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APPROXIMATION PROPERTIES OF MODIFIED SZASZ-SCHURER BASKAKOV TYPE OPERATORS

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ABSTRACT

In the present paper, we study some approximation properties of modified Szász-Schurer-Baskakov type operators. We estimate the moments for these operators using the Hypergeometric series, which are related to Laguerre polynomials. We give approximation properties of derivatives of these operators. Finally, we obtain the Voronovskaya type theorem for derivatives of these operators.

Keywords: Modified Szász-Schurer-Baskakov operator; Szász operator; point-wise convergence; approximation of derivatives; Modulus of continuity; Voronovskaya type theorem.

2000 Mathematics Subject Classification. 41A10, 41A36, 41A25

1. INTRODUCTION

In 1962, Schurer defined the Bernstein-Schurer operators for any $n \in \mathbb{N}$, $f \in C[0, 1 + p]$ and non-negative integers p using well-known Bernstein operators [7]. He defined Bernstein-Schurer operators for $B_{n,p}: C[0, 1 + p] \rightarrow C[0, 1]$ as follows

$$B_{n,p}(f)(x) = \sum_{k=0}^{n+p} f\left(\frac{k}{n}\right) \binom{n+p}{k} x^k (1-x)^{n+p-k}, x \in [0, 1]$$

and studied the approximation properties of these operators. It should be noted that the special case of $p = 0$ here gives classical Bernstein operators.

Later in 1965, [6], Schurer generalized the well-known Szász operator and the Baskakov operator for every $p \in \mathbb{N}_0$, $f \in E_2$, $x \in [0, \infty)$, $n \geq 1$, the n -th Schurer-Szász-Mirakjan operator $M_{n,p}: E_2 \rightarrow C[0, \infty)$ and the n -th Baskakov-Schurer operator $A_{n,p}: E_2 \rightarrow C[0, \infty)$ as

$$M_{n,p}(f)(x) := \exp[-(n+p)x] \sum_{k=0}^{\infty} \frac{[(n+p)x]^k}{k!} f\left(\frac{k}{n}\right)$$

and

$$A_{n,p}(f)(x) := (1+x)^{-n-p} \sum_{k=0}^{\infty} \binom{n+p+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right)$$

respectively. Where $E_2 := \left\{f \in C[0, \infty): \frac{f(x)}{1+x^2} \text{ is convergent as } x \rightarrow \infty\right\}$

Many studies have been made on various generalizations of the above operators and their convergence properties, and such studies are still ongoing, [1], [2], [10], [11], [13], [16], [17], [20], [22].

In 1967, J. L. Durrmeyer defined one-dimensional Bernstein-Durrmeyer operators as

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$$D_n(f; x) := (n+1) \sum_{k=0}^{\infty} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 \binom{n}{k} u^k (1-u)^{n-k} f(u) du, \quad x \in [0,1], n = 1, 2, \dots$$

and studied the approximation properties of these operators. Where f is any real-valued function on $[0,1]$, which is integrable with respect to the kernel [8].

In 1995 [21], Gupta and Srivastava studied the convergence properties of the derivatives of the Szász-Mirakyan-Baskakov type operators defined as follows;

$$M_n(f; x) := (n-1) \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{(n+k-1)!}{k!(n-1)!} \int_0^{\infty} \frac{t^k}{(1+t)^{n+k}} f(t) dt, \quad x \in [0, \infty). \quad (1.1)$$

Later, many researchers defined Durrmeyer-type extensions and many modifications of other well-known operators and studied their convergence properties, [3], [9], [12], [14], [15].

Also, in 2006, for $f \in C_{\gamma}[0, \infty) \equiv \{f \in C[0, \infty): |f(t)| \leq Mt^{\gamma}, \gamma > 0\}$, Gupta et al. defined the Baskakov-Durrmeyer operators as an integral modification of the Baskakov operators as follows

$$B_n(f; x) := \sum_{k=1}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \frac{1}{B(n+1, k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t) dt + \frac{f(0)}{(1+x)^n}, \quad (1.2)$$

and studied the convergence properties of the derivatives of these operators, [8].

In 2012, Verma et al. defined a Stancu type extension of operator of (1.1) type, gave the approximation properties of these operators, the error estimation and obtained some recurrence relations [4]. In 2012, Gupta et al. examined the point convergence properties of the Stancu extension of the operators (1.2) and gave Voronovskaya type theorems, [19]. In 2014, Agrawal et al. gave a different generalization of the (1.2) operators and examined the properties of uniform convergence and point convergence, [12].

In 2015, Pandey et al. a Szász-Baskakov Stancu type generalization of operators is defined as follows:

$$M_n(f; x) := (n-1) \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{(n+k-1)!}{k!(n-1)!} \int_0^{\infty} \frac{t^k}{(1+t)^{n+k}} f\left(\frac{nt+\alpha}{n+\beta}\right) dt, \quad x \in [0, \infty). \quad (1.3)$$

and examined various approximation properties, [5]. They also benefited from the properties of hypergeometric functions while doing these investigations.

By defining the Schurer generalization of the (1.1) operator, we wanted to achieve similar results.

For $f \in C[0, \infty)$, we define the Schurer type generalization of the Szász-Mirakyan-Baskakov type operator given in (1.1) as follows

$$A_{n,a,\beta_n}(f; x) := (n+a-1) \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^{\infty} q_{n,a,k}(t) f(t) dt; \quad a \in N_0, x \in [0, \infty), \quad (1.4)$$

where β_n is a sequence of positive numbers such that

$$\lim_{n \rightarrow \infty} \beta_n = 0 \quad (1.5)$$

and

$$p_{n,\beta_n,k}(x) = e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!}, \quad q_{n,a,k}(t) = \frac{(n+a+k-1)!}{k!(n+a-1)!} \frac{t^k}{(1+t)^{n+a+k}} \quad (1.6)$$

It is clear that for $x \in [0, \infty)$ the operators A_{n,a,β_n} are linear and positive.

Now, let's give some important information that we will benefit from when examining the convergence properties of the operator we have defined.

Well known Beta and Gamma functions provides the equality

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$$B(x, y) = \int_0^1 \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!(y-1)!}{(x+y-1)!}. \quad (1.7)$$

We can represent the operator defined by (1.4) in a different way by using the hyper-geometric functions defined as

$${}_1F_1(\alpha; \beta; t) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{t^k}{k!} \quad (1.8)$$

Where $(\alpha)_k$ is called Pochhammer symbol defined as

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2) \dots (\alpha+k-1) \quad (1.9)$$

and $(1)_k = k!$. Using hyper-geometric functions, we can give an equivalent definition of the operators $A_{n,a,\beta_n}(f; x)$ as follows;

$$A_{n,a,\beta_n}(f; x) = (n+a-1) \int_0^1 \frac{e^{-(n+\beta_n)x}}{(1+t)^{n+a}} f(t) {}_1F_1\left(n+a; 1; \frac{(n+\beta_n)xt}{1+t}\right) dt.$$

Our aim in this study is to examine the convergence properties of the operator (1.4), which we define as a generalization of the (1.1) operator, and its derivatives, taking into account the study of Pandey et al., and evaluate the results. Let us give some recurrence relations and lemmas that we need for this.

