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Generalized Bertrand and Mannheim Curves in $3D$ Lie Groups

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Abstract

In this paper, we give a new approach for Bertrand and Mannheim curves in $3D$ Lie groups with bi-invariant metrics. In this way, some conditions including the known results have been given for a curve to be Bertrand or Mannheim curve in $3D$ Euclidean space and in $3D$ Lie groups.

1. Introduction

The curve and surface theory is a comprehensive field in differential geometry. Especially associated curves whose the Frenet apparatus satisfy some geometric conditions in Euclidean 3-space. For examples, general helix is a curve, whose tangent vector makes a constant angle with a fixed straight line. Lancret gave the condition for a given curve to be a general helices by the ratio of its curvatures to be constant [1]. In [2], a different approach is given to a general helix lying on a sphere. Also, slant helix was defined as a curve whose normal vector makes a constant angle with a fixed straight line in $3D$ Euclidean space by Izumiya and Takeuchi [3]. They showed that a curve is a slant helix iff the geodesic curvature of spherical image of principal normal indicatrix of the curve is a constant function. On the other hand, there exist some examples of associated curves such as Bertrand and Mannheim curve couples whose the Frenet apparatus satisfy some geometric conditions in $3D$ Euclidean space. Bertrand curve couples defined by J. M. Bertrand in 1845 [4]. "If the normal vectors of the two curves are coincide at the corresponding points of the curves, we say that these curves are Bertrand curve couple". Liu and Wang defined Mannheim curve couples in 2008 [5]. "If the normal vector of a given curve is coincide with an other curve's bi-normal vector at the corresponding points of the curves, we say that these curves are Mannheim curve couple." They gave a condition for a given curve to be a Mannheim curve. Also, [6]-[9] can be looked at for examining Bertand and Mannheim curves in different spaces. Recently, Ç. Camcı et.all and A. Uçum et.all gave a generalization for Bertrand and Mannheim curves in $3D$ Euclidean space, respectively [10, 11].
Lie groups are an important mathematical form because they have three different structures in mathematics such that S^3, $SO(3)$ and Abelian Lie groups. In addition, Lie groups have a wide range of theory and application in physics and mechanics, as well as their importance in mathematics. Some associated curves such as general helices, slant helices, Bertrand and Mannheim curves are introduced in the Lie groups [12]-[15]. On the other hand, different structures such as spinor representations, curve flows in Lie groups and the conjugate mate structures of curves were investigated in [16]-[19]. And it has been shown that the conditions obtained in $3D$ Lie groups are a generalization of the conditions obtained in $3D$ Euclidean space. In [20], Lie algebras and their applications related to dynamical structures are given.
In this paper, we introduce a generalization for Bertrand and Mannheim curves in $3D$ Lie groups, respectively. Also, we obtain some characterizations of these curves. Moreover, we give some results about this curves for special cases of $3D$ Lie groups.

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2. Preliminaries

Assume that $\mathbb{G}$ be the $3D$ Lie group with bi-invariant metric $\langle , \rangle$ and ∇ be the Levi-Civita connection of Lie group $\mathbb{G}$. The Lie algebra of $\mathbb{G}$ denotes with $\mathfrak{g}$, which is isomorphic to $T_f\mathbb{G}$, where f is neutral element of $\mathbb{G}$. As the metric is bi-invariant, we have the following equations for all $\mathbb{P}, \mathbb{Q}, \mathbb{R} \in \mathfrak{g}$;

$$\langle \mathbb{P}, [\mathbb{Q}, \mathbb{R}] \rangle = \langle [\mathbb{P}, \mathbb{Q}], \mathbb{R} \rangle$$

and

$$\nabla_{\mathbb{P}}\mathbb{Q} = \frac{1}{2}[\mathbb{P}, \mathbb{Q}].$$

Let $E_1, E_2, ..., E_n$ be an orthonormal basis of $\mathfrak{g}$ and γ be an an arc-lengthed curve on $\mathbb{G}$. Then, we can write any two vector fields $\mathbb{Y}_1$ and $\mathbb{Y}_2$ along γ as $\mathbb{Y}_1 = \sum_{i=1}^{n} a_i E_i$ and $\mathbb{Y}_2 = \sum_{i=1}^{n} b_i E_i$ where a_i and b_i are real-valued smooth functions. Also, the Lie bracket of $\mathbb{Y}_1$ and $\mathbb{Y}_2$ is given

