)^2(4a - 4x + \lambda a - \lambda b)$$

and

$$\begin{aligned} \int_a^{a+\lambda\frac{b-a}{2}} \left(a + \lambda\frac{b-a}{2} - t\right)(x-t)dt &= \int_{b-\lambda\frac{b-a}{2}}^b \left(t - b + \lambda\frac{b-a}{2}\right)(t - a - b + x)dt \\ &= \frac{1}{48}\lambda^2(b-a)^2(6x - 6a + \lambda a - \lambda b). \end{aligned}$$

So, the proof is completed.

Remark 1 In Theorem 1, if we choose just $\lambda = 0$ and $\lambda = 0$ with $x = \frac{a+b}{2}$ we have Theorem 5 and Corollary 2 – (2) respectively which were proved in [2].

Theorem 2 Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable and absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. If $|f'|$ is s -convex on $[a, b]$ and $f' \in L[a, b]$ we have

$$\begin{aligned} &\left| (b-a) \left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) \frac{f(x)+f(a+b-x)}{2} \right] - \int_a^b f(t)dt \right| \\ &\leq \left[\frac{2}{(x-a)^s(s+1)(s+2)} \left(\lambda \frac{b-a}{2} \right)^{s+2} + \frac{(x-a)^2}{s+2} - \left(\lambda \frac{b-a}{2} \right) \frac{(x-a)}{s+1} \right. \\ &\quad \left. + \frac{s2^{s+1}}{2^{s+1}(s+1)(s+2)} (a+b-2x)^2 \right] [|f'(x)| + |f'(a+b-2x)|] \\ &\quad + \left[\frac{2}{(x-a)^s(s+1)(s+2)} \left(x-a-\lambda \frac{b-a}{2} \right)^{s+2} + \frac{(x-a)^2}{s+2} \right. \\ &\quad \left. + \left(a + \lambda \frac{b-a}{2} - x \right) \frac{(x-a)}{s+1} \right] [|f'(a)| + |f'(b)|] \end{aligned}$$

for all $\lambda \in [0,1]$ and $a + \lambda \frac{b-a}{2} \leq x \leq \frac{a+b}{2}$.

Proof. From Lemma 1 and using the properties of the modulus, we have

$$\begin{aligned} |M| &= \left| (b-a) \left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) \frac{f(x)+f(a+b-x)}{2} \right] - \int_a^b f(t)dt \right| \\ &\leq \int_a^b |K(x,t)| |f'(t)| dt \\ &= \int_a^x \left| t - \left(a + \lambda \frac{b-a}{2} \right) \right| |f'(t)| dt + \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| |f'(t)| dt \\ &\quad + \int_{a+b-x}^b \left| t - \left(b - \lambda \frac{b-a}{2} \right) \right| |f'(t)| dt. \end{aligned}$$

Since $|f'|$ is s -convex on $[a, b] = [a, x] \cup (x, a+b-x) \cup (a+b-x, b]$, therefore we have

$$|f'(t)| \leq \left(\frac{t-a}{x-a}\right)^s |f'(x)| + \left(\frac{x-t}{x-a}\right)^s |f'(a)|, \quad t \in [a, x];$$

$$|f'(t)| \leq \left(\frac{t-x}{a+b-2x}\right)^s |f'(a+b-x)| + \left(\frac{a+b-x-t}{a+b-2x}\right)^s |f'(x)|, \quad t \in (x, a+b-x]$$

and

$$|f'(t)| \leq \left(\frac{t-a-b+x}{x-a}\right)^s |f'(b)| + \left(\frac{b-t}{x-a}\right)^s |f'(a+b-x)|, \quad t \in (a+b-x, b]$$

