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A Note on Horadam Hybrinomials

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Abstract

This paper ensures an extensive survey of the generalization of the various hybrid numbers and hybrid polynomials especially as part of its enhancing importance in the disciplines of mathematics and physics. In this paper, by using the Horadam polynomials, we define the Horadam hybrid polynomials called Horadam hybrinomials. We obtain some special cases and algebraic properties of the Horadam hybrinomials such as recurrence relation, generating function, exponential generating function, Binet formula, summation formulas, Catalan's identity, Cassini's identity and d'Ocagne's identity, respectively. Moreover, we give some applications related to the Horadam hybrinomials in matrices.

1. Introduction

Horadam defined the sequence $w_n = w_n(a,b;p,q)$ by the recurrence relation

$$w_n = pw_{n-1} + qw_{n-2}, \quad n \geq 2$$

with the initial values $w_0 = a$ and $w_1 = b$. For different values $p,q,a,b \in \mathbb{Z}$, Horadam sequence turns into several well-known sequences such as Fibonacci, Lucas, Pell and so on. These sequences are studied in many areas such as physics, number theory, algebra, geometry, and combinatorics. For more details, we refer to [1]-[6].
In [7], the Horadam polynomials $h_n(x) = h_n(x;a,b;p,q)$ are defined by the recurrence relation

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad n \geq 3 \tag{1.1}$$

with the initial values $h_1(x) = a$ and $h_2(x) = bx$. Let $\alpha = \frac{px+\sqrt{p^2x^2+4q}}{2}$ and $\beta = \frac{px-\sqrt{p^2x^2+4q}}{2}$ be the real roots of the characteristic equation $t^2 - pxt - q = 0$. The Binet formula for the polynomial $h_n(x)$ is given by

$$h_n(x) = A\alpha^{n-1} + B\beta^{n-1}, \tag{1.2}$$

where $A = \frac{bx-a\beta}{\sqrt{p^2x^2+4q}}$ and $B = \frac{a\alpha-bx}{\sqrt{p^2x^2+4q}}$.
The generating function of the Horadam polynomials is

$$\sum_{n=0}^{\infty} h_n(x)t^n = \frac{a + xt(b-ap)}{1 - pxt - qt^2}. \tag{1.3}$$

Hybrid numbers were studied by Ozdemir in [8], extensively. A hybrid number is defined as

$$\mathbb{K} = \left\{ a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a,b,c,d \in \mathbb{R},\ \mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = \mathbf{hi} = \varepsilon + \mathbf{i} \right\}.$$

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Addition and subtraction of hybrid numbers are done by adding and subtracting corresponding terms. Two hybrid numbers are equal if all their components are equal, one by one.
Using the equalities $\mathbf{i}^2 = -1$, $\varepsilon^2 = 0$, $\mathbf{h}^2 = 0$, $\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}$, the multiplication table of the basis of hybrid numbers is as follows:

Table 1: Multiplication table for $\mathbb{K}$

.	$\mathbf{1}$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{1}-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$\mathbf{h}+\mathbf{1}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	$\mathbf{1}$

Recently, many researchers have studied related to hybrid numbers. For example, in [9] Szynal-Liana and Wloch considered the Fibonacci hybrid numbers and obtained some properties of this numbers. In [10, 11] the authors also defined and examined the Jacosthal and Jacosthal–Lucas hybrid numbers and the Pell and Pell–Lucas hybrid numbers respectively. In [12] Szynal-Liana generalized their results and defined the Horadam hybrid numbers. In [13] Kızılateş introduced the another generalization of hybrid numbers and gave miscellaneous properties of these numbers. For more details, we refer to [8]-[23].
We now turn to a recent investigation by Szynal-Liana and Wloch [24], who defined and studied a family of the special polynomials and the special numbers which are related to the Fibonacci hybrinomials and Lucas hybrinomials. The Fibonacci hybrinomials and Lucas hybrinomials are defined as follows:

$$FH_n(x) = F_n(x) + F_{n+1}(x)\mathbf{i} + F_{n+2}(x)\varepsilon + F_{n+3}(x)\mathbf{h},$$

and

$$LH_n(x) = L_n(x) + L_{n+1}(x)\mathbf{i} + L_{n+2}(x)\varepsilon + L_{n+3}(x)\mathbf{h}.$$

