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Some new Pascal sequence spaces

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Abstract

The main purpose of the present paper is to study of some new Pascal sequence spaces p_∞, p_c and p_0. New Pascal sequence spaces p_∞, p_c and p_0 are found as BK-spaces and it is proved that the spaces p_∞, p_c and p_0 are linearly isomorphic to the spaces l_∞, c and c_0 respectively. Afterward, α-, β- and γ-duals of these spaces p_c and p_0 are computed and their bases are consructed. Finally, matrix the classes $(p_c : l_p)$ and $(p_c : c)$ have been characterized.

1. Preliminaries, background and notation

By w, we shall denote the space all real or complex valued sequences. Any vector subspace of w is called a sequence space. We shall write l_∞, c, and c_0 for the spaces of all bounded, convergent and null sequence are given by $l_\infty = \left\{ x = (x_k) \in w : \sup_{k\to\infty} |x_k| < \infty \right\}$, $c = \left\{ x = (x_k) \in w : \lim_{k\to\infty} x_k \text{ exists} \right\}$ and $c_0 = \left\{ x = (x_k) \in w : \lim_{k\to\infty} x_k = 0 \right\}$. Also by bs, cs, l_1 and l_p we denote the spaces of all bounded, convergent, absolutely convergent and p-absolutely convergent series, respectively.

A sequence space λ with a linear topology is called an K-space provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the set of complex field and $\mathbb{N} = \{0, 1, 2, ...\}$. An K-space λ is called an FK- space provided λ is a complete linear metric space. An FK-space provided whose topology is normable is called a BK- space [1].

Let X, Y be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we write $Ax = ((Ax)_n)$, the A-transform of x, if $A_n(x) = \sum_k a_{nk} x_k$ converges for each $n \in \mathbb{N}$. If $x \in X$ implies that $Ax \in Y$, then we say that A defines a matrix transformation from X into Y and denote it by $A : X \to Y$. By $(X : Y)$ we denote the class of all infinite matrices A such that $A : X \to Y$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞.

Let F denote the collection of all finite subsets on $\mathbb{N}$ and $K, \mathbb{N} \subset F$. The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.1}$$

which is a sequence space.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method was used by authors [2, 3, 4, 5, 6, 7, 8]. They introduced the sequence spaces $(c_0)_{T^r} = t_0^r$ and $(c)_{T^r} = t_c^r$ in [2], $(c_0)_{E^r} = e_0^r$ and $(c)_{E^r} = e_c^r$ in [3], $(c_0)_C = \bar{c}_0$ and $c_C = \bar{c}$ in [4], $(l_p)_{E^r} = e_p^r$ in [5], $(l_\infty)_{R^t} = r_\infty^t$, $c_{R^t} = r_c^t$ and $(c_0)_{R^t} = r_0^t$ in [6], $(l_p)_C = X_p$ in [7] and $(l_p)_{N_q}$ in [8] where T^r, E^r, C, R^t and N_q denote the Taylor, Euler, Cesaro, Riesz and Nörlund means, respectively.

Following [2, 3, 4, 5, 6, 7, 8], this way, the purpose of this paper is to introduce the new Pascal sequence spaces p_∞, p_c and p_0 and derive some results related to those sequence spaces. Furthermore, we have constructed the basis and computed the α-, β- and γ-duals of the spaces p_∞, p_c and p_0. Finally, we have characterized the matrix mappings from the space p_c to l_p and from the space p_c to c.

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2. The Pascal matrix of inverse formula and Pascal sequence spaces

Let P denote the Pascal means defined by the Pascal matrix [9] as is defined by

$$P=[p_{nk}]=\begin{cases} \binom{n}{n-k}, (0\le k\le n) \\ 0,\ (k>n) \end{cases},(n,k\in\mathbb{N})$$

and the inverse of Pascal's matrix $P_n=[p_{nk}]$ [10] is given by

$$P^{-1}=[p_{nk}]^{-1}=\begin{cases} (-1)^{n-k}\binom{n}{n-k},\ (0\le k\le n) \\ 0\ ,\ (k>n) \end{cases},(n,k\in\mathbb{N}). \tag{2.1}$$