which follows that

$$\begin{aligned} |M| &\leq \int_a^x \left| t - \left(a + \lambda \frac{b-a}{2} \right) \right| \left[\left(\frac{t-a}{x-a} \right)^s |f'(x)| + \left(\frac{x-t}{x-a} \right)^s |f'(a)| \right] dt \\ &+ \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| \left[\left(\frac{t-x}{a+b-2x} \right)^s |f'(a+b-x)| + \left(\frac{a+b-x-t}{a+b-2x} \right)^s |f'(x)| \right] dt \\ &+ \int_{a+b-x}^b \left| t - \left(b - \lambda \frac{b-a}{2} \right) \right| \left[\left(\frac{t-a-b+x}{x-a} \right)^s |f'(b)| + \left(\frac{b-t}{x-a} \right)^s |f'(a+b-x)| \right] dt \\ &= \int_a^{a+\lambda \frac{b-a}{2}} \left(a + \lambda \frac{b-a}{2} - t \right) \left[\left(\frac{t-a}{x-a} \right)^s |f'(x)| + \left(\frac{x-t}{x-a} \right)^s |f'(a)| \right] dt \\ &+ \int_{a+\lambda \frac{b-a}{2}}^x \left(t - a - \lambda \frac{b-a}{2} \right) \left[\left(\frac{t-a}{x-a} \right)^s |f'(x)| + \left(\frac{x-t}{x-a} \right)^s |f'(a)| \right] dt \\ &+ \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) \left[\left(\frac{t-x}{a+b-2x} \right)^s |f'(a+b-x)| + \left(\frac{a+b-x-t}{a+b-2x} \right)^s |f'(x)| \right] dt \\ &+ \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2} \right) \left[\left(\frac{t-x}{a+b-2x} \right)^s |f'(a+b-x)| + \left(\frac{a+b-x-t}{a+b-2x} \right)^s |f'(x)| \right] dt \\ &+ \int_{a+b-x}^{b-\lambda \frac{b-a}{2}} \left(b - \lambda \frac{b-a}{2} - t \right) \left[\left(\frac{t-a-b+x}{x-a} \right)^s |f'(b)| + \left(\frac{b-t}{x-a} \right)^s |f'(a+b-x)| \right] dt \\ &+ \int_{b-\lambda \frac{b-a}{2}}^b \left(t - b + \lambda \frac{b-a}{2} \right) \left[\left(\frac{t-a-b+x}{x-a} \right)^s |f'(b)| + \left(\frac{b-t}{x-a} \right)^s |f'(a+b-x)| \right] dt \\ &= \left[\frac{2}{(x-a)^s(s+1)(s+2)} \left(\lambda \frac{b-a}{2} \right)^{s+2} + \frac{(x-a)^2}{s+2} - \left(\lambda \frac{b-a}{2} \right) \frac{(x-a)}{s+1} \right. \\ &+ \left. \frac{s2^{s+1}+1}{2^{s+1}(s+1)(s+2)} (a+b-2x)^2 \right] \times [|f'(x)| + |f'(a+b-2x)|] \\ &+ \left[\frac{2}{(x-a)^s(s+1)(s+2)} \left(x - a - \lambda \frac{b-a}{2} \right)^{s+2} + \frac{(x-a)^2}{s+2} \right. \\ &\times \left. + \left(a + \lambda \frac{b-a}{2} - x \right) \frac{(x-a)}{s+1} \right] \times [|f'(a)| + |f'(b)|] \end{aligned}$$

where we use the facts

$$\begin{aligned} \int_a^{a+\lambda \frac{b-a}{2}} \left(a + \lambda \frac{b-a}{2} - t \right) \left(\frac{t-a}{x-a} \right)^s dt &= \int_{b-\lambda \frac{b-a}{2}}^b \left(t - b + \lambda \frac{b-a}{2} \right) \left(\frac{b-t}{x-a} \right)^s dt \\ &= \frac{1}{(x-a)^s(s+1)(s+2)} \left(\lambda \frac{b-a}{2} \right)^{s+2}, \end{aligned}$$

$$\begin{aligned} \int_{a+\lambda \frac{b-a}{2}}^x \left(t - a - \lambda \frac{b-a}{2} \right) \left(\frac{x-t}{x-a} \right)^s dt &= \int_{a+b-x}^{b-\lambda \frac{b-a}{2}} \left(b - \lambda \frac{b-a}{2} - t \right) \left(\frac{t-a-b+x}{x-a} \right)^s dt \\ &= \frac{1}{(x-a)^s(s+1)(s+2)} \left(x - a - \lambda \frac{b-a}{2} \right)^{s+2}, \end{aligned}$$

$$\begin{aligned} \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) \left(\frac{t-x}{a+b-2x} \right)^s dt &= \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2} \right) \left(\frac{a+b-x-t}{a+b-2x} \right)^s dt \\ &= \frac{1}{2^{s+2}(s+1)(s+2)} (a+b-2x)^2, \end{aligned}$$

$$\begin{aligned} \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) \left(\frac{a+b-x-t}{a+b-2x} \right)^s dt &= \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2} \right) \left(\frac{t-x}{a+b-2x} \right)^s dt \\ &= \frac{s2^{s+1}+1}{2^{s+2}(s+1)(s+2)} (a+b-2x)^2, \end{aligned}$$

$$\begin{aligned} \int_{a+b-x}^{b-\lambda\frac{b-a}{2}} \left(b - \lambda\frac{b-a}{2} - t \right) \left(\frac{b-t}{x-a} \right)^s dt \\ &= \int_{a+\lambda\frac{b-a}{2}}^x \left(t - a - \lambda\frac{b-a}{2} \right) \left(\frac{t-a}{x-a} \right)^s dt \\ &= \frac{1}{(x-a)^s} \left[\frac{(x-a)^{s+2}}{s+2} - \lambda\frac{b-a}{2} \frac{(x-a)^{s+1}}{s+1} + \frac{1}{(s+1)(s+2)} \left(\lambda\frac{b-a}{2} \right)^{s+2} \right] \end{aligned}$$

and

$$\begin{aligned} \int_a^{a+\lambda\frac{b-a}{2}} \left(a + \lambda\frac{b-a}{2} - t \right) \left(\frac{x-t}{x-a} \right)^s dt \\ &= \int_{b-\lambda\frac{b-a}{2}}^b \left(t - b + \lambda\frac{b-a}{2} \right) \left(\frac{t-a-b+x}{x-a} \right)^s dt \\ &= \frac{1}{(x-a)^s} \left[\frac{(x-a)^{s+2}}{s+2} + \left(a + \lambda\frac{b-a}{2} - x \right) \frac{(x-a)^{s+1}}{s+1} + \frac{1}{(s+1)(s+2)} \left(x - a - \lambda\frac{b-a}{2} \right)^{s+2} \right]. \end{aligned}$$

Remark 2 In Theorem 2, if we choose $s = 1$, Theorem 2 reduces to Theorem 1.