For $n \geq 2$, the recurrence relations of the Fibonacci hybrinomials and the Lucas hybrinomials are

$$FH_n(x) = xFH_{n-1}(x) + FH_{n-2}(x),$$

and

$$LH_n(x) = xLH_{n-1}(x) + LH_{n-2}(x),$$

with the initial values $FH_0(x) = \mathbf{i} + x\varepsilon + (x^2+1)\mathbf{h}$, $FH_1(x) = 1 + x\mathbf{i} + (x^2+1)\varepsilon + (x^3+2x)\mathbf{h}$, $LH_0(x) = 2 + x\mathbf{i} + (x^2+2)\varepsilon + (x^3+3x)\mathbf{h}$ and $LH_1(x) = x + (x^2+2)\mathbf{i} + (x^3+3x)\varepsilon + (x^4+4x^2+2)\mathbf{h}$, respectively. The Fibonacci hybrinomials and the Lucas hybrinomials, namely polynomials, are a generalization of the Fibonacci hybrid and Lucas hybrid numbers.
Motivated by some of the above-mentioned recent papers, we introduce here new polynomials which are called Horadam hybrinomials. This definition brings about a more general hybrid polynomial sequence by taking components from Horadam polynomials. Thanks to this generalization, we obtain the Fibonacci hybrinomials $FH_n(x)$, the Lucas hybrinomials $LH_{n-1}(x)$, the Pell hybrinomials $PH_n(x)$, the Pell-Lucas hybrinomials $QH_{n-1}(x)$, the Chebyshev hybrinomials of the first kind $TH_{n-1}(x)$, the Chebyshev hybrinomials of the second kind $UH_{n-1}(x)$ and the Balancing hybrinomials $BH_n(x)$. We also obtain various results for the Horadam hybrinomials. Moreover, we give some applications of Horadam hybrinomials in matrices.

2. Horadam hybrinomials

In this section, we define the Horadam hybrinomials. Then we give some special cases of Horadam hybrinomials such as the Fibonacci hybrinomials, the Fibonacci hybrid numbers, the Lucas hybrinomials, the Lucas hybrid numbers, the Pell hybrinomials, the Pell hybrid numbers, the Pell-Lucas hybrinomials, the Pell-Lucas hybrid numbers, the Chebyshev hybrinomials of the first kind, the Chebyshev hybrid numbers of the first kind, the Chebyshev hybrinomials of the second kind, the Chebyshev hybrid numbers of the second kind, the Balancing hybrinomials and the Balancing hybrid numbers. Finally we obtain some algebraic properties of Horadam hybrinomials.

Definition 2.1. *For $n \geq 1$, the n^{th} Horadam hybrinomials are defined by*

$$\mathbb{H}_n(x) = h_n(x) + h_{n+1}(x)\mathbf{i} + h_{n+2}(x)\varepsilon + h_{n+3}(x)\mathbf{h}. \tag{2.1}$$

Some special cases of Horadam hybrinomials are as follows:

1. For $a = b = p = q = 1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Fibonacci hybrinomials $FH_n(x)$,
2. For $a = 2$ and $b = p = q = 1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Lucas hybrinomials $LH_{n-1}(x)$,

3. For $a=q=1$ and $b=p=2$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Pell hybrinomials $PH_n(x)$,
4. For $a=b=p=2$ and $q=1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Pell-Lucas hybrinomials $QH_{n-1}(x)$,
5. For $a=b=1$, $p=2$, and $q=-1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Chebyshev hybrinomials of the first kind $TH_{n-1}(x)$,
6. For $a=1$, $b=p=2$, and $q=-1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Chebyshev hybrinomials of the second kind $UH_{n-1}(x)$,
7. For $a=1$, $b=p=6$, and $q=-1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Balancing hybrinomials $BH_n(x)$,
8. For $x=1$, the Fibonacci hybrinomials $FH_n(x)$, reduce to the Fibonacci hybrid numbers FH_n,
9. For $x=1$, the Lucas hybrinomials $LH_{n-1}(x)$, reduce to the Lucas hybrid numbers LH_{n-1},
10. For $x=1$, the Pell hybrinomials $PH_n(x)$, reduce to the Pell hybrid numbers PH_n,
11. For $x=1$, the Pell-Lucas hybrinomials $QH_{n-1}(x)$, reduce to the Pell-Lucas hybrid numbers QH_{n-1},
12. For $x=1$, the Chebyshev hybrinomials of the first kind $TH_{n-1}(x)$, reduce to the Chebyshev hybrid numbers of the first kind TH_{n-1},
13. For $x=1$, the Chebyshev hybrinomials of the second kind $UH_{n-1}(x)$, reduce to the Chebyshev hybrid numbers of the second kind UH_{n-1},
14. For $x=1$, the Balancing hybrinomials $BH_n(x)$, reduce to the Balancing hybrid numbers BH_n.