$$[\mathbb{Y}_1, \mathbb{Y}_2] = \sum_{i=1}^{n} a_i b_i [E_i, E_j]$$

On the other hand, the covariant derivative of $\mathbb{Z}$ along γ is given by

$$\nabla_{\gamma'}\mathbb{Z} = \dot{\mathbb{Z}} + \frac{1}{2}[\mathbf{t}, \mathbb{Z}] \tag{2.1}$$

where $\mathbf{t} = \gamma'$ and $\dot{\mathbb{Z}} = \sum_{i=1}^{n} \frac{dz}{dt} E_i$. Also, if $\mathbb{Z}$ is the left-invariant vector field, then $\dot{\mathbb{Z}} = 0$ [21].
Let γ be a curve with Frenet apparatus $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau\}$ in Lie group $\mathbb{G}$. Then the Frenet-Serret formulas are expressed by

$$\nabla_{\mathbf{t}}\mathbf{t} = \kappa\mathbf{n}, \quad \nabla_{\mathbf{t}}\mathbf{n} = -\kappa\mathbf{t} + \tau\mathbf{b}, \quad \nabla_{\mathbf{t}}\mathbf{b} = -\tau\mathbf{n}$$

where ∇ is connection of Lie group $\mathbb{G}$ and $\kappa = \|\dot{\mathbf{t}}\|$.

Proposition 2.1. *[12] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a curve in Lie group $\mathbb{G}$ with the Frenet apparatus $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau\}$. Then Lie curvature $\tau_{\mathbb{G}}$ is defined by*

$$\tau_{\mathbb{G}} = \frac{1}{2}\langle [\mathbf{t}, \mathbf{n}], \mathbf{b} \rangle.$$

Proposition 2.2. *[13] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be an arc length parametrized curve with the Frenet apparatus $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. Then the following equalities*

$$[\mathbf{t}, \mathbf{n}] = \langle [\mathbf{t}, \mathbf{n}], \mathbf{b} \rangle \mathbf{b} = 2\tau_{\mathbb{G}}\mathbf{b}$$

$$[\mathbf{t}, \mathbf{b}] = \langle [\mathbf{t}, \mathbf{b}], \mathbf{n} \rangle \mathbf{n} = -2\tau_{\mathbb{G}}\mathbf{n}$$

hold.

Remark 2.3. *[12, 22] The follows hold for Lie group $\mathbb{G}$ with bi-invariant metric in special cases:*
(i) Let $\mathbb{G}$ is an Abelian group, then $\tau_{\mathbb{G}} = 0$,
(ii) Let $\mathbb{G}$ is SU^2, then $\tau_{\mathbb{G}} = 1$,
(iii) Let $\mathbb{G}$ is SO^3, then $\tau_{\mathbb{G}} = \frac{1}{2}$.

Theorem 2.4. *[12] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a curve in Lie group $\mathbb{G}$ with the curvatures κ, τ and Lie curvature $\tau_{\mathbb{G}}$. Then, γ is a general helix iff*

$$\frac{\tau - \tau_{\mathbb{G}}}{\kappa} = constant$$

Theorem 2.5. *[13] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a curve in Lie group $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures κ, τ and Lie curvature $\tau_{\mathbb{G}}$. Then γ is a slant helix iff*

$$\frac{\kappa\left(1 + \left(\frac{\tau - \tau_{\mathbb{G}}}{\kappa}\right)^2\right)^{\frac{3}{2}}}{\left(\frac{\tau - \tau_{\mathbb{G}}}{\kappa}\right)'} = constant$$

Theorem 2.6. *[14] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a curve in Lie group $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures κ, τ and Lie curvature $\tau_{\mathbb{G}}$. Then, γ is Mannheim curve iff*

$$\lambda\kappa\left(1+\left(\frac{\tau-\tau_{\mathbb{G}}}{\kappa}\right)^2\right)=1$$

where λ is constant.

Theorem 2.7. *[15] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in Lie group $\mathbb{G}$ with the curvatures κ, τ and Lie curvature $\tau_{\mathbb{G}}$. Then, γ satisfy the following equality*

$$\lambda\kappa+\mu(\tau-\tau_{\mathbb{G}})=1$$

where λ, μ are constants.