2. SOME MOMENTS AND RECURRENCE RELATIONS

In this section, we give some lemmas that we will use in this study.

Lemma 1. For all $x \in [0, \infty)$, the operators A_{n,a,β_n} defined by (1.4) satisfy the followings:

$$A_{n,a,\beta_n}(1; x) = 1 \quad (2.1)$$

$$A_{n,a,\beta_n}(t; x) = \frac{(n+\beta_n)x+1}{(n+a-2)} \quad (2.2)$$

$$A_{n,a,\beta_n}(t^2; x) = \frac{[(n+\beta_n)x]^2 + 4[(n+\beta_n)x] + 2}{(n+a-2)(n+a-3)} \quad (2.3)$$

The proof can be easily done from the definition of the operator A_{n,a,β_n} .

Lemma 2. Let $r \in \mathbb{N}_0$. If the r -th order moment is defined as

$$u_{n+\beta_n,r}(x) = \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \left(\frac{k}{n+\beta_n} - x\right)^r = \sum_{k=0}^{\infty} e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!} \left(\frac{k}{n+\beta_n} - x\right)^r \quad (2.4)$$

then there exists a recurrence relation

$$(n+\beta_n)u_{n+\beta_n,r+1}(x) = x[u'_{n+\beta_n,r}(x) + ru_{n+\beta_n,r-1}(x)]. \quad (2.5)$$

Consequently

1. $u_{n+\beta_n,r}(x)$ is a polynomial in x of degree $\leq r$.

2. $u_{n+\beta_n,r}(x) = O\left(\frac{1}{n^{\left[\frac{r+1}{2}\right]}}\right)$; as $n \rightarrow \infty$.

The proof can be done using the definition.

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Lemma 3. If we define the central moments as

$$T_{n,a,\beta_n,r}(x) = A_{n,a,\beta_n}((t-x)^2; x) = (n+a-1) \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^{\infty} q_{n,a,k}(t)(t-x)^r dt, \quad (2.6)$$

then

$$\begin{aligned} T_{n,a,\beta_n,0}(x) &= 1 \\ T_{n,a,\beta_n,1}(x) &= \frac{(\beta_n - a + 2)x + 1}{(n + a - 2)} \\ T_{n,a,\beta_n,2}(x) &= \frac{[(n - a - 3) + (\beta_n - a + 3)^2]x^2 + (2n + 4\beta_n - 2a + 6)x + 2}{(n + a - 2)(n + a - 3)} \end{aligned}$$

and for $n > 1$ we have the following recurrence relation;

$$(n + a - r - 2)T_{n,a,\beta_n,r+1}(x) = xT'_{n,a,\beta_n,r}(x) + [(2r + a - \beta_n + 2)x + r + 1]T_{n,a,\beta_n,r}(x) + (x^2 + 2x)rT_{n,a,\beta_n,r-1}(x). \quad (2.7)$$

From the recurrence relation, it can be easily verified that for all $x \in [0, \infty)$, we have $T_{n,a,\beta_n,r}(x) = O\left(\frac{1}{n^{\lfloor \frac{r+1}{2} \rfloor}}\right)$.

Proof. Since the operator A_{n,a,β_n} is linear, we get $T_{n,a,\beta_n,0}(x) = 1$, $T_{n,a,\beta_n,1}(x) = \frac{(\beta_n - a + 2)x + 1}{(n + a - 2)}$,

$$T_{n,a,\beta_n,2}(x) = \frac{[(n - a - 3) + (\beta_n - a + 3)^2]x^2 + (2n + 4\beta_n - 2a + 6)x + 2}{(n + a - 2)(n + a - 3)}. \text{ From (2.6), we can write}$$

$$T'_{n,a,\beta_n,r}(x) = \frac{k}{x}T_{n,a,\beta_n,r}(x) - (n + \beta_n)T_{n,a,\beta_n,r}(x) - rT_{n,a,\beta_n,r-1}(x)$$

$\Rightarrow$

$$xT'_{n,a,\beta_n,r}(x) = [k - (n + \beta_n)]T_{n,a,\beta_n,r}(x) - xrT_{n,a,\beta_n,r-1}(x).$$

On the other hand, using $t(1+t)q'_{n,a,k}(t) = [k - (n + a)t]q_{n,a,k}(t)$ and $xp'_{n,\beta_n,k}(x) = [k - (n + \beta_n)x]p_{n,\beta_n,k}(x)$, we obtain

$$\begin{aligned} xT'_{n,a,\beta_n,r}(x) + rxT_{n,a,\beta_n,r-1}(x) &= (n + a - 1) \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^{\infty} q'_{n,a,k}(t)(t-x)^{r+2} dt \\ &\quad + (n + a - 1)(2x + 1) \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^{\infty} q'_{n,a,k}(t)(t-x)^{r+1} dt \\ &\quad + (n + a - 1)(3x^2 + x) \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^{\infty} q'_{n,a,k}(t)(t-x)^r dt \\ &\quad + (n + a)T_{n,a,\beta_n,r+1}(x) - (\beta_n - a)xT_{n,a,\beta_n,r}(x). \end{aligned}$$

If partial integration is applied to the integrals here, we get

$$\begin{aligned} xT'_{n,a,\beta_n,r}(x) + rxT_{n,a,\beta_n,r-1}(x) \\ = (n + a - r - 2)T_{n,a,\beta_n,r+1}(x) - (2xr + 2x + r + 1 - \beta_n x + ax)T_{n,a,\beta_n,r}(x) - (x^2 + x)rT_{n,a,\beta_n,r-1}(x). \end{aligned}$$

Then we obtain (2.7) and

$$T_{n,a,\beta_n,r}(x) = O\left(n^{-\lfloor \frac{r+1}{2} \rfloor}\right).$$

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Lemma 4. For all $r \in N_0$,

$$A_{n,a,\beta_n}(t^r; x) = \frac{r!(n+a-r-2)!}{(n+a-2)!} \sum_{j=0}^r \binom{r}{j} \frac{[(n+\beta_n)x]^j}{j!}$$

is a polynomial in x of degree exactly r .