Remark 3 In Theorem 2, if we choose $\lambda = 0$, we obtain Theorem 4.10.1 which is proved in [4].

3. Applications to Some Special Means

We now consider the applications of our theorems to the following special means

a) The arithmetic mean:

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0,$$

b) The harmonic mean:

$$H = H(a, b) := \frac{2ab}{a+b}, \quad a, b \geq 0,$$

c) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b \geq 0$$

d) The p -logarithmic mean:

$$L_p = L_p(a, b) := \begin{cases} a & \text{if } a = b \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}; \quad a, b \geq 0$$

We now derive some sophisticated bounds of the above means.

Proposition 1 For all $a, b, x > 0$, $\lambda \in [0, 1]$ and $a + \lambda \frac{b-a}{2} \leq x \leq \frac{a+b}{2}$, we have

$$\begin{aligned} & |(b-a)[\lambda A(a^n, b^n) + (1-\lambda)A(x^n, (a+b-x)^n)] - (b-a)L_n^n(a, b)| \\ & \leq K \times A(x^{n-1}, (a+b-x)^{n-1}) + L \times A(a^{n-1}, b^{n-1}) \end{aligned}$$

where

$$\begin{aligned} K &= \frac{n}{12} \left[3(a+b-2x)^2 + 8(a-x)^2 - 6(a-b)(a-x)\lambda + \frac{(a-b)^3 \lambda^3}{a-x} \right] \\ L &= \frac{n}{12} \left[4(a-x)^2 - 6(a-b)(a-x)\lambda + 6(a-b)^2 \lambda^2 - \frac{(a-b)^3 \lambda^3}{a-x} \right]. \end{aligned}$$

Proof. By applying $f(x) = x^n$ for $n \geq 2$ to Theorem 1, we get the result.

Proposition 2 For all $a, b, x > 0$, $\lambda \in [0, 1]$ and $a + \lambda \frac{b-a}{2} \leq x \leq \frac{a+b}{2}$, we have

$$\begin{aligned} & |(b-a)[\lambda H^{-1}(a, b) + (1-\lambda)H^{-1}(x, a+b-x)] - (b-a)L^{-1}(a, b)| \\ & \leq M \times H^{-1}(x^2, (a+b-x)^2) + N \times H^{-1}(a^2, b^2) \end{aligned}$$

where

$$\begin{aligned} M &= \frac{1}{12} \left[3(a+b-2x)^2 + 8(a-x)^2 - 6(a-b)(a-x)\lambda + \frac{(a-b)^3 \lambda^3}{a-x} \right] \\ N &= \frac{1}{12} \left[4(a-x)^2 - 6(a-b)(a-x)\lambda + 6(a-b)^2 \lambda^2 - \frac{(a-b)^3 \lambda^3}{a-x} \right]. \end{aligned}$$

Proof. By applying $f(x) = \frac{1}{x}$ to Theorem 1, we get the result.

Proposition 3 For all $a, b > 0$, $x, \lambda \in [0, 1]$, $s \in (0, 1)$ and $a + \lambda \frac{b-a}{2} \leq x \leq \frac{a+b}{2}$. we have

$$|(b-a)[\lambda A(a^s, b^s) + (1-\lambda)A(x^s, (a+b-x)^s)] - (b-a)L_s^s(a, b)| \\ \leq P \times A(x^{s-1}, (a+b-2x)^{s-1}) + R \times A(a^{s-1}, b^{s-1})$$

where

$$P = \frac{2}{(x-a)^s(s+1)(s+2)} \left(\lambda \frac{b-a}{2} \right)^{s+2} + \frac{(x-a)^2}{s+2} - \left(\lambda \frac{b-a}{2} \right) \frac{(x-a)}{s+1} \\ + \frac{s2^s+1}{2^{s+1}(s+1)(s+2)} (a+b-2x)^2 \\ R = \frac{2}{(x-a)^s(s+1)(s+2)} \left(x-a-\lambda \frac{b-a}{2} \right)^{s+2} + \frac{(x-a)^2}{s+2} + \left(a+\lambda \frac{b-a}{2}-x \right) \frac{(x-a)}{s+1}$$

Proof. The assertion follows from Theorem 2 applied for $f(x) = x^s$.

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