Using (2.1) and (1.1), we obtain that for $n>2$,

$$\begin{aligned}\mathbb{H}_n(x) &= pxh_{n-1}(x)+qh_{n-2}(x)+(pxh_n(x)+qh_{n-1}(x))\,\mathbf{i}\\ &\quad +(pxh_{n+1}(x)+qh_n(x))\,\varepsilon+(pxh_{n+2}(x)+qh_{n+1}(x))\,\mathbf{h}\\ &= px\mathbb{H}_{n-1}(x)+q\mathbb{H}_{n-2}(x)\end{aligned}$$

and so

$$\mathbb{H}_n(x)=px\mathbb{H}_{n-1}(x)+q\mathbb{H}_{n-2}(x),$$

with the initial values $\mathbb{H}_1(x)=a+bx\mathbf{i}+(bpx^2+aq)\varepsilon+(bp^2x^3+(apq+bq)x)\mathbf{h}$ and $\mathbb{H}_2(x)=bx+(bpx^2+aq)\mathbf{i}+(bp^2x^3+(apq+bq)x)\varepsilon+(bp^3x^4+(ap^2q+2bpq)x^2+aq^2)\mathbf{h}$.

Theorem 2.2. *The Binet formula for the Horadam hybrinomial $\mathbb{H}_n(x)$ is*

$$\mathbb{H}_n(x)=A\alpha^{n-1}\tilde{\alpha}+B\beta^{n-1}\tilde{\beta}, \tag{2.2}$$

where $\tilde{\alpha}=1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3\mathbf{h}$ and $\tilde{\beta}=1+\beta\mathbf{i}+\beta^2\varepsilon+\beta^3\mathbf{h}$.

Proof. Due to (1.2) and (2.1), we find that

$$\begin{aligned}\mathbb{H}_n(x) &= (A\alpha^{n-1}+B\beta^{n-1})+(A\alpha^n+B\beta^n)\mathbf{i}+(A\alpha^{n+1}+B\beta^{n+1})\varepsilon+(A\alpha^{n+2}+B\beta^{n+2})\mathbf{h}\\ &= A\alpha^{n-1}(1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3\mathbf{h})+B\beta^{n-1}(1+\beta\mathbf{i}+\beta^2\varepsilon+\beta^3\mathbf{h})\\ &= A\alpha^{n-1}\tilde{\alpha}+B\beta^{n-1}\tilde{\beta}.\end{aligned}$$

□

We now give the generating function and exponential generating function for the Horadam hybrinomials.

Theorem 2.3. *The generating function for the Horadam hybrinomial $\mathbb{H}_n(x)$ is*

$$\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=\frac{\mathbb{H}_0(x)+(\mathbb{H}_1(x)-px\mathbb{H}_0(x))\,t}{1-pxt-qt^2}. \tag{2.3}$$

Proof. Suppose that the generating function for the Horadam hybrinomials $\left\{\mathbb{H}_n(x)_{n=0}^{\infty}\right\}$, has the following formal power series

$$\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=\mathbb{H}_0(x)+\mathbb{H}_1(x)t+\cdots+\mathbb{H}_k(x)t^k+\cdots. \tag{2.4}$$

Hence

$$pxt\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=px\mathbb{H}_0(x)t+px\mathbb{H}_1(x)t^2+\cdots+px\mathbb{H}_k(x)t^{k+1}+\cdots, \tag{2.5}$$

$$qt^2\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=q\mathbb{H}_0(x)t^2+q\mathbb{H}_1(x)t^3+\cdots+q\mathbb{H}_k(x)t^{k+2}+\cdots. \tag{2.6}$$