Proof. By using (1.4), (1.9) and the equations $\Gamma(k+r+1) = \Gamma(r+1)(r+1)_k = r!(r+1)_k$, $k! = (1)_k$, we have

$$\begin{aligned} A_{n,a,\beta_n}(t^r; x) &= (n+a-1) \sum_{k=0}^{\infty} e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!} \frac{(n+a+k-1)!}{k!(n+a-1)!} \int_0^{\infty} \frac{t^{k+r}}{(1+t)^{n+a+k}} dt \\ &= (n+a-1) \sum_{k=0}^{\infty} e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!} \frac{(n+a+k-1)!}{k!(n+a-1)!} \frac{r!(r+1)_k(n+a-r-2)!}{(n+a+k-1)!} \\ &= e^{-(n+\beta_n)x} \frac{r!(n+a-r-2)!}{(n+a-2)!} \sum_{k=0}^{\infty} \frac{[(n+\beta_n)x]^k}{k!} \frac{(r+1)_k}{k!} \\ &= e^{-(n+\beta_n)x} \frac{r!(n+a-r-2)!}{(n+a-2)!} {}_1F_1(r+a; 1; (n+\beta_n)x) \end{aligned}$$

Using the Kummer transformation

$${}_1F_1(\alpha; \beta; x) = e^x {}_1F_1(\beta - \alpha; \beta; -x)$$

and

$${}_1F_1(1; 1; x) = e^x$$

We have

$$e^{-(n+\beta_n)x} {}_1F_1(r+a; 1; (n+\beta_n)x) = {}_1F_1(-r; 1; -(n+\beta_n)x).$$

Then, we can write

$$A_{n,a,\beta_n}(t^r; x) = \frac{r!(n+a-r-2)!}{(n+a-2)!} {}_1F_1(-r; 1; -(n+\beta_n)x).$$

Since the confluent hypergeometric function is related with generalized Laguerre polynomial with the relation

$$L_n^m(x) = \binom{m+n}{n} {}_1F_1(-n; m+1; x) = \frac{(m+n)!}{m!n!} {}_1F_1(-n; m+1; x),$$

and

$$L_r^0(x) = L_r(x) = {}_1F_1(-r; 1; x) \Rightarrow L_r(-(n+\beta_n)x) = {}_1F_1(-r; 1; -(n+\beta_n)x)$$

we obtain

$$A_{n,a,\beta_n}(t^r; x) = \frac{r!(n+a-r-2)!}{(n+a-2)!} L_r(-(n+\beta_n)x),$$

where

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$$L_r(-(n + \beta_n)x) = \sum_{j=0}^r (-1)^j \binom{r}{r-j} \frac{[-(n+\beta_n)x]^j}{j!} = \sum_{j=0}^r \binom{r}{j} \frac{[(n+\beta_n)x]^j}{j!}.$$

Then we have

$$A_{n,a,\beta_n}(t^r; x) = \frac{r!(n+a-r-2)!}{(n+a-2)!} \sum_{j=0}^r \binom{r}{j} \frac{[(n+\beta_n)x]^j}{j!} = \frac{(n+\beta_n)^r (n+a-r-2)!}{(n+a-2)!} x^r + \frac{r^2 (n+\beta_n)^{r-1} (n+a-r-2)!}{(n+a-2)!} x^{r-1} + \frac{r(r-1)(n+\beta_n)^{r-2} (n+a-r-2)!}{(n+a-2)!} x^{r-2} + O(n^{-2}).$$

So the proof is completed.

Lemma 5. [21] There exist a polynomial $\phi_{i,j,r}(x)$ independent of n and k such that

$$x^r \frac{d^r}{dx^r} [e^{-(n+\beta_n)x} [(n + \beta_n)x]^k] = \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n + \beta_n)^i [k - (n + \beta_n)x]^j \phi_{i,j,r}(x) e^{-(n+\beta_n)x} [(n + \beta_n)x]^k.$$

The proof can be easily done using the induction method.

Lemma 6. Let f be r -times differentiable on $[0, \infty)$ such that $f^{(r-1)} = O(t^\alpha)$, for some $\alpha > 0$ as $t \rightarrow \infty$. Then for $r = 1, 2, \dots$ and $n > \alpha + r$, we have

$$A_{n,a,\beta_n}^{(r)}(f; x) = \frac{(n + \beta_n)^r (n + a - r - 1)!}{(n + a - 2)!} \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^{\infty} (-1)^r q_{n-r,a,k+r}(t) f^{(r)}(t) dt.$$

Proof. Using Leibnitz theorem, we can write

$$\begin{aligned} A_{n,a,\beta_n}^{(r)}(f; x) &= \\ (n + a - 1) \sum_{i=0}^r \sum_{k=i}^{\infty} \binom{r}{i} \frac{(-1)^{r-i} (n + \beta_n)^r [(n + \beta_n)x]^{k-i}}{(k-i)!} e^{-(n+\beta_n)x} \frac{(n + a + k - 1)!}{k! (n + a - 1)!} \int_0^{\infty} \frac{t^k}{(1+t)^{n+a+k}} f(t) dt \\ &= (n + a - 1) \sum_{k=0}^{\infty} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{r-i} (n + \beta_n)^r [(n + \beta_n)x]^k}{k!} e^{-(n+\beta_n)x} \frac{(n + a + k + i - 1)!}{(k+i)! (n + a - 1)!} \int_0^{\infty} \frac{t^k}{(1+t)^{n+a+k+i}} f(t) dt \\ &= (n + a - 1) \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^{\infty} (-1)^r \sum_{i=0}^r \binom{r}{i} (-1)^i (n + \beta_n)^r (-1)^r q_{n,a,k+i}(t) f(t) dt. \end{aligned}$$

By Leibnitz theorem, we have

$$q_{n-r,a,k+r}^{(r)}(t) = \frac{(n + a - 1)!}{(n + a - r - 1)!} \sum_{i=0}^r \binom{r}{i} (-1)^i q_{n,a,k+i}(t)$$

If this expression is substituted in $A_{n,a,\beta_n}^{(r)}(f; x)$, we get

$$A_{n,a,\beta_n}^{(r)}(f; x) = \frac{(n + \beta_n)^r (n + a - r - 1)!}{(n + a - 2)!} \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^{\infty} (-1)^r q_{n-r,a,k+r}^{(r)}(t) f(t) dt.$$

If partial integration is applied to this expression, we obtain the desired result

$$A_{n,a,\beta_n}^{(r)}(f; x) = \frac{(n+\beta_n)^r (n+a-r-1)!}{(n+a-2)!} \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^{\infty} (-1)^r q_{n-r,a,k+r}(t) f^{(r)}(t) dt. \quad (2.8)$$

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Remark 1. A simple consequence of (2.8) is

$$A_{n,a,\beta_n}(t^r; x) = \frac{r!(n+\beta_n)^r(n+a-r-2)!}{(n+a-2)!} \quad (2.9)$$

