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Exact Sequences of BCK-Modules

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Abstract

BCK-modules were introduced as an action of a BCK-algebra over an Abelian group. Homomorphisms of BCK-modules form an exact sequence which is called BCK-sequence. In this paper, we study homomorphisms of BCK-modules. We show that this homomorphisms have a module structure. Moreover, we show that sequences of Hom functors are BCK-sequences.

1. Introduction

BCK/BCI-algebras were introduced by Imai and Iseki [1, 2]. BCK/BCI-algebras have been studied by many authors, extensively. In 1994, the BCK-module structure of BCK-algebras was introduced as an action on an Abelian group [3]. In [4], exact sequences of BCK-modules were studied. Further, in [5],the authors studied the homomorphisms between BCK-modules and they showed that the set of homomorphisms of BCK-modules form a BCK-module. Later, in [6], homology theory of BCK-modules was investigated. In [7], the authors studied BCK-sequences and finitely presented BCK-modules.
The paper organized as follows; in section 2, we give general theory of BCK-algebras and BCK-modules. In section 3, we study the exactness of modules of homomorphisms between BCK-modules.

2. Preliminaries

In this section we introduce the background informations about BCK-algebras, BCK-modules and X-homomorphisms.

Definition 2.1. *[8] A BCK-algebra is an algebra $(X;*,0)$ of type $(2,0)$ which satisfies the following axioms: for all $p,q,r \in X$,*

1. $((p*q)*(p*r))*(r*q)=0$,
2. $(p*(p*q))*q=0$,
3. $(p*p)=0$,
4. $p*q=0=q*p$ *implies* $p=q$.
5. $0*p=0$.

Moreover, the relation $\leq$ can be defined as $p\leq q$ if and only if $p*q=0$, for any $p,q\in X$, is a partial-order on X which is called *BCK-ordering* of X.

Definition 2.2. *[6] Let $(X;*,0)$ be a BCK-algebra and M be an Abelian group under addition $+$, then M is said to be an (left) X-module, if there is a mapping $(x,m)\mapsto xm$ from $X\times M\to M$ such that it satisfies the following conditions for all $x,x_1,x_2\in X$ and $m,m_1,m_2\in M$:*

1. $(x_1\wedge x_2)m = x_1(x_2m)$,

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2. $x(m_1+m_2)=xm_1+xm_2$,
3. $0m=0$

where, $x_1\wedge x_2=x_2*(x_2*x_1)$. *If* X *is bounded with maximal element* 1, *then*

4. $1m=m$.

The right X-module can be defined similarly. This X-module M is an BCK-module. If a subgroup N of the X-module M is also an X-module, then N is called a *submodule*.
Let M and N be X-modules. A mapping $\phi: M\to N$ is said to be an X-homomorphism, if for any $x\in X$ and $m_1,m_2\in M$ the followings hold:

1. $\phi(m_1+m_2)=\phi(m_1)+\phi(m_2)$,
2. $\phi(xm_1)=x\phi(m_1)$.

If ϕ is both injective and surjective, then ϕ is an X-isomorphism. We say M is isomorphic to N if ϕ is an X-isomorphism and denote it by $M\cong N$.
The bounded implicative BCK-algebras form a BCK-module over itself (Abujabal et al., 1994). This section devoted to the examples of BCK-modules.

Example 2.3. *Let* $(X;*,0)$ *be a bounded implicative BCK-algebra with* $X=\{0,x,y,1\}$. *Let* $M=\{0,x\}$ *be a subset of* X. *If we define addition operation* $+$ *as* $x+y=(x*y)\vee(y*x)$ *and* $xm=x\wedge m$ *for all* $x\in X$, $m\in M$, *then* M *is an* X-*module. Cayley table of these operations are as follows:*

*	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

+	0	x
0	0	x
x	x	0

∧	0	x
0	0	0
x	0	x
y	0	0
1	0	x

3. Exact BCK-sequences

Definition 3.1. *[7] The sequence of* X-*module homomorphisms* $M_1\xrightarrow{f}M_2\xrightarrow{g}M_3$ *is said to be exact at* M_2, *if* $\mathrm{Im}(f)=\mathrm{Ker}(g)$. *A sequence of* X-*module homomorphisms,* $M_1\xrightarrow{f_1}M_2\xrightarrow{f_2}\dots\xrightarrow{f_{n-1}}M_n$ *is called exact sequence of* X-*modules, if* $\mathrm{Im}(f_i)=\mathrm{Ker}(f_{i+1})$ *for all* $i\in\{1,2,\dots,n\}$.