There is some interesting properties of Pascal matrix. For example; we can form three types of matrices: symmetric, lower triangular, and upper triangular, for any integer $n>0$. The symmetric Pascal matrix of order n is defined by

$$S_n=(s_ij)=\binom{i+j-2}{j-1}i,\ j=1,2,....,n. \tag{2.2}$$

We can define the lower triangular Pascal matrix of order n by

$$L_n=(l_{ij})=\begin{cases} \binom{i-1}{j-1}, (0\le j\le i) \\ 0, \qquad (j>i) \end{cases}, \tag{2.3}$$

and the upper triangular Pascal matrix of order n is defined by

$$U_n=(u_{ij})=\begin{cases} \binom{j-1}{i-1}, (0\le i\le j) \\ 0, \qquad (j>i) \end{cases}. \tag{2.4}$$

We notice that $U_n=(L_n)^T$, for any positive integer n.
i. Let S_n be the symmetric Pascal matrix of order n defined by (2.1), L_n be the lower triangular Pascal matrix of order n defined by (2.3), and U_n be the upper triangular Pascal matrix of order n defined by (2.4), then $S_n=L_nU_n$ and $det(S_n)=1$ [11].
ii. Let A and B be $n\times n$ matrices. We say that A is similar to B if there is an invertible $n\times n$ matrix P such that $P^{-1}AP=B$ [12].
iii. Let S_n be the symmetric Pascal matrix of order n defined by (2.2), then S_n is similar to its inverse S_n^{-1} [11].
iv. Let L_n be the lower triangular Pascal matrix of order n defined by (2.3), then $L_n^{-1}=((-1)^{i-j}l_{ij})$ [13].
We wish to introduce the Pascal sequence spaces p_∞, p_c and p_0, as the set of all sequences such that P-transforms of them are in the spaces l_∞, c and c_0, respectively, that is

$$p_\infty=\left\{x=(x_k)\in w:\sup_n\left|\sum_{k=0}^{n}\binom{n}{n-k}x_k\right|<\infty\right\},$$

$$p_c=\left\{x=(x_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{n-k}x_k \text{ exists}\right\}$$

and

$$p_0=\left\{x=(x_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{n-k}x_k=0\right\}.$$

With the notation of (1.1), we may redefine the spaces p_∞, p_c and p_0 as follows:

$$p_\infty=(l_\infty)_P,\ p_c=(c)_P \text{ and } p_0=(c_0)_P. \tag{2.5}$$

If λ is an normed or paranormed sequence space, then matrix domain λ_P is called an Pascal sequence space. We define the sequence $y=(y_n)$ which will be frequently used, as the P-transform of a sequence $x=(x_n)$ i.e.,

$$y_n=\sum_{k=0}^{n}\binom{n}{n-k}x_k,\quad (n\in N). \tag{2.6}$$

It can be shown easily that p_∞, p_c and p_0 are linear and normed spaces by the following norm:

$$\|x\|_{p_0}=\|x\|_{p_c}=\|x\|_{p_\infty}=\|Px\|_{l_\infty}. \tag{2.7}$$

Theorem 2.1. *The sequence spaces p_∞, p_c and p_0 endowed with the norm (2.7) are Banach spaces.*

Proof. Let sequence $\{x^t\} = \{x_0^{(t)}, x_1^{(t)}, x_2^{(t)}, ...\}$ at p_∞ a Cauchy sequence for every fixed $t \in \mathbb{N}$. Then, there exists an $n_0 = n_0(\varepsilon)$ for every $\varepsilon > 0$ such that $\|x^t - x^r\|_\infty < \varepsilon$ for all t, $r > n_0$. Hence, $|P(x^t - x^r)| < \varepsilon$ for all t, $r > n_0$ and for each $k \in \mathbb{N}$.
Therefore, $\{Px_k^t\} = \{(Px^0)_k, (Px^1)_k, (Px^2)_k, ...\}$ is a Cauchy sequence in the set of complex numbers $\mathbb{C}$. Since $\mathbb{C}$ is complete, it is convergent say $\lim_{t\to\infty}(Px^t)_k = (Px)_k$ and $\lim_{m\to\infty}(Px^m)_k = (Px)_k$ for each $k \in \mathbb{N}$. Hence, we have

$$\lim_{m\to\infty}\left|Px_k^t - x_k^m\right| = \left|P\left(x_k^t - x_k\right) - P\left(x_k^m - x_k\right)\right| \le \varepsilon \text{ for all } n \ge n_0.$$