From (2.4), (2.5) and (2.6), we find that

$$(1-pxt-qt^2)\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=\mathbb{H}_0(x)+(\mathbb{H}_1(x)-px\mathbb{H}_0(x))t.$$

So

$$\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=\frac{\mathbb{H}_0(x)+(\mathbb{H}_1(x)-px\mathbb{H}_0(x))t}{1-pxt-qt^2}.$$

□

Corollary 2.4. *([24, Theorem 2.10]) The generating function for the Fibonacci hybrinomial $FH_n(x)$ is*

$$\sum_{n=0}^{\infty}FH_n(x)t^n=\frac{\mathbf{i}+x\varepsilon+(x^2+1)\mathbf{h}+(1+\varepsilon+x\mathbf{h})t}{1-xt-t^2}.$$

Proof. If we take $a=b=p=q=1$ in Equation (2.3), the proof is completed. □

Corollary 2.5. *([24, Theorem 2.11]) The generating function for the Lucas hybrinomial $LH_n(x)$ is*

$$\sum_{n=0}^{\infty}LH_n(x)t^n=\frac{LH_0(x)+(LH_1(x)-xLH_0(x))t}{1-xt-t^2}.$$

Proof. If we take $a=2$ and $b=p=q=1$ in Equation (2.3), the proof is completed. □

Theorem 2.6. *The exponential generating function for the Horadam hybrinomial $\mathbb{H}_n(x)$ is*

$$\sum_{n=0}^{\infty}\mathbb{H}_n(x)\frac{t^n}{n!}=A\alpha^{-1}\tilde{\alpha}e^{\alpha t}+B\beta^{-1}\tilde{\beta}e^{\beta t}.$$

Proof. Using the Equation (2.2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{H}_n(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}(A\alpha^{n-1}\tilde{\alpha}+B\beta^{n-1}\tilde{\beta})\frac{t^n}{n!}\\ &= \frac{A\tilde{\alpha}}{\alpha}\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!}+\frac{B\tilde{\beta}}{\beta}\sum_{n=0}^{\infty}\frac{(\beta t)^n}{n!}\\ &= \frac{A\tilde{\alpha}}{\alpha}e^{\alpha t}+\frac{B\tilde{\beta}}{\beta}e^{\beta t}\\ &= A\alpha^{-1}\tilde{\alpha}e^{\alpha t}+B\beta^{-1}\tilde{\beta}e^{\beta t}.\end{aligned}$$

So the proof is completed. □

We now give the following interesting identities.

Theorem 2.7. *(Catalan's Identity). For positive integers n and r, with $n\geq r$, the following identity is true:*

$$\mathbb{H}_{n+r}(x)\mathbb{H}_{n-r}(x)-\mathbb{H}_n^2(x)=(-q)^{n-1}AB\left(\tilde{\alpha}\tilde{\beta}\left(\left(\frac{\beta}{\alpha}\right)^r-1\right)+\tilde{\beta}\tilde{\alpha}\left(\left(\frac{\alpha}{\beta}\right)^r-1\right)\right). \tag{2.7}$$

Proof. Using the Equation (2.2), we obtain the LHS of the equality (2.7),

$$\begin{aligned}\mathbb{H}_{n+r}(x)\mathbb{H}_{n-r}(x)-\mathbb{H}_n^2(x) &= \left(A\alpha^{n-r-1}\tilde{\alpha}+B\beta^{n-r-1}\tilde{\beta}\right)\left(A\alpha^{n+r-1}\tilde{\alpha}+B\beta^{n+r-1}\tilde{\beta}\right)\\ &\quad-\left(A\alpha^{n-1}\tilde{\alpha}+B\beta^{n-1}\tilde{\beta}\right)^2\\ &= AB(\alpha\beta)^{n-1}\alpha^{-r}\beta^{r}\tilde{\alpha}\tilde{\beta}+BA(\beta\alpha)^{n-1}\beta^{-r}\alpha^{r}\tilde{\beta}\tilde{\alpha}\\ &\quad-AB(\alpha\beta)^{n-1}\tilde{\alpha}\tilde{\beta}-BA(\beta\alpha)^{n-1}\tilde{\beta}\tilde{\alpha}.\end{aligned}$$