3. Generalized Bertrand curves in $3D$ Lie groups

In this section, we investigate generalized Bertrand curves in $3D$ Lie groups and we give some characterizations.

Definition 3.1. *A curve $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ is a Bertrand curve if there exists a special curve $\overline{\gamma} : \overline{J} \subset \mathbb{R} \to \mathbb{G}$ and a bijection $\zeta : \gamma \to \overline{\gamma}$ where $\mathbf{n}(s)$ and $\overline{\mathbf{n}}(\overline{s})$ at $s \in J$, $\overline{s} \in \overline{J}$ coincide. Also, $\overline{\gamma}(\overline{s})$ is called the Bertrand mate of $\gamma(s)$ in Lie group $\mathbb{G}$.*

Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the Frenet apparatus $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ and the curvatures $\kappa,\tau \neq 0$ and $\overline{\gamma}(\overline{s})$ be a Bertrand mate curve of γ with the Frenet apparatus $\{\overline{\mathbf{t}},\overline{\mathbf{n}},\overline{\mathbf{b}}\}$ and the curvatures $\overline{\kappa},\overline{\tau} \neq 0$. We can write as

$$\overline{\gamma}(\overline{s})=\overline{\gamma}(\sigma(s))=\gamma(s)+a(s)\mathbf{t}(s)+b(s)\mathbf{n}(s)+c(s)\mathbf{b}(s) \tag{3.1}$$

where $a(s), b(s)$ and $c(s)$ are differentiable functions on J.

Theorem 3.2. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa,\tau \neq 0$. γ is a Bertrand curve with Bertrand mate $\overline{\gamma}$ iff one of the followings holds:*
i. The differentiable functions a, b and c satisfy the following equations:

$$a\kappa+b'-c(\tau-\tau_G)=0 \quad and \quad c'+b(\tau-\tau_{\mathbb{G}})=0 \tag{3.2}$$

ii. The differentiable functions a, b, c and real number ℓ satisfy the following equations:

$$a\kappa+b'-c(\tau-\tau_{\mathbb{G}})=0, \quad c'+b(\tau-\tau_{\mathbb{G}})\neq 0 \tag{3.3}$$

$$1+a'-b\kappa=\ell(c'+b(\tau-\tau_{\mathbb{G}})), \quad \ell\kappa-(\tau-\tau_{\mathbb{G}})\neq 0, \quad \kappa+\ell(\tau-\tau_{\mathbb{G}})\neq 0$$

Proof. Let us assume that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa,\tau \neq 0$. By differentiating equation (3.1), we get

$$\frac{d\overline{\gamma}(\overline{s})}{d\overline{s}}\sigma'=\frac{d\gamma(s)}{ds}+a'(s)\mathbf{t}(s)+a(s)\dot{\mathbf{t}}(s)+b'(s)\mathbf{n}(s)+b(s)\dot{\mathbf{n}}(s)+c'(s)\mathbf{b}(s)+c(s)\dot{\mathbf{b}}(s). \tag{3.4}$$

By using equation (2.1) and Proposition 2.1, we have

$$\overline{\mathbf{t}}\sigma'=(1+a'-b\kappa)\mathbf{t}+(a\kappa+b'-c(\tau-\tau_{\mathbb{G}}))\mathbf{n}+(c'+b(\tau-\tau_{\mathbb{G}}))\mathbf{b} \tag{3.5}$$

By taking the inner product of equation (3.5) with $\mathbf{n}$, we have

$$a\kappa+b'-c(\tau-\tau_{\mathbb{G}})=0$$

Therefore, we get

$$\overline{\mathbf{t}}\sigma'=(1+a'-b\kappa)\mathbf{t}+(c'+b(\tau-\tau_{\mathbb{G}}))\mathbf{b} \tag{3.6}$$

It is clear that,

$$(\sigma')^2=(1+a'-b\kappa)^2+(c'+b(\tau-\tau_{\mathbb{G}}))^2 \tag{3.7}$$

Then, we can write as

$$\overline{\mathbf{t}}=\lambda_1\mathbf{t}+\lambda_2\mathbf{b} \tag{3.8}$$

for

$$\lambda_1 = \frac{1+a^{'}-b\kappa}{\sigma^{'}} \quad and \quad \lambda_2 = \frac{c^{'}+b(\tau-\tau_{\mathbb{G}})}{\sigma^{'}} \tag{3.9}$$