Theorem 3.2. *Let* X *be a BCK-algebra and* K,L *and* M *be* X-*modules. If* A *is an* X-*module and* $0\to K\xrightarrow{\psi}L\xrightarrow{\phi}M$ *is exact, then*

$$0\to \mathrm{Hom}(A,K)\xrightarrow{\psi_*}\mathrm{Hom}(A,L)\xrightarrow{\phi_*}\mathrm{Hom}(A,M)$$

is an exact sequence of X-*modules.*

Proof. First we show that ψ_* is a monomorphism. Let $\theta: A\to K$ be a X-homomorphism with $\psi_*\theta=0$. Since ψ is a monomorphism, then for any $a\in A$, the identity $\psi_*\theta(a)=0$ implies that $\theta(a)=0$. Thus $\theta=0$. Hence ψ_* is a monomorphism. Let $b\in\mathrm{Im}(\psi_*)\subseteq\mathrm{Hom}(A,L)$. Then there exists $a\in\mathrm{Hom}(A,K)$ such that $\psi_*(a)=b=\psi a$. Since $\phi_*(b)=\phi_*(\psi a)=\phi\psi a=0a=0$, we have $b\in\mathrm{Ker}(\phi_*)$. Hence $\mathrm{Im}(\psi_*)\subseteq\mathrm{Ker}(\phi_*)$. Let $u\in\mathrm{Ker}(\phi_*)\subseteq\mathrm{Hom}(A,L)$. Then $\phi_*(u)=0$ and $\phi u(a)=0$ for any $a\in A$. The exactness of the sequence gives that $\mathrm{Ker}(\phi)=\psi(K)$. Thus there exists an $x\in K$ which satisfies $\psi(x)=u(a)$. Then $v(a)=x$ defines a homomorphism $v: A\to K$ with $\psi_*(v)=u$. Thus $\mathrm{Ker}(\phi_*)\subseteq\mathrm{Im}(\psi_*)$. Therefore $\mathrm{Ker}(\phi_*)=\mathrm{Im}(\psi_*)$. $\square$

Theorem 3.3. *Let* X *be a BCK-algebra and* K,L *and* M *be* X-*modules. If* A *is an* X-*module and* $K\xrightarrow{\psi}L\xrightarrow{\phi}M\to 0$ *is exact, then*

$$0 \to \mathrm{Hom}(M,A) \xrightarrow{\phi_*} \mathrm{Hom}(L,A) \xrightarrow{\psi_*} \mathrm{Hom}(K,A)$$

is an exact sequence of X-modules.

Proof. First we show that ϕ_* is a monomorphism. Let $\theta : M \to A$ be an X-homomorphism and $\theta \in \mathrm{Ker}(\phi_*)$. Since $0 = \phi_*\theta = \theta\phi$, this implies that $\theta(\phi(l)) = 0$ for all $l \in L$. Thus $\theta(m) = 0$ for all $m \in \mathrm{Im}(\phi)$. The fact that ϕ is epimorphism implies that $\mathrm{Im}(\phi) = M$ and $\theta = 0$. Hence ϕ_* is a monomorphism.
Let $b \in \mathrm{Im}(\phi_*) \subseteq \mathrm{Hom}(L,A)$. Then there exists $a \in \mathrm{Hom}(M,A)$ such that $\phi_*(a) = b = a\phi$. Since $\psi_*(b) = \psi_*(a\phi)$ and $\psi_*(a\phi) = a\phi\psi = a0 = 0$, this implies that $b \in \mathrm{Ker}(\psi_*)$. Hence $\mathrm{Im}(\phi_*) \subseteq \mathrm{Ker}(\psi_*)$. Let $u \in \mathrm{Ker}(\psi_*) \subseteq \mathrm{Hom}(L,A)$. Then $\psi_*(u) = 0 = u\psi$. Following the diagram,

$$\begin{tikzcd} K \arrow[r, "\psi"] & L \arrow[r, "\phi"] \arrow[d, "u"'] & M \arrow[r] \arrow[dl, "p"] & 0 \\ & A & & \end{tikzcd}$$

There exists $p \in \mathrm{Hom}(M,A)$ such that $u = p\phi = \phi_*(p)$. This implies that $u \in \mathrm{Im}(\phi_*)$. Thus $\mathrm{Ker}(\psi_*) \subseteq \mathrm{Im}(\phi_*)$. Therefore $\mathrm{Ker}(\psi_*) = \mathrm{Im}(\phi_*)$.