Then, we have

$$\mathbb{H}_{n+r}(x)\mathbb{H}_{n-r}(x)-\mathbb{H}_n^2(x)=(-q)^{n-1}AB\left(\tilde{\alpha}\tilde{\beta}\left(\left(\frac{\beta}{\alpha}\right)^r-1\right)+\tilde{\beta}\tilde{\alpha}\left(\left(\frac{\alpha}{\beta}\right)^r-1\right)\right).$$

□

Theorem 2.8. *(Cassini's Identity). For $n \geq 1$, the following equality holds:*

$$\mathbb{H}_{n+1}(x)\mathbb{H}_{n-1}(x) - \mathbb{H}_n^2(x) = (-q)^{n-1}AB\left(\tilde{\alpha}\tilde{\beta}\left(\frac{\beta}{\alpha}-1\right)+\tilde{\beta}\tilde{\alpha}\left(\frac{\alpha}{\beta}-1\right)\right). \tag{2.8}$$

Proof. If we take $r = 1$, in (2.7), we obtain the assertion of the theorem. □

Theorem 2.9. *(d'Ocagne's Identity) Let $m \geq 0$ and $n \geq 0$ be integers such that $m > n+1$. Then we have*

$$\mathbb{H}_m(x)\mathbb{H}_{n+1}(x) - \mathbb{H}_{m+1}(x)\mathbb{H}_n(x) = \sqrt{\Delta}AB(-q)^{n-1}\left(\beta^{m-n}\tilde{\beta}\tilde{\alpha} - \alpha^{m-n}\tilde{\alpha}\tilde{\beta}\right), \tag{2.9}$$

where $\Delta = p^2x^2 + 4q$.

Proof. By virtue of Equation (2.2), we get

$$\begin{aligned}
\mathbb{H}_m(x)\mathbb{H}_{n+1}(x) - \mathbb{H}_{m+1}(x)\mathbb{H}_n(x) &= \left(A\alpha^{m-1}\tilde{\alpha} + B\beta^{m-1}\tilde{\beta}\right)\left(A\alpha^{n}\tilde{\alpha} + B\beta^{n}\tilde{\beta}\right) \\
&\quad - \left(A\alpha^{m}\tilde{\alpha} + B\beta^{m}\tilde{\beta}\right)\left(A\alpha^{n-1}\tilde{\alpha} + B\beta^{n-1}\tilde{\beta}\right) \\
&= AB\alpha^{m-1}\beta^{n}\tilde{\alpha}\tilde{\beta} - AB\alpha^{m}\beta^{n-1}\tilde{\alpha}\tilde{\beta} \\
&\quad + BA\alpha^{n}\beta^{m-1}\tilde{\beta}\tilde{\alpha} - BA\alpha^{n-1}\beta^{m}\tilde{\beta}\tilde{\alpha}.
\end{aligned}$$

After some calculations, we can easily see that

$$\mathbb{H}_m(x)\mathbb{H}_{n+1}(x) - \mathbb{H}_{m+1}(x)\mathbb{H}_n(x) = \sqrt{\Delta}AB(-q)^{n-1}\left(\beta^{m-n}\tilde{\beta}\tilde{\alpha} - \alpha^{m-n}\tilde{\alpha}\tilde{\beta}\right).$$

□

If we take $a = b = p = q = 1$ in (2.7), (2.8) and (2.9), we obtain the Catalan, the Cassini and the d'Ocagne identities for the Fibonacci hybrinomials [24, Theorem 2.4], [24, Corollary 2.6] and [24, Theorem 2.7], respectively. Similarly, if we take $a = 2$ and $b = p = q = 1$ in (2.7), (2.8) and (2.9), we obtain the Catalan, the Cassini and the d'Ocagne identities for the Lucas hybrinomials [24, Theorem 2.5], [24, Corollary 2.6] and [24, Theorem 2.9], respectively.