Lemma 7. If we define

$$\lambda_r(n) = \frac{\prod_{j=0}^r (n+\beta_n)^j (n+a-j-2)!}{(n+a-2)!} = \frac{(n+\beta_n)^r (n+a-r-2)!}{(n+a-2)!} \quad (2.10)$$

then we have the following recurrence relations

$$[\lambda_{r+1}(n) - \lambda_r(n)]x = \frac{r+1}{n+\beta_n} \lambda_{r+1}(n) = \lambda_r(n) \left[\frac{(\beta_n - a + r + 2)x + (r+1)}{n+a-r-2} \right] \quad (2.11)$$

$$[\lambda_r(n) - 2\lambda_{r+1}(n) - \lambda_{r+2}(n)] = \lambda_r(n) \left[\frac{(r-a)^2 + (r+\beta_n)^2 - r^2 + 5(r-a) + 6(1+\beta_n) - 2a\beta_n + n}{(n+a-r-2)(n+a-r-3)} \right] \quad (2.12)$$

$$\frac{r+2}{n+\beta_n} \lambda_{r+1}(n) - \frac{r+1}{n+\beta_n} \lambda_{r+2}(n) = \lambda_r(n) \left[\frac{r(a-\beta_n) - r^2 - 5r + 2a - 6 + \beta_n + n}{(n+a-r-2)(n+a-r-3)} \right]. \quad (2.13)$$

The proof can be easily done with simple operations.

Let us now give some definitions that we will use.

Definition 1. [19] The m -th order modulus of continuity $\omega_m(f; \delta; [a, b])$ for a function f which is continuous on $[a, b]$ is defined by

$$\omega_m(f; \delta; [a, b]) = \sup\{|\Delta_h^m f(x)|; |h| < \delta; x, x+h \in [a, b]\}. \quad (2.14)$$

For $m = 1$, $\omega_m(f; \delta)$ is usual modulus of continuity.

Definition 2. [5] Let $C_\gamma[0, \infty) = \{f \in [0, \infty); |f(t)| \leq Mt^\gamma, \gamma > 0\}$. The norm $(\|\cdot\|)$ on $C_\gamma[0, \infty)$ is defined by;

$$\|f\|_\gamma = \sup_{0 \leq t < \infty} |f(t)|t^{-\gamma}. \quad (2.15)$$

Definition 3. [19] Let $0 < a < a_1 < b_1 < b < \infty$. For sufficiently small $\eta > 0$, the $2 - nd$ order Steklov mean $f_{\eta,2}$ corresponding to $f \in C_\gamma[a, b]$ and $t \in I_1$ is defined as;

$$f_{\eta,2}(t) = \eta^{-2} \int_{\frac{\eta}{2}}^{\frac{\eta}{2}} \int_{\frac{\eta}{2}}^{\frac{\eta}{2}} [f(t) - \Delta_h^2 f(t)] dt_1 dt_2, \quad (2.16)$$

Where $h = \frac{t_1+t_2}{2}$ and Δ_h^2 is the second order forward difference operator with step length h . For $f \in C[a, b]$, $f_{\eta,2}$ satisfy the following properties:

1. $f_{\eta,2}$, has continuous derivatives up to order 2 over $[a_1, b_1]$.
2. $\|f_{\eta,2}\|_{C[a_1, b_1]} \leq C\omega_r(f; \delta; [a, b]), r = 1, 2$
3. $\|f - f_{\eta,2}\|_{C[a_1, b_1]} \leq C\omega_2(f; \delta; [a, b])$
4. $\|f_{\eta,2}\|_{C[a_1, b_1]} \leq C\eta^{-2}\|f\|_{C[a, b]}$
5. $\|f_{\eta,2}\|_{C[a_1, b_1]} \leq C\|f\|_\gamma$

where C are certain constants which are different in each occurrence and are independent of f and η .

3. DIRECT RESULTS

Theorem 1. (Pointwise convergence) Let $a \in N_0$, $x \in [0, \infty)$ and β_n be a sequence satisfying (1.5). If $r \in N_0$, $f \in C_\gamma[0, \infty)$ for some $\gamma > 0$ and $f^{(r)}$ exists at a point $x \in (0, \infty)$, then

$$\lim_{n \rightarrow \infty} |A_{n,a,\beta_n}^{(r)}(f; x)| = f^{(r)}(x).$$

Proof. By Taylor expansion of f , we have

$$f(t) = \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} (t-x)^i + \varepsilon(t, x)(t-x)^r$$

where $\varepsilon(t, x) \rightarrow 0$ as $t \rightarrow x$. If the operator $A_{n,a,\beta_n}(f; x)$ is applied to this expression, we get

$$A_{n,a,\beta_n}(f; x) = \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} A_{n,a,\beta_n}((t-x)^i; x) + A_{n,a,\beta_n}(\varepsilon(t, x)(t-x)^r; x).$$

By taking the derivative of this expression r times, we obtain

$$\frac{d^r}{dx^r} [A_{n,a,\beta_n}(f; x)] = \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} A_{n,a,\beta_n}^{(r)}((t-x)^i; x) + A_{n,a,\beta_n}^{(r)}(\varepsilon(t, x)(t-x)^r; x) = I_1 + I_2$$

Using Lemma 4, Lemma 5 and (2.9),

$$\begin{aligned} I_1 &= \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} A_{n,a,\beta_n}^{(r)}((t-x)^i; x) \\ &= \sum_{i=0}^{r-1} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^i \binom{i}{j} (-x)^{i-j} \frac{i! (n+\beta_n)^i (n+a-i-2)!}{(n+a-2)!} + \frac{f^{(r)}(x)}{r!} \sum_{j=0}^{r-1} \binom{r}{j} (-x)^{r-j} A_{n,a,\beta_n}^{(r)}(t^j; x) + \frac{f^{(r)}(x)}{r!} A_{n,a,\beta_n}^{(r)}(t^r; x) \\ &= I_1 + I_2 + I_3 \end{aligned}$$

is obtained. Since $I_3 = O((1/n))$, $I_4 = O((1/n))$, $I_5 = \frac{f^{(r)}(x)}{r!} A_{n,a,\beta_n}^{(r)}(t^r; x)$, we have

$$I_1 = \frac{f^{(r)}(x)}{r!} A_{n,a,\beta_n}^{(r)}(t^r; x) = \frac{f^{(r)}(x)}{r!} \frac{r! (n+\beta_n)^r (n+a-r-2)!}{(n+a-2)!} = \frac{(n+\beta_n)^r (n+a-r-2)!}{(n+a-2)!} f^{(r)}(x).$$

thus $f^{(r)}(x)$ for $n \rightarrow \infty$. Now let's set an upper bound for I_2 by using Lemma 6.

$$\begin{aligned} I_2 &= A_{n,a,\beta_n}^{(r)}(\varepsilon(t, x)(t-x)^r; x) \\ &= (n+a-1) \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \frac{(n+\beta_n)^i}{x^r} |\phi_{i,j,r}(x)| \sum_{k=0}^{\infty} [k - (n+\beta_n)x]^j e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!} \\ &\quad \times \frac{(n+a+k-1)!}{k!(n+a-1)!} \int_0^{\infty} \varepsilon(t, x) \frac{t^k (t-x)^r}{(1+t)^{n+a+k}} dt \end{aligned}$$