□

Definition 3.4. *Let X be a BCK-algebra and M,N and K be X-modules. If the following sequence of X-modules is exact. Then*

$$0 \to M \to N \to K \to 0$$

is called short exact sequence.

Theorem 3.5. *Let X be a BCK-algebra and M,N and K be X-modules. If the short sequence of X-homomorphisms is exact;*

$$0 \to M \underset{\eta}{\overset{\psi}{\rightleftarrows}} N \underset{\theta}{\overset{\phi}{\rightleftarrows}} K \to 0$$

then followings are equivalent;

1. *There exists an X-homomorphism $\eta : N \to M$ such that $\eta\psi = 1_M$.*
2. *Submodule $\mathrm{Im}(\psi)$ is a direct summand of N.*
3. *There exists an X-homomorphism $\theta : K \to N$ suct that $\phi\theta = 1_K$.*

Moreover, we have $N \cong M \oplus K$.

Proof. $1 \Rightarrow 2$ Let $x \in N$ be any element. Since $\eta(x - \psi\eta(x)) = \eta(x) - ((\eta\psi)\eta(x)) = \eta(x) - \eta(x) = 0$, then we have $x - \psi\eta(x) \in \mathrm{Ker}(\eta)$. This implies that $x = \psi(\eta(x)) + (x - \psi\eta(x)) \in \mathrm{Im}(\psi) + \mathrm{Ker}(\eta)$.
Let $\psi(m) \in \mathrm{Im}(\psi) \cap \mathrm{Ker}(\eta)$. Since $m = \eta\psi(m) = \eta(\psi(m)) = 0$, one can conclude that $\mathrm{Im}(\psi) \cap \mathrm{Ker}(\eta) = 0$. Hence $N = \mathrm{Im}(\psi) \oplus \mathrm{Ker}(\eta)$.
$2 \Rightarrow 3$ Let N' be a submodule of N and $N = \mathrm{Im}(\psi) \oplus N'$. Now since $N' \cap \mathrm{Ker}(\phi) = N' \cap \mathrm{Im}(\psi) = 0$, the $\phi|_{N'}$ is a monomorphism. The fact that ϕ is a epimorphism implies that there exists x in N for every $y \in K$ such that $\phi(x) = y$. If we set $x = \psi(a) + b$ for $a \in M, b \in N'$. Then $y = \phi(x) = \phi(\psi(a) + b) = \phi\psi(a) + \phi(b) = \phi(b)$. This implies that $\phi|_{N'}$ is an epimorphism. Thus $\phi|_{N'}$ is an isomorphism. Since $\phi|_{N'}$is an isomorphism, we can conclude that $\phi|_{N'}$ has an inverse $(\phi|_{N'})^{-1} : K \to N$ for $\theta := (\phi|_{N'})^{-1} : K \to N$ then we have $\phi\theta = 1_K$.
$3 \Rightarrow 1$ Since $\phi(n - \theta\phi(n)) = \phi(n) - \phi(\theta\phi(n)) = 0$, we have $n - \theta\phi(n) \in \mathrm{Ker}(\phi) = \mathrm{Im}(\psi)$. Then there exists $m \in M$ such that $\psi(m) = n - \theta\phi(n)$. This m is unique, since ψ is a monomorphism. Set $\eta : N \to M$ and $\eta(n) = m$ with η is a homomorphism. The equality,

$$\psi(m) - \theta\phi(\psi(m)) = \psi(m) - \theta(\phi\psi(m)) = \psi(m) - \theta(0) = \psi(m), \text{ for every } m \text{ in } M.$$

holds, since $\phi\psi(n) = 0$. It follows that $\psi(m) = \psi(m) - \theta\phi(\psi(m))$, and combining this equality with $\psi(m) = n - \theta\phi(n)$, we can deduce that $\psi(m) = n$. Thus $\eta(\psi(m)) = m$, so we have $\eta\psi = 1_M$. Since ψ is a monomorphism, then $\mathrm{Im}(\psi) \cong M$. Therefore, $N \cong M \oplus K$.

□

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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