Theorem 2.10. *Let $n \geq 2$ be an integer. Then we obtain*

$$\sum_{k=1}^{n-1}\mathbb{H}_k(x) = \frac{\mathbb{H}_1(x) - \mathbb{H}_n(x) + q\left(\mathbb{H}_0(x) - \mathbb{H}_{n-1}(x)\right)}{1-px-q}. \tag{2.10}$$

Proof. By virtue of Equation (2.2), we find that

$$\begin{aligned}
\sum_{k=1}^{n-1}\mathbb{H}_k(x) &= \sum_{k=1}^{n-1}\left(A\alpha^{k-1}\tilde{\alpha} + B\beta^{k-1}\tilde{\beta}\right) \\
&= A\tilde{\alpha}\sum_{k=1}^{n-1}\alpha^{k-1} + B\tilde{\beta}\sum_{k=1}^{n-1}\beta^{k-1} \\
&= A\tilde{\alpha}\left(\frac{1-\alpha^{n-1}}{1-\alpha}\right) + B\tilde{\beta}\left(\frac{1-\beta^{n-1}}{1-\beta}\right) \\
&= \frac{A\tilde{\alpha}(1-\beta)(1-\alpha^{n-1}) + B\tilde{\beta}(1-\alpha)\left(1-\beta^{n-1}\right)}{1-px-q}.
\end{aligned}$$

Utilizing the last equation, we have

$$\sum_{k=1}^{n-1}\mathbb{H}_k(x) = \frac{\mathbb{H}_1(x) - \mathbb{H}_n(x) + q\left(\mathbb{H}_0(x) - \mathbb{H}_{n-1}(x)\right)}{1-px-q}.$$

□

Corollary 2.11. *([24, Theorem 2.13]) Let $n \geq 2$ be an integer. Then we have*

$$\sum_{k=1}^{n-1}FH_k(x) = \frac{FH_n(x) + FH_{n-1}(x) - FH_0(x) - FH_1(x)}{x}.$$

Proof. If we take $a = b = p = q = 1$ in Equation (2.10), the proof is completed. □

Corollary 2.12. *([24, Theorem 2.15]) Let $n \geq 2$ be an integer. Then we have*

$$\sum_{k=1}^{n-1} LH_k(x) = \frac{LH_n(x) + LH_{n-1}(x) - LH_0(x) - LH_1(x)}{x}.$$

Proof. If we take $a = 2$ and $b = p = q = 1$ in Equation (2.10), the proof is completed. □

Theorem 2.13. *For $n \geq 0$, we have*

$$q^n \sum_{i=0}^{n} \binom{n}{i} \left(\frac{px}{q}\right)^{n-i} \mathbb{H}_{n-i}(x) = \mathbb{H}_{2n}(x). \tag{2.11}$$

Proof. Because of the Binet formula of the Horadam hybrinomials, we have the LHS of the equality (2.11),

$$\begin{aligned} & q^n \sum_{i=0}^{n} \binom{n}{i} (px)^{n-i} q^i \left(A\alpha^{n-i-1}\tilde{\alpha} + B\beta^{n-i-1}\tilde{\beta}\right) \\ = \; & A\tilde{\alpha}\alpha^{-1} \sum_{i=0}^{n} \binom{n}{i} (px\alpha)^{n-i} q^i + B\tilde{\beta}\beta^{-1} \sum_{i=0}^{n} \binom{n}{i} (px\beta)^{n-i} q^i \\ = \; & A\tilde{\alpha}\alpha^{-1} (px\alpha + q)^n + B\tilde{\beta}\beta^{-1} (px\beta + q)^n \\ = \; & A\tilde{\alpha}\alpha^{2n-1} + B\tilde{\beta}\beta^{2n-1} \\ = \; & \mathbb{H}_{2n}(x). \end{aligned}$$

Thus the proof is completed. □

Corollary 2.14. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} \binom{n}{i} x^{n-i} FH_{n-i}(x) = FH_{2n}(x).$$

Proof. If we take $a = b = p = q = 1$ in Equation (2.11), the proof is completed. □

Corollary 2.15. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} \binom{n}{i} (2x)^{n-i} PH_{n-i}(x) = PH_{2n}(x).$$

Proof. If we take $a = q = 1$ and $b = p = 2$ in Equation (2.11), the proof is completed. □

Corollary 2.16. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} (-1)^n \binom{n}{i} (-6x)^{n-i} BH_{n-i}(x) = BH_{2n}(x).$$

Proof. If we take $a = 1$, $b = p = 6$, and $q = -1$ in Equation (2.11), the proof is completed. □

3. An application of Horadam hybrinomials in matrices

In this section, we derive the matrix representation of the Horadam hybrinomials. Then we obtain closed formula for the Horadam hybrinomials $\mathbb{H}_n(x)$, in terms of tridiagonal determinant (see [26]-[28]).