So we can write I_2 as:

$$\begin{aligned} |I_2| &\leq (n+a-1) \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \frac{(n+\beta_n)^i}{x^r} |\phi_{i,j,r}(x)| \sum_{k=0}^{\infty} |[k - (n+\beta_n)x]^j| e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!} \\ &\quad \times \frac{(n+a+k-1)!}{k!(n+a-1)!} \int_0^{\infty} |\varepsilon(t, x)| \frac{t^k |t-x|^r}{(1+t)^{n+a+k}} dt \end{aligned}$$

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Since $\varepsilon(t, x) \rightarrow 0$ as $t \rightarrow x$, for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\varepsilon(t, x)|$ whenever $|t - x| < \delta$, moreover if $\lambda \geq \max\{\gamma, r\}$ is any integer, then we find a constant $K > 0$ such that $|\varepsilon(t, x)||t - x|^r \leq K|t - x|^\gamma$. Thus we obtain

$$|I_2| \leq (n + a - 1)C_1 \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n + \beta_n)^i \sum_{k=0}^{\infty} |[k - (n + \beta_n)x]^j| e^{-(n+\beta_n)x} \frac{[(n + \beta_n)x]^k}{k!} \\ \times \left\{ \frac{(n+a+k-1)!}{k!(n+a-1)!} \int_{|t-x|<\delta} |\varepsilon(t, x)| \frac{t^k |t-x|^r}{(1+t)^{n+a+k}} dt + \frac{(n+a+k-1)!}{k!(n+a-1)!} \int_{|t-x|\geq\delta} K \frac{t^k |t-x|^r}{(1+t)^{n+a+k}} dt \right\} = I_6 + I_7$$

where

$$C_1 = \sup_{\substack{2i+j \leq r \\ i,j \geq 0}} \frac{|\phi_{i,j,r}(x)|}{x^r} > 0$$

and K is a constant independent of C_1 . If we use the Schwarz inequality first for the integration and then for the summation to calculate I_6 , we have

$$I_6 \leq \varepsilon C_1 \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n + \beta_n)^i \left[\sum_{k=0}^{\infty} [k - (n + \beta_n)x]^{2j} e^{-(n+\beta_n)x} \frac{[(n + \beta_n)x]^k}{k!} \right]^{\frac{1}{2}} \\ \times \left[(n + a - 1) \sum_{k=0}^{\infty} [k - (n + \beta_n)x]^{2j} e^{-(n+\beta_n)x} \frac{[(n + \beta_n)x]^k}{k!} [(n + \beta_n)x]^k \frac{(n+a+k-1)!}{k!(n+a-1)!} \int_0^{\infty} \frac{t^k (t-x)^r}{(1+t)^{n+a+k}} dt \right]^{\frac{1}{2}},$$

as

$$\int_0^{\infty} q_{n,a,k}(t) dt = \frac{1}{(n + a - 1)}$$

By using Lemma 2, we get

$$\sum_{k=0}^{\infty} [k - (n + \beta_n)x]^{2j} e^{-(n+\beta_n)x} \frac{[(n + \beta_n)x]^k}{k!} = (n + \beta_n)^{2j} \sum_{k=0}^{\infty} \left[\frac{k}{(n + \beta_n)} \right]^{2j} e^{-(n+\beta_n)x} \frac{[(n + \beta_n)x]^k}{k!} \\ = (n + \beta_n)^{2j} [O((n + \beta_n)^{-j})] = O((n + \beta_n)^j).$$

Since

$$\left[(n + a - 1) \sum_{k=0}^{\infty} [k - (n + \beta_n)x]^{2j} e^{-(n+\beta_n)x} \frac{[(n + \beta_n)x]^k}{k!} [(n + \beta_n)x]^k \frac{(n+a+k-1)!}{k!(n+a-1)!} \int_0^{\infty} \frac{t^k (t-x)^r}{(1+t)^{n+a+k}} dt \right]^{\frac{1}{2}} = O((n + \beta_n)^{-r})$$

can be written as a result of Lemma 2, we have

$$I_6 \leq \varepsilon C_1 \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n + \beta_n)^i O\left((n + \beta_n)^{\frac{j}{2}}\right) O\left((n + \beta_n)^{-\frac{r}{2}}\right) = \varepsilon O(1).$$

Next, using Lemma 2 and Schwarz inequality for the integration and summation, in view of the above results, we have

$$I_7 \leq (n + a - 1)C_2 \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n + \beta_n)^i \sum_{k=0}^{\infty} |[k - (n + \beta_n)x]^j| e^{-(n+\beta_n)x} \frac{[(n + \beta_n)x]^k}{k!} \\ \times \left[\frac{(n + a + k - 1)!}{k!(n + a - 1)!} \int_{|t-x|\geq\delta} \frac{t^k}{(1+t)^{n+a+k}} dt \right]^{\frac{1}{2}} \left[\frac{(n + a + k - 1)!}{k!(n + a - 1)!} \int_{|t-x|\geq\delta} \frac{t^k (t-x)^{2r}}{(1+t)^{n+a+k}} dt \right]^{\frac{1}{2}} \\ \leq C_2 \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n + \beta_n)^i \left[\sum_{k=0}^{\infty} [k - (n + \beta_n)x]^{2j} e^{-(n+\beta_n)x} \frac{[(n + \beta_n)x]^k}{k!} \right]^{\frac{1}{2}}$$

$$\times \left[\sum_{k=0}^{\infty} e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!} \frac{(n+a+k-1)!}{k!(n+a-1)!} \int_{|t-x| \geq \delta} \frac{t^k(t-x)^{2r}}{(1+t)^{n+a+k}} dt \right]^{\frac{1}{2}}$$

$$= \sum_{\substack{2l+j \leq r \\ l, j \geq 0}} (n+\beta_n)^i O\left((n+\beta_n)^{\frac{j}{2}}\right) O\left((n+\beta_n)^{-\frac{r}{2}}\right) = O(1).$$

Thus the proof is completed by using the results obtained for I_1 and I_2 .