This implies that $\|x^t - x^m\| \to \infty$ for t, $m \to \infty$. Now, we should that $x \in p_\infty$. We have

$$\|x\|_\infty = \|Px\|_\infty = \sup_n\left|\sum_{k=0}^{n}\binom{n}{n-k}x_k\right| = \sup_n\left|\sum_{k=0}^{n}\binom{n}{n-k}(x_k - x_k^t + x_k^t\right|$$

$$\le \sup_n\left|P\left(x_k^t - x_k\right)\right| + \sup_n\left|Px_k^t\right|$$

$$\le \left\|x^t - x\right\|_\infty + \left|Px_k^t\right| < \infty$$

for t, $k \in \mathbb{N}$. This implies that $x = (x_k) \in p_\infty$. Thus, p_∞ the space is a Banach space with the norm (2.7). It can be shown that p_0 and p_c are closed subspaces of p_∞ which leads us to the consequence that the spaces p_0 and p_c are also the Banach spaces with the norm (2.7). Furthermore, since p_∞ is a Banach space with continuous coordinates, i.e., $\left\|P\left(x_k^t - x\right)\right\|_\infty \to \infty$ imples $\left|P\left(x_k^t - x_k\right)\right| \to \infty$ for all $k \in \mathbb{N}$, it is also a BK-space. □

Theorem 2.2. *The sequence spaces p_∞, p_c and p_0 are linearly isomorphic to the spaces l_∞, c and c_0 respectively, i.e $p_\infty \cong l_\infty$, $p_c \cong c$ and $p_0 \cong c_0$.*

Proof. To prove the fact $p_0 \cong c_0$, we should show the existence of a linear bijection between the spaces p_0 and c_0. Consider the transformation T defined, with the notation (2.6), from p_0 to c_0. The linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.
Let $y \in c_0$. We define the sequence $x = (x_k)$ as follows:

$$x_k = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{k-i}y_i.$$

Then

$$\lim_{n\to\infty}(Px)_n = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{n-k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{k-i}y_i = \lim_{n\to\infty}y_n = 0.$$

Thus, we have that $x \in p_0$. In addition, note that

$$\|x\|_{p_0} = \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\binom{n}{n-k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{k-i}y_i\right| = \sup_{n\in\mathbb{N}}|y_n| = \|y\|_{c_0} < \infty.$$

Consequently, T is surjective and is norm preserving. Hence, T is a linear bijection which therefore says us that the spaces p_0 to c_0 are linearly isomorphic. In the same way, it can be shown that p_c and p_∞ are linearly isomorphic to c and l_∞, respectively, and so we omit the detail. □

Before giving the basis of of the sequence spaces p_c and p_0, we define the Schauder basis. A sequence $(b_n)_{n\in\mathbb{N}}$ in a normed sequence space λ is called a Schauder basis (or briefly basis) [14], if for every $x \in \lambda$ there is a unique sequence (α_n) of scalars such that

$$\lim_{n\to\infty}\|x - (\alpha_0 x_0 + \alpha_1 x_1 + ... + \alpha_n x_n)\| = 0.$$

In the following theorem, we shall give the Schauder basis for the spaces p_c and p_0.

Theorem 2.3. *Let $k \in \mathbb{N}$ a fixed natural number and $b^{(k)} = \left\{b_n^{(k)}\right\}_{n\in\mathbb{N}}$ where*

$$b_n^{(k)} = \begin{cases} 0, & (0 \le n < k) \\ (-1)^{n-k}\binom{n}{n-k}, & (n \ge k \end{cases}.$$

Then the following assertions are true:
i. *The sequence $\left\{b_n^{(k)}\right\}$ is a basis for the space p_0 and every $x \in p_0$ has a unique representation of the from $x = \sum_k \lambda_k b^{(k)}$ where $\lambda_k = (Px)_k$ for all $k \in \mathbb{N}$.*

ii. *The set $\left\{e, b^{(0)}, b^{(1)}, ..., b^{(k)}, ...\right\}$ is a basis for the space p_c and every $x \in p_c$ has a unique representation of the form $x = le + \sum_k(\lambda_k - l)b^{(k)}$, where $l = \lim_{k\to\infty}(Px)_k$ and $\lambda_k = (Px)_k$ for all $k \in \mathbb{N}$.*

3. The $\alpha-$, $\beta-$ and $\gamma-$ duals of the spaces p_∞, p_c and p_0

In this section, we state and prove the theorems determining the α-, β- and γ-duals of the sequence spaces p_∞, p_c and p_0. For the sequence spaces X and Y define the set $S(X,Y)$ by

$$S(X,Y)=\{z=(z_k)\in w: xz=(x_kz_k)\in Y \text{ for all } x\in X\}.$$

The α-, β- and γ-duals of the sequence spaces λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by Garling [15] , by $\lambda^\alpha=S(\lambda,l_1)$, $\lambda^\beta=S(\lambda,cs)$ and $\lambda^\gamma=S(\lambda,bs)$. We shall begin with the Lemmas due to Stieglitz and Tietz [16], which are needed in the proof of the Theorems 3.4-3.6.