Theorem 3.1. *Let $n \geq 1$ be an integer. The following equality holds:*

$$\begin{bmatrix} \mathbb{H}_{n+3}(x) & \mathbb{H}_{n+2}(x) \\ \mathbb{H}_{n+2}(x) & \mathbb{H}_{n+1}(x) \end{bmatrix} = \begin{bmatrix} \mathbb{H}_3(x) & \mathbb{H}_2(x) \\ \mathbb{H}_2(x) & \mathbb{H}_1(x) \end{bmatrix} \begin{bmatrix} px & 1 \\ q & 0 \end{bmatrix}^n. \tag{3.1}$$

Proof. For the proof, we use induction method on n. The equality holds for $n = 1$. Now suppose that the equality is true for $n > 1$. Then we can verify it for $n + 1$ as follows:

$$\begin{aligned} \begin{bmatrix} \mathbb{H}_3(x) & \mathbb{H}_2(x) \\ \mathbb{H}_2(x) & \mathbb{H}_1(x) \end{bmatrix} \begin{bmatrix} px & 1 \\ q & 0 \end{bmatrix}^{n+1} &= \begin{bmatrix} \mathbb{H}_3(x) & \mathbb{H}_2(x) \\ \mathbb{H}_2(x) & \mathbb{H}_1(x) \end{bmatrix} \begin{bmatrix} px & 1 \\ q & 0 \end{bmatrix}^{n} \begin{bmatrix} px & 1 \\ q & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{H}_{n+3}(x) & \mathbb{H}_{n+2}(x) \\ \mathbb{H}_{n+2}(x) & \mathbb{H}_{n+1}(x) \end{bmatrix} \begin{bmatrix} px & 1 \\ q & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{H}_{n+4}(x) & \mathbb{H}_{n+3}(x) \\ \mathbb{H}_{n+3}(x) & \mathbb{H}_{n+2}(x) \end{bmatrix}. \end{aligned}$$

So the proof is completed. □

Corollary 3.2. *([24, Theorem 2.16]) Let $n \geq 1$ be an integer. The following equality holds:*

$$\begin{bmatrix} FH_{n+3}(x) & FH_{n+2}(x) \\ FH_{n+2}(x) & FH_{n+1}(x) \end{bmatrix} = \begin{bmatrix} FH_3(x) & FH_2(x) \\ FH_2(x) & FH_1(x) \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Proof. If we take $a = b = p = q = 1$ in Equation (3.1), the proof is completed. □

Corollary 3.3. *([24, Theorem 2.17]) Let $n \geq 1$ be an integer. The following equality holds:*

$$\begin{bmatrix} LH_{n+3}(x) & LH_{n+2}(x) \\ LH_{n+2}(x) & LH_{n+1}(x) \end{bmatrix} = \begin{bmatrix} LH_3(x) & LH_2(x) \\ LH_2(x) & LH_1(x) \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Proof. If we take $a = 2$ and $b = p = q = 1$ in Equation (3.1), the proof is completed. □

The n^{th} term of Horadam hybrinomial can be obtained via the computation of the determinant of the tridiagonal matrix $M_{n-1}(x)$.

Proposition 3.4. *The $n \times n$ tridiagonal matrices*

$$M\mathbb{H}_n(x) = \begin{pmatrix} \mathbb{H}_2(x) & \mathbb{H}_1(x) & & & & & \\ -q & px & 1 & & & & \\ & -q & px & 1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -q & px & 1 \\ & & & & & -q & px \end{pmatrix}, \tag{3.2}$$

satisfy

$$|M\mathbb{H}_n(x)| = \mathbb{H}_{n+1}(x).$$

Corollary 3.5. *The $n \times n$ tridiagonal matrices*

$$MF_n(x) = \begin{pmatrix} FH_2(x) & FH_1(x) & & & & & \\ -1 & x & 1 & & & & \\ & -1 & x & 1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & x & 1 \\ & & & & & -1 & x \end{pmatrix},$$

satisfy

$$|MF_n(x)| = FH_{n+1}(x).$$

Proof. If we take $a = b = p = q = 1$ in Equation (3.2), the proof is completed. □