Theorem 2. (Asymptotic expansion) Let $f \in C_\gamma[0, \infty)$ be bounded for every finite sub-interval of $[0, \infty)$ and has the derivative of order $(r+2)$ at a fixed $x \in (0, \infty)$. For some $\gamma > 0$, let $f(t) = O(t^\gamma)$ as $t \rightarrow \infty$. Then we have,

$$\lim_{n \rightarrow \infty} n \left[A_{n,a,\beta_n}^{(r)}(f; x) - f^{(r)}(x) \right] = [(2+r-a)x + r + 1] f^{(r+1)}(x) + \left(x + \frac{x^2}{2}\right) f^{(r+2)}(x).$$

Proof. By Taylor expansion of f , we have

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} (t-x)^i + \varepsilon(t, x)(t-x)^{r+2}$$

where $\varepsilon(t, x) \rightarrow 0$ as $t \rightarrow x$. If the operator $A_{n,a,\beta_n}^{(r)}(f; \cdot)$ is applied to this expression, we have

$$A_{n,a,\beta_n}^{(r)}(f; x) = \left\{ \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} A_{n,a,\beta_n}^{(r)}((t-x)^i; x) \right\} + A_{n,a,\beta_n}^{(r)}[\varepsilon(t, x)(t-x)^{r+2}; x] = I_1 + I_2$$

Using Lemma 7, we get

$$\begin{aligned} I_1 &= \left\{ \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} A_{n,a,\beta_n}^{(r)}((t-x)^i; x) \right\} = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \left\{ A_{n,a,\beta_n}^{(r)} \left(\sum_{j=1}^i \binom{i}{j} (-x)^{i-j} t^j; x \right) \right\} \\ &= \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=1}^i \binom{i}{j} (-x)^{i-j} A_{n,a,\beta_n}^{(r)}(t^j; x) \\ &= \sum_{i=0}^{r-1} \frac{f^{(i)}(x)}{i!} \sum_{j=1}^i \binom{i}{j} (-x)^{i-j} A_{n,a,\beta_n}^{(r)}(t^j; x) + \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} \sum_{j=1}^i \binom{i}{j} (-x)^{i-j} A_{n,a,\beta_n}^{(r)}(t^r; x) \\ &\quad + \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{j=1}^{r+1} \binom{r+1}{j} (-x)^{r+1-j} A_{n,a,\beta_n}^{(r)}(t^r; x) + \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{j=1}^{r+2} \binom{r+2}{j} (-x)^{r+2-j} A_{n,a,\beta_n}^{(r)}(t^r; x) \\ &= \sum_{i=0}^{r-1} \frac{f^{(i)}(x)}{i!} O\left(\frac{1}{n}\right) + \frac{f^{(r)}(x)}{r!} A_{n,a,\beta_n}^{(r)}(t^r; x) + \frac{f^{(r+1)}(x)}{(r+1)!} \left[(r+1)(-x) A_{n,a,\beta_n}^{(r)}(t^r; x) + A_{n,a,\beta_n}^{(r)}(t^{r+1}; x) \right] \\ &\quad + \frac{f^{(r+2)}(x)}{(r+2)!} \left[\frac{(r+1)(r+2)}{2} x^2 A_{n,a,\beta_n}^{(r)}(t^r; x) + (r+2)(-x) A_{n,a,\beta_n}^{(r)}(t^{r+1}; x) + A_{n,a,\beta_n}^{(r)}(t^{r+2}; x) \right] \\ &= \frac{f^{(r)}(x)}{r!} \frac{(n+\beta_n)^r (n+a-r-2)! r!}{(n+a-2)!} + O(n^{-1}) + \frac{f^{(r+1)}(x)}{(r+1)!} \left[\frac{(r+1)(-x)(n+\beta_n)^r (n+a-r-2)! r!}{(n+a-2)!} \right. \\ &\quad \left. + \frac{(n+\beta_n)^{r+1} (n+a-r-3)! (r+1)!}{(n+a-2)!} x + (r+1)^2 \frac{(n+\beta_n)^{r+1} (n+a-r-3)! r!}{(n+a-2)!} x + O(n^{-1}) \right] \\ &\quad + \frac{f^{(r+2)}(x)}{(r+2)!} \left\{ \frac{(r+1)(r+2)}{2} x^2 \frac{(n+\beta_n)^r (n+a-r-2)! r!}{(n+a-2)!} + (r+2)(-x) \left[\frac{(n+\beta_n)^{r+1} (n+a-r-3)! (r+1)!}{(n+a-2)!} x \right. \right. \\ &\quad \left. \left. + (r+1)^2 \frac{(n+\beta_n)^r (n+a-r-3)! r!}{(n+a-2)!} \right] + \frac{(n+\beta_n)^{r+2} (n+a-r-4)! (r+2)!}{(n+a-2)!} x \right\} \end{aligned}$$

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$$+ \frac{(r+1)^2(n+\beta_n)^{r+1}(n+a-r-4)!(r+1)!}{(n+a-2)!}x + (r+1)(r+2)\frac{(n+\beta_n)^r(n+a-r-4)!r!}{(n+a-2)!} + O(n^{-1})\}.$$

and using (2.10), (2.11), (2.12) and (2.13) we obtain

$$\begin{aligned} I_1 &= \frac{f^{(r)}(x)}{r!} \left[\lambda_r(n)r! + O\left(\frac{1}{n}\right) \right] + \frac{f^{(r+1)}(x)}{(r+1)!} \left[(r+1)(-x)\lambda_r(n)r! + \lambda_{r+1}(n)(r+1)!x + \frac{r+1}{n+\beta_n}\lambda_{r+1}(n) + O\left(\frac{1}{n}\right) \right] \\ &+ \frac{f^{(r+2)}(x)}{(r+2)!} \left[\frac{(r+2)!}{2}x^2\lambda_r(n) - (r+2)!x^2\lambda_{r+1}(n) + \frac{(r+2)!}{2}x^2\lambda_{r+2}(n) + \left(\frac{r+2}{n+\beta_n}\lambda_{r+1}(n) - \frac{r+1}{n+\beta_n}\lambda_{r+2}(n) \right)x \right. \\ &\left. + \frac{1}{(n+\beta_n)^2}\lambda_{r+2}(n) + O\left(\frac{1}{n}\right) \right] \\ &= f^{(r)}(x)\lambda_r(n) + f^{(r+1)}(x) \left[(\lambda_{r+1}(n) - \lambda_r(n))x - \frac{r+1}{n+\beta_n}\lambda_{r+1}(n) \right] + f^{(r+2)}(x) \left\{ [\lambda_r(n) - 2\lambda_{r+1}(n) + \lambda_{r+2}(n)] \frac{x^2}{2} \right. \\ &\left. + \left(\frac{r+2}{n+\beta_n}\lambda_{r+1}(n) - \frac{r+1}{n+\beta_n}\lambda_{r+2}(n) \right)x + \frac{1}{(n+\beta_n)^2}\lambda_{r+2}(n) + O\left(\frac{1}{n}\right) \right\} \\ &= f^{(r)}(x)\lambda_r(n) + f^{(r+1)}(x) \left[\lambda_r(n) \frac{(\beta_n - a + r + 2)x + (r+1)}{n+a-r-2} \right] \\ &+ f^{(r+2)}(x) \left\{ \lambda_r(n) \left[\frac{(r-a)^2 + (r+\beta_n)^2 - r^2 + 5(r-a) + 6(1+\beta_n) - 2a\beta_n + n}{(n+a-r-2)(n+a-r-3)} \right] \frac{x^2}{2} + \lambda_r(n) \left[\frac{r(a-\beta_n) - r^2 - 5r + 2a - 6 + \beta_n + n}{(n+a-r-2)(n+a-r-3)} \right] x \right. \\ &\left. + \frac{\lambda_r(n)}{(n+a-r-2)(n+a-r-3)} \right\} + O\left(\frac{1}{n}\right) \end{aligned}$$