By differentiating equation (3.8) in $\mathbb{G}$, we get

$$\sigma^{'}\overline{\kappa}\overline{\mathbf{n}} = \lambda_1^{'}\mathbf{t} + (\lambda_1\kappa - \lambda_2(\tau-\tau_{\mathbb{G}}))\mathbf{n} + \lambda_2^{'}\mathbf{b} \tag{3.10}$$

This shows that $\lambda_1^{'}=0$ and $\lambda_2^{'}=0$.
i. Let us suppose that $\lambda_2 = 0$. Therefore, we have $c^{'}+b(\tau-\tau_{\mathbb{G}})=0$.
ii. Let us suppose that $\lambda_2 \neq 0$. Then, we can write

$$1+a^{'}-b\kappa = \ell(c^{'}+b(\tau-\tau_{\mathbb{G}})) \tag{3.11}$$

where $\dfrac{\lambda_1}{\lambda_2} = \ell =$ constant. By according to equation (3.10), we write

$$\sigma^{'}\overline{\kappa}\overline{\mathbf{n}} = (\lambda_1\kappa - \lambda_2(\tau-\tau_{\mathbb{G}}))\mathbf{n}$$

By taking the norm of both sides and by using equations (3.7) and (3.9), we get

$$(\sigma^{'})^2(\overline{\kappa})^2 = \frac{(\ell\kappa-(\tau-\tau_{\mathbb{G}}))^2}{\ell^2+1} \tag{3.12}$$

where $\ell\kappa-(\tau-\tau_{\mathbb{G}}) \neq 0$. If we denote by $\lambda = \dfrac{\lambda_1\kappa-\lambda_2(\tau-\tau_{\mathbb{G}})}{\sigma^{'}\overline{\kappa}}$, we have

$$\overline{\mathbf{n}} = \lambda\mathbf{n} \tag{3.13}$$

By differentiating equation (3.13), we get

$$(-\overline{\kappa}\overline{\mathbf{t}} + (\overline{\tau}-\overline{\tau}_{\mathbb{G}})\overline{\mathbf{b}})\sigma^{'} = -\lambda\kappa\mathbf{t} + \lambda(\tau-\tau_{\mathbb{G}})\mathbf{b} \tag{3.14}$$

where $\lambda^{'}=0$. If we rewrite equation (3.14) by using equation (3.6), we get,

$$-\sigma^{'}(\overline{\tau}-\overline{\tau}_{\mathbb{G}})\overline{\mathbf{b}} = \mu_1(s)\mathbf{t} + \mu_2(s)\mathbf{b}$$

where

$$\mu_1(s) = -\frac{(c^{'}+b(\tau-\tau_{\mathbb{G}}))(\ell\kappa-(\tau-\tau_{\mathbb{G}}))}{(\sigma^{'})^2(\ell^2+1)\overline{\kappa}}(\kappa+\ell(\tau-\tau_{\mathbb{G}}))$$

and

$$\mu_2(s) = \frac{(c^{'}+b(\tau-\tau_{\mathbb{G}}))(\ell\kappa-(\tau-\tau_{\mathbb{G}}))\ell}{(\sigma^{'})^2(\ell^2+1)\overline{\kappa}}(\kappa+\ell(\tau-\tau_{\mathbb{G}}))$$

It is clear that $\kappa+\ell(\tau-\tau_{\mathbb{G}}) \neq 0$.
Conversely, assume that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$.
i. Let's assume that the condition (3.2) is satisfied for the differentiable functions a, b and c. Therefore, we write the derivative of equation (3.1) as follows:

$$\frac{d\overline{\gamma}}{ds} = (1+a^{'}-b\kappa)\mathbf{t} \tag{3.15}$$

From equation (3.15), we get

$$\sigma^{'} = \frac{d\overline{s}}{ds} = \left\|\frac{d\overline{\gamma}}{ds}\right\| = \varepsilon_1(1+a^{'}-b\kappa) > 0$$

where $\varepsilon_1 = sgn(1+a^{'}-b\kappa)$. Therefore, we have

$$\overline{\mathbf{t}} = \varepsilon_1\mathbf{t}, \quad \overline{\mathbf{n}} = \varepsilon_1\mathbf{n}, \quad \overline{\mathbf{b}} = \mathbf{b}$$

and

$$\overline{\kappa}=\frac{\kappa}{\sigma'},\quad \overline{\tau}-\overline{\tau}_{\mathbb{G}}=\frac{\varepsilon_1(\tau-\tau_{\mathbb{G}})}{\sigma'}$$