Corollary 3.6. *The $n \times n$ tridiagonal matrices*

$$MP_n(x) = \begin{pmatrix} PH_2(x) & PH_1(x) & & & & & \\ -1 & 2x & 1 & & & & \\ & -1 & 2x & 1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2x & 1 \\ & & & & & -1 & 2x \end{pmatrix},$$

satisfy

$$|MP_n(x)| = PH_{n+1}(x).$$

Proof. If we take $a = q = 1$ and $b = p = 2$ in Equation (3.2), the proof is completed. □

Corollary 3.7. *The $n \times n$ tridiagonal matrices*

$$MB_n(x) = \begin{pmatrix} BH_2(x) & BH_1(x) & & & & & \\ 1 & 6x & 1 & & & & \\ & 1 & 6x & 1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 6x & 1 \\ & & & & & 1 & 6x \end{pmatrix},$$

satisfy

$$|MB_n(x)| = BH_{n+1}(x).$$

Proof. If we take $a = 1$, $b = p = 6$ and $q = -1$ in Equation (3.2), the proof is completed. □

Note that, Horadam hybrinomial can be obtained using the another tridiagonal matrix.

Proposition 3.8. *For $n \geq 1$, we have*

$$\mathbb{H}_n(x) = \begin{vmatrix} \mathbb{H}_1(x) & \mathbb{H}_2(x) & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & q & 0 & \cdots & 0 & 0 \\ 0 & -1 & px & q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & px & q \\ 0 & 0 & 0 & 0 & \cdots & -1 & px \end{vmatrix}_{n\times n}. \tag{3.3}$$

Corollary 3.9. *For $n \geq 1$, we have*

$$FH_n(x) = \begin{vmatrix} FH_1(x) & FH_2(x) & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & x \end{vmatrix}_{n\times n}.$$

Proof. This follows from setting $a = b = p = q = 1$ in the Equation (3.3). □

Corollary 3.10. *For $n \geq 1$, we have*

$$PH_n(x) = \begin{vmatrix} PH_1(x) & PH_2(x) & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2x & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2x \end{vmatrix}_{n\times n}.$$

Proof. This follows from taking $a = q = 1$ and $b = p = 2$ in the Equation (3.3). □

Corollary 3.11. *For $n \geq 1$, we have*

$$BH_n(x) = \begin{vmatrix} BH_1(x) & BH_2(x) & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 6x & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 6x & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 6x \end{vmatrix}_{n\times n}.$$

Proof. This follows from setting $a = 1$, $b = p = 6$, and $q = -1$ in the Equation (3.3). □

Remark 3.12. *This paper is a slightly corrected and revised version of the electronic preprint [29].*

4. Conclusion

In our present research, we have studied Horadam hybrinomials which are defined by dint of the Horadam polynomials. We have obtained some properties of Horadam hybrinomials. Finally in Section 3, with the help of the two different tridiagonal matrix, we have obtained the n^{th} term of Horadam hybrinomials. According to the special cases of a, b, p and q, all the results given in Section 2 and Section 3 are applicable to all hybrinomials and hybrid numbers mentioned in this paper. The Horadam hybrinomials that we have defined include previously introduced the Fibonacci hybrinomials $FH_n(x)$, the Fibonacci hybrid numbers FH_n, the Lucas hybrinomials $LH_{n-1}(x)$, the Lucas hybrid numbers LH_{n-1}, the Pell hybrinomials $PH_n(x)$, the Pell hybrid numbers PH_n, the Pell-Lucas hybrinomials $QH_{n-1}(x)$, the Pell-Lucas hybrid numbers QH_{n-1} (see, [24, 25]). From the definition of the Horadam hybrinomials, we also have obtained the Chebyshev hybrinomials of the first kind $TH_{n-1}(x)$, the Chebyshev hybrid numbers of the first kind TH_{n-1}, the Chebyshev hybrinomials of the second kind $UH_{n-1}(x)$, the Chebyshev hybrid numbers of the second kind UH_{n-1}, the Balancing hybrinomials $BH_n(x)$ and the Balancing hybrid numbers BH_n.

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