Taking limit for $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} n[A_{n,a,\beta_n}^{(r)}(f; x) - f^{(r)}(x)] = [(2+r-a)x + r+1]f^{(r+1)}(x) + \left(x + \frac{x^2}{2}\right)f^{(r+2)}(x).$$

Theorem 3. (Error estimation) Let $f \in C_\gamma[0, \infty)$ for some $\gamma > 0$ and $0 < a < a_1 < b_1 < b < \infty$. Then for sufficiently large n , we have

$$\|A_{n,a,\beta_n}^{(r)}(f; \cdot) - f^{(r)}\|_{C[a_1, b_1]} \leq C_1\omega_2\left(f^{(r)}, n^{-\frac{1}{2}}[a, b]\right) + C_2n^{-1}\|f\|_\gamma,$$

where $C_1 = C_1(r)$ and $C_2 = C_2(r, f)$.

Proof. The equation

$$\begin{aligned} f(t) &= f^{(r)}(x) + f(t) - f_{\eta,2}(t) + f_{\eta,2}(t) - f_{\eta,2}^{(r)}(x) + f_{\eta,2}^{(r)}(x) - f^{(r)}(x) \\ f(t) - f^{(r)}(x) &= f(t) - f_{\eta,2}(t) + f_{\eta,2}(t) - f_{\eta,2}^{(r)}(x) + f_{\eta,2}^{(r)}(x) - f^{(r)}(x) \end{aligned}$$

can be written by considering (2.16). If we apply the operator $A_{n,a,\beta_n}^{(r)}$ to each side of this equation and using the linearity of the operator, we can write

$$\begin{aligned} A_{n,a,\beta_n}^{(r)}(f; \cdot) - f^{(r)}(x)A_{n,a,\beta_n}^{(r)}(1; \cdot) \\ = A_{n,a,\beta_n}^{(r)}(f - f_{\eta,2}; \cdot) + A_{n,a,\beta_n}^{(r)}(f_{\eta,2}; \cdot) - f_{\eta,2}^{(r)}(x)A_{n,a,\beta_n}^{(r)}(1; \cdot) + f_{\eta,2}^{(r)}(x) - f^{(r)}(x)A_{n,a,\beta_n}^{(r)}(1; \cdot) \end{aligned}$$

If we first take the absolute value of the equation and then add both sides of the equation on $C[a_1, b_1]$, we obtain

$$\begin{aligned} \|A_{n,a,\beta_n}^{(r)}(f; \cdot) - f^{(r)}\|_{C[a_1, b_1]} &\leq \|A_{n,a,\beta_n}^{(r)}(f - f_{\eta,2}; \cdot)\|_{C[a_1, b_1]} + \|A_{n,a,\beta_n}^{(r)}(f_{\eta,2}; \cdot) - f_{\eta,2}^{(r)}\|_{C[a_1, b_1]} + \|f^{(r)} - f_{\eta,2}^{(r)}\|_{C[a_1, b_1]} \\ &:= H_1 + H_2 + H_3 \end{aligned}$$

By property (3) of Steklov mean, we get

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$$H_3 \leq K\omega_2(f, [a, b]).$$

Next, using Theorem 2, we obtain

$$H_2 \leq K_1 n^{-1} \sum_{j=r}^{r+2} \|f_{\eta,2}^{(j)}\|_{C[a,b]} \leq K_1 n^{-1} [\|f_{\eta,2}\|_{C[a,b]} + \|f_{\eta,2}^{(r+2)}\|_{C[a,b]}].$$

Now by properties (2) and (4) of Steklov mean, we have

$$H_2 \leq K_2 n^{-1} [\eta^{-2} \|f\|_{C[a,b]} + \eta^{-2} \omega_r(f^{(r)}; \eta; [a, b])] \leq K_3 n^{-1} [\|f\|_{\gamma} + \eta^{-2} \omega_r(f^{(r)}; \eta; [a, b])].$$

Finally, we estimate H_1 choosing a, b satisfying the condition $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$. Let $\chi(t)$ denotes the characteristic function in the interval $[a^*, b^*]$, then

$$H_1 \leq \|A_{n,a,\beta_n}^{(r)}[\chi(t)(f - f_{\eta,2}; \cdot)]\|_{C[a_1,b_1]} + \|A_{n,a,\beta_n}^{(r)}[(1 - \chi(t))(f - f_{\eta,2}; \cdot)]\|_{C[a_1,b_1]} := H_4 + H_5$$

From (2.8), we have

$$\begin{aligned} & A_{n,a,\beta_n}^{(r)}[\chi(t)(f(t) - f_{\eta,2}(t); x)] \\ &= \frac{(n + \beta_n)^r (n + a - r - 1)!}{(n + a - 2)!} \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^{\infty} q_{n-r,a,k+r}(t) (\chi(t)(f^{(r)}(t) - f_{\eta,2}^{(r)}(t); x)) dt. \end{aligned}$$

Hence

$$H_4 \leq K_4 \|f^{(r)} - f_{\eta,2}^{(r)}\|_{C[a^*,b^*]}.$$

Now for $x \in [a_1, b_1]$ and $t \in [0, \infty) \setminus [a^*, b^*]$, we choose a $\delta_1 > 0$ satisfying $|t - x| > \delta_1$. By Lemma 5 and Schwarz inequality, we have