Lemma 3.1. *$A\in(c_0:l_1)=(c:l_1)$ if and only if*

$$\sup_{K\in F}\sum_n\left|\sum_{k\in K}a_{nk}\right|<\infty. \tag{3.1}$$

Lemma 3.2. *$A\in(c_0:c)$ if and only if*

$$\sup_n\sum_k|a_{nk}|<\infty, \tag{3.2}$$

$$\lim_{n\to\infty}a_{nk}=\alpha_k,\ (k\in\mathbb{N}). \tag{3.3}$$

Lemma 3.3. *$A\in(c_0:l_\infty)$ if and only if (3.2) holds.*

Theorem 3.4. *The $\alpha-$ dual of the sequence spaces p_∞, p_c and p_0 is the set*

$$D=\left\{a=(a_k)\in w:\sup_{K\in F}\sum_n\left|\sum_{k\in K}(-1)^{n-k}\binom{n}{n-k}a_n\right|<\infty\right\}.$$

Proof. Let $a=(a_n)\in w$ and consider the matrix B whose rows are the products of the rows of the matrix P^{-1} and sequence $a=(a_n)$. Bearing in mind the relation (2.3), we immediately derive that

$$a_nx_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{n-k}a_ny_k=\sum_{k=0}^{n}b_{nk}y_k=(By)_n,\ (n\in\mathbb{N}). \tag{3.4}$$

Therefore by (3.4) we observe that that $ax=(a_nx_n)\in l_1$ whenever $x\in p_\infty$, p_c and p_0 if and only if $By\in l_1$ whenever $y\in l_\infty$, c, and c_0. Then, we derive by Lemma 3.1 that

$$\sup_{K\in F}\sum_n\left|\sum_{k\in K}(-1)^{n-k}\binom{n}{n-k}a_n\right|<\infty$$

which yields the consequences that $\{p_\infty\}^\alpha=\{p_c\}^\alpha=\{p_0\}^\alpha=D.$ □

Theorem 3.5. *Consider the sets D_1, D_2 and D_3 defined as follows:*

$$D_1=\left\{a=(a_k)\in w:\ \sup_{n\in N}\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right|<\infty\right\},$$

$$D_2=\left\{a=(a_k)\in w:\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{i-k}a_i \text{ exists for each } k\in\mathbb{N}\right\},$$

and

$$D_3=\left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i \text{ exists}\right\}.$$

Then $\{p_0\}^\beta=D_1\cap D_2$, $\{p_c\}^\beta=D_1\cap D_2\cap D_3$ and $\{p_\infty\}^\beta=D_2\cap D_3$.

Proof. We give the proof only for the space p_0. Since the proof may be given by a similar way for the spaces p_c and p_∞, we omit it. Consider the equation

$$\sum_{k=0}^{n}a_kx_k=\sum_{k=0}^{n}\left[\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}y_i\right]a_k=\sum_{k=0}^{n}\left[\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right]y_k=(Dy)_n, \tag{3.5}$$

where

$$D=(d_{nk})=\begin{cases}\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i, & (0\le k\le n)\\ 0 & ,(k>n)\end{cases},(n,k\in\mathbb{N}). \tag{3.6}$$

Thus, we deduce from Lemma 3.2 with (3.5) that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in p_0$ if and only if $Dy\in c$ whenever $y=(y_k)\in c_0$. Therefore, using relations (3.2) and (3.3), we conclude that $\lim_{n\to\infty}d_{nk}$ exists fo each $k\in\mathbb{N}$ and

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right|<\infty$$

which shows that $\{p_0\}^{\beta}=D_1\cap D_2$. □

Theorem 3.6. *The $\gamma-$ dual of the sequence spaces p_∞, p_c and p_0 are D_1.*