Consequently, γ is a Bertrand curve in Lie group $\mathbb{G}$.
ii. Let's assume that the condition (3.3) is satisfied for the differentiable functions a,b,c and real function ℓ. Therefore, we write the derivative of equation (3.1) as follows:

$$\frac{d\overline{\gamma}}{ds}=(1+a^{'}-b\kappa)\mathbf{t}+(c^{'}+b(\tau-\tau_{\mathbb{G}}))\mathbf{b} \tag{3.16}$$

From equation, we get

$$\sigma^{'}=\left\|\frac{d\overline{\gamma}}{ds}\right\|=\xi_1(c^{'}+b(\tau-\tau_{\mathbb{G}}))\sqrt{\ell^2+1}$$

where $\xi_1=sgn(c^{'}+b(\tau-\tau_{\mathbb{G}}))$. By according to equation (3.16), we have

$$\bar{\mathbf{t}}=\frac{\xi_1}{\sqrt{\ell^2+1}}(\ell\mathbf{t}+\mathbf{b}),\quad \langle\bar{\mathbf{t}},\bar{\mathbf{t}}\rangle=1 \tag{3.17}$$

By differentiating (3.17) with respect to s, we get

$$\begin{aligned}\dot{\bar{\mathbf{t}}}\sigma^{'} &= \frac{\xi_1}{\sqrt{\ell^2+1}}(\ell\dot{\mathbf{t}}+\dot{\mathbf{b}})\\ \dot{\bar{\mathbf{t}}} &= \frac{\xi_1(\ell\kappa-(\tau-\tau_{\mathbb{G}}))\mathbf{n}}{\sigma'\sqrt{\ell^2+1}}\end{aligned} \tag{3.18}$$

Then, from equation (3.18), we get

$$\overline{\kappa}=\|\dot{\bar{\mathbf{t}}}\|=\frac{\xi_2(\ell\kappa-(\tau-\tau_{\mathbb{G}}))}{\sigma'\sqrt{\ell^2+1}} \tag{3.19}$$

and

$$\overline{\mathbf{n}}=\xi_1\xi_2\mathbf{n},\quad \langle\overline{\mathbf{n}},\overline{\mathbf{n}}\rangle=1 \tag{3.20}$$

where $\xi_2=sgn(\ell\kappa-(\tau-\tau_{\mathbb{G}}))$. Then, we have

$$\overline{\mathbf{b}}=\bar{\mathbf{t}}\wedge\overline{\mathbf{n}}=\frac{\xi_2}{\sqrt{\ell^2+1}}(-\mathbf{t}+\ell\mathbf{b}),\quad \langle\overline{\mathbf{b}},\overline{\mathbf{b}}\rangle=1 \tag{3.21}$$

By differentiating equation (3.21), we get

$$\overline{\tau}-\overline{\tau}_{\mathbb{G}}=-\langle\dot{\overline{\mathbf{b}}},\overline{\mathbf{n}}\rangle=\frac{\xi_1(\kappa+(\tau-\tau_{\mathbb{G}})\ell)}{\sigma'\sqrt{\ell^2+1}} \tag{3.22}$$

Thus, γ is a Bertrand curve in Lie group $\mathbb{G}$. □

Proposition 3.3. *Let $\gamma:J\subset\mathbb{R}\to\mathbb{G}$ and $\overline{\gamma}:\overline{J}\subset\mathbb{R}\to\mathbb{G}$ be Bertrand curve pair with the Frenet vectors $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ and $\{\bar{\mathbf{t}},\overline{\mathbf{n}},\overline{\mathbf{b}}\}$, respectively. Then $\tau_{\mathbb{G}}=\overline{\tau}_{\mathbb{G}}$ for $\tau_{\mathbb{G}}=\frac{1}{2}\langle[\mathbf{t},\mathbf{n}],\mathbf{b}\rangle$ and $\overline{\tau}_{\mathbb{G}}=\frac{1}{2}\langle[\bar{\mathbf{t}},\overline{\mathbf{n}}],\overline{\mathbf{b}}\rangle$.*

Proof. The proof is easily seen from