$$\begin{aligned} & \left| \frac{d^r}{dx^r} A_{n,a,\beta_n}[(1 - \chi(t))(f(t) - f_{\eta,2}(t); x)] \right| \\ & \leq (n + a - 1) \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \frac{(n + \beta_n)^i}{x^r} \phi_{i,j,r}(x) \sum_{k=0}^{\infty} [k - (n + \beta_n)x]^j e^{-(n+\beta_n)x} [(n + \beta_n)x]^k \\ & \times \int_0^{\infty} q_{n,a,k}(t) ((1 - \chi(t))|f(t) - f_{\eta,2}(t)|) dt. \\ & \leq K_5 \|f\|_{\gamma} (n + a - 1) \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n + \beta_n)^i \sum_{k=0}^{\infty} [k - (n + \beta_n)x]^j e^{-(n+\beta_n)x} [(n + \beta_n)x]^k \int_{|t-x| \geq \delta_1} q_{n,a,k}(t) dt \\ & \leq \frac{K_5}{\delta^{2l}} \|f\|_{\gamma} (n + a - 1) \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n + \beta_n)^i \sum_{k=0}^{\infty} [k - (n + \beta_n)x]^j e^{-(n+\beta_n)x} [(n + \beta_n)x]^k \\ & \times \left[\int_0^{\infty} q_{n,a,k}(t) dt \right]^{\frac{1}{2}} \left[\int_0^{\infty} q_{n,a,k}(t) (t - x)^{4l} dt \right]^{\frac{1}{2}} \\ & \leq \frac{K_5}{\delta^{2l}} \|f\|_{\gamma} (n + a - 1) \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n + \beta_n)^i \left[\sum_{k=0}^{\infty} [k - (n + \beta_n)x]^j e^{-(n+\beta_n)x} [(n + \beta_n)x]^k \right]^{\frac{1}{2}} \\ & \times \left[(n + a - 1) \sum_{k=0}^{\infty} e^{-(n+\beta_n)x} [(n + \beta_n)x]^k \int_0^{\infty} q_{n,a,k}(t) (t - x)^{4l} dt \right]^{\frac{1}{2}}. \end{aligned}$$

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Hence by Lemma 1 and Lemma 2 , we get

$$H_5 \leq K_6 \|f\|_\gamma \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n + \beta_n)^{i+\frac{j}{2}-1} \leq K_6 \|f\|_\gamma (n + \beta_n)^{-v} \leq K_6 \|f\|_\gamma (n + \beta_n)^{-1},$$

where $v = \left(l - \frac{r}{2}\right) > 1, \eta > 0$.

Therefore, by property (3) of Steklov mean, we obtain

$$H_1 \leq K_7 \|f^{(r)} - f_{\eta,2}^{(r)}\|_{C[a^*,b^*]} + K_6 \|f\|_\gamma (n + \beta_n)^{-1} \leq K_8 \omega_2(f; \eta; [a, b]) K_6 \|f\|_\gamma (n + \beta_n)^{-1}.$$

The proof is done by choosing $\eta = (n + \beta_n)^{-\frac{1}{2}}$.

As a result, when the results obtained for the operators defined by (1.4) are compared with the results given for the (1.1) operator, it is seen that the convergence properties are preserved and the error rate decreases by changing compared to β_n .

REFERENCES

- [1] A. Aral, D. Inoan, I. Raşa, "On the Generalized Szász-Mirakjan Operators." *Result Math.* 65(3-4), 441-452 (2014).
- [2] A. Erencin, G. Bascanbaz-Tunca, "Approximation Properties of a Class of Linear Positive Operators in Weighted Spaces" *C. R. Acad. Bulg. Sci.* 63(10), 1397-1404, (2010)
- [3] A. Erencin, "Durrmeyer Type Modification of Generalized Baskakov Operators." *Appl. Math. comp.* 218 (3), 4384-4390 (2011)
- [4] D.K. Verma, V. Gupta, P.N. Agrawal, "Some Approximation Properties of Baskakov-Durrmeyer-Stancu Operators" *Appl. Math. and Comp.* 218, 6549-6556, (2012)
- [5] E. Pandey, R. K. Mishra, S.P. Pandey, "Approximation Properties of Some Modified Summation-Integral Type Operator" *Inter. Jour. of Soft Comp. and Eng. (IJSCE)* ISSN:2231-2307, Volume-5, Issue-1, (2015)
- [6] F. Altomare, M. Campiti, "Korovkin-type Approximation Theory and its Applications" Walter de Gruyter, Berlin. New York 1994
- [7] F. Schurer, "Linear Positive Operators in Approximation Theory" *Math. Inst. Techn. univ. Delft. report.* (1962)
- [8] J.L. Durrmeyer, Une formule d'inversion de la transformée de Laplace: Applications à la théorie des moments. Thèse de 3e cycle, Paris, (1967)
- [9] Kumar, S., Acar, T. "Approximation by generalized Baskakov-Durrmeyer-Stancu type operators", *Rend. Circ. Mat. Palermo*, 65(3), 2016. (E-SCI)
- [10] N. Ispir, "Rate of Convergence of Generalized Rational Type Baskakov Operators" *Math. Comp. Model* 46, 625-631 (2007)
- [11] O. Gürel Yılmaz, M. Bodur, A. Aral, "On Approximation Properties of Baskakov-Schurer-Szász Operators Preserving Exponential Functions" *Filomat*, 32:15, 5433-5440 (2018)
- [12] P.N. Agrawal, V. Gupta, A.S. Kumar, "Generalized Baskakov-Durrmeyer Type Operators" *Rend. Circ. Mat. Palermo*, 63:193-209 (2014)
- [13] R. Aktaş, B. Çekim, F. Taşdelen, "A Kantorovich-Stancu Type Generalization of Szász operators Including Brenke-type Polynomials", *Journal of Function Spaces and Applications*, (2013), 9 pages
- [14] R. Aktaş, D. Söylemez, F. Taşdelen, "Stancu type generalization of Szász-Durrmeyer operators involving Brenke-type polynomials", *Filomat*, 33 (3), 855-868 (2019).
- [15] S. Arpağuş, A. Olgun "Approximation properties of Modified Baskakov Gamma Operators" *Facta Universitatis Mathematics and Informatics*, (2020) (in press)
- [16] S. Goldberg and V. Meir, "Minimum Moduli of ordinary differential operators" *Proc. London Math. Soc.*, 23, 1-15, (1971)
- [17] T. Acar, V. Gupta, A. Aral, "Rate of Convergence for Generalized Szász Type Operators" *Bull. Math. Sci.* 1(1), 99-113 (2011)
- [18] V. Gupta, M.A. Noor, M.S. Beniwal and M.K. Gupta "On Simultaneous Approximation for Certain Baskakov Durrmeyer Type Operators" *Jour. of Ineq. in Pure and Appl. Math.* Volum 7 Issue 4 Article 125 (2006)

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- [19] V. Gupta, D.K. Verma, P.N. Agrawal, "Simultaneous Approximation by Certain Baskakov-Durrmeyer-Stancu Operators", *Jour. of the Egyp. Math. Soc.* 20-183-187(2012)
- [20] V. N. Mishra, P. Sharma, "On Approximation Properties of Baskakov-Szász Operators", *appl. Math. and. comp.* v. 281, pp;381-393(2016)
- [21] V. Gupta and G. S. Srivastava, "On the Convergence of Derivative by Szász-Mirakyan Baskakov Type Operators", *The Math. Student* 64(1-49, 195-205(1995)
- [22] V. Gupta and G. Tachev, "Approximation by Szász-Mirakyan Baskakov Operators" *J. Appl. Funct. Anal.*, 9(3-4) ,308-319(2014)