Proof. We give the proof only for the space p_0. Consider the equality

$$\begin{aligned}\left|\sum_{k=0}^{n}a_kx_k\right| &= \left|\sum_{k=0}^{n}a_k\left[\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{k-i}y_i\right]\right|\\ &= \left|\sum_{k=0}^{n}\left[\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right]y_k\right|\\ &\le \sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right||y_k|.\end{aligned}$$

Taking supremum over $n\in\mathbb{N}$, we get

$$\begin{aligned}\sup_{n\in N}\left|\sum_{k=0}^{n}a_kx_k\right| &\le \sup_{n\in N}\left(\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right||y_k|\right)\\ &\le \|y\|_{c_0}\sup_{n}\left(\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right|\right)\le\infty.\end{aligned}$$

This means that $a=(a_k)\in\{p_0\}^{\gamma}$. Hence,

$$D_1\subset\{p_0\}^{\gamma}. \tag{3.7}$$

Conversely, let $a=(a_k)\in\{p_0\}^{\gamma}$ and $x\in p_0$. Then one can easily see that

$$\left(\sum_{k=0}^{n}\left[\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right]y_k\right)\in l_\infty$$

whenever $ax=(a_kx_k)\in bs$. This implies that the matrix D given at the (3.6) is in the class $(c_0:l_\infty)$. Hence, the condition

$$\sup_{n}\left(\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right|\right)<\infty$$

is satisfied, which implies that $a=(a_k)\in D_1$. In other words,

$$\{p_0\}^{\gamma}\subset D_1. \tag{3.8}$$

Therefore, by combining inclusions (3.7) and (3.8), we estahlish that the γ-dual of the sequence spaces p_0 is D_1, which completes the proof. □

4. Some matrix mappings related to Pascal sequence spaces

Lemma 4.1. *[16, p. 57] The matrix mappings between BK-spaces are continuous.*

Lemma 4.2. *[16, p. 128]* $A \in \left(c : l_p\right)$ *if and only if*

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} a_{nk} \right|^p < \infty,\ 1 \le p < \infty. \tag{4.1}$$

Theorem 4.3. $A \in \left(p_c : l_p\right)$ *if and only if the following conditions are satisfied: For* $1 \le p < \infty$,

$$\sup_{K \in F} \sum_k \left| \sum_{k \in K} \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{i-k} a_{ni} \right|^p < \infty\,, \tag{4.2}$$

$$\sum_{i=k}^{n} (-1)^{i-k} \binom{i}{i-k} a_{ni} \text{ exists for all } k,\ n \in \mathbb{N}, \tag{4.3}$$

$$\sum_k \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{i-k} a_{ni} \text{ converges for all } n \in \mathbb{N}, \tag{4.4}$$

$$\sup_{m \in N} \sum_{k=0}^{m} \left| \sum_{i=k}^{m} (-1)^{i-k} \binom{i}{i-k} a_{ni} \right| < \infty\,,\ n \in \mathbb{N}, \tag{4.5}$$

and for $p = \infty$, *conditions* (4.3) *and* (4.5) *are satisfied and*

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{i-k} a_{ni} \right| < \infty. \tag{4.6}$$

Proof. Let $1 \le p < +\infty$. Assume that conditions (4.2) - (4.6) are satisfied and take any $x \in p_c$. Then $(a_{nk}) \in (p_c)^{\beta}$ for all $k, n \in \mathbb{N}$, which implies that Ax exists. We define the matrix $G = (g_{nk})$ with

$$g_{nk} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{i-k} a_{ni}$$

for all $k, n \in \mathbb{N}$. Then, since condition (4.1) is satisfied for the matrix G, we have $G \in \left(c : l_p\right)$. Now consider the following equality obtained from the s. th partial sum of the series $\sum_k a_{nk} x_k$:

$$\sum_{k=0}^{s} a_{nk} x_k = \sum_{k=0}^{s} \sum_{i=k}^{s} (-1)^{i-k} \binom{i}{i-k} a_{ni} y_k,\ m, n \in \mathbb{N}. \tag{4.7}$$

Therefore, we derive from (4.7) as $s \to \infty$ that

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{i-k} a_{ni} y_k,\ n \in \mathbb{N}. \tag{4.8}$$

Whence taking l_p-norm we get

$$\|Ax\|_{l_p} = \|Gy\|_{l_p} < \infty. \tag{4.9}$$

This means that $A \in \left(p_c : l_p\right)$. Now let $p = \infty$. Assume that conditions (4.2) - (4.6) are satisfied and take any $x \in p_c$. Then $(a_{nk}) \in (p_c)^{\beta}$ for all k, $n \in \mathbb{N}$, which implies that Ax exists. Whence taking l_∞-norm (4.8)

$$\|Ax\|_{l_\infty} = \sup_{n \in N} \left| \sum_k g_{nk} \right| \le \|y\|_{l_\infty} \sup_{n \in N} \sum_k |g_{nk}| < \infty.$$

Then, we have $A \in (p_c : l_\infty)$.

Conversely, assume that $A \in \left(p_c : l_p\right)$. Then, since p_c and l_p are BK-spaces, it follows from Lemma 4 that there exists a real constant $K > 0$ such that

$$\|Ax\|_{l_p} = K\, \|x\|_{h_c} \tag{4.10}$$

for all $x \in p_c$. Since inequality (4.10) also holds for the sequence

$$x = (x_k) = \sum_{k \in F} b^{(k)} \in p_c,$$

where

$$b^{(k)} = \{b_n^{(k)}\} = \begin{cases} 0, & (0 \leq n < k) \\ (-1)^{n-k}\binom{n}{n-k}, & (n \geq k \end{cases}$$

for every fixed $k \in \mathbb{N}$. We have

$$\|Ax\|_{l_p} = \left[\sum_n \left|\sum_{k\in F}\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}\right|^p\right]^{\frac{1}{p}} \leq K\,\|x\|_{p_c} = K,$$

which shows the necessity of (4.2). □

Theorem 4.4. *$A \in (p_c : c)$ if and only if conditions (4.3), (4.5) and (4.6) are satisfied,*

$$\lim_{n\to\infty}\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni} = \alpha_k \text{ for all } k \in \mathbb{N} \tag{4.11}$$

and

$$\lim_{n\to\infty}\sum_k\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni} = \alpha. \tag{4.12}$$

Proof. Assume that A satisfies conditions (4.3), (4.5), (4.6), (4.11) and (4.12). Let us take an arbitrary an $x = (x_k)$ in p_c such that $x_k \to l$ as $k \to \infty$. Then Ax exists, and it is trivial that the sequence $y = (y_k)$ associated with the sequence $x = (x_k)$ by relation (2.3) belongs to c and is such that $y_k \to l$ as $k \to \infty$. At this stage, it follows from (4.11) and (4.6) that

$$\sum_{j=0}^{k}|\alpha_j| \leq \sup_{n\in N}\sum_j\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}\right| < \infty$$

for every $n \in \mathbb{N}$. This yield $\alpha_n \in l_1$. Considering (4.8), we write

$$\sum_k a_{nk}x_k = \sum_k\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}(y_k - l) + l\sum_k\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}y_k. \tag{4.13}$$

In this situation, letting $n \to \infty$ in (4.13), we establish that the first term on the right-hand side tends to $\sum_k \alpha_k(y_k - l)$ by (4.6) and(4.11), and the second term tends to $l\alpha$ by (4.11). Taking these facts into account, we deduce from (4.13) as $n \to \infty$ that

$$(Ax)_n \to \sum_k \alpha_k(y_k - l) + l\alpha$$

which shows that $A \in (p_c : c)$.
Conversely, assume that $A \in (p_c : c)$. Then, since the inclusion $c \subset l_\infty$ holds the necessity of (4.3), (4.5) and (4.6) is immediately obtained from

$$\sup_n\sum_k\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}\right| < \infty.$$

To prove the necessity of (4.11) consider the sequence $x = b^{(k)} = \left\{b_n^{(k)}\right\}_{n\in\mathbb{N}}$ in p_c. Where

$$b^{(k)} = \{b_n^{(k)}\} = \begin{cases} 0, & (0 \leq n < k) \\ (-1)^{n-k}\binom{n}{n-k}, & (n \geq k \end{cases}$$

for every fixed $k \in \mathbb{N}$. Since Ax exists and belongs to c for every $x \in p_c$, one can easily see that

$$Ab^{(k)} = \left\{\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}\right\}_{n\in\mathbb{N}}$$

for each $k \in \mathbb{N}$, which yields the necessity of (4.11).
Similarly, by setting $x = e = (1, 1, \ldots)$ in (4.8), we obtain

$$Ax = \left\{\sum_k\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}\right\}_{n\in\mathbb{N}},$$

which belongs to the space c, and this shows the necessity of (4.12). This step conludes the proof.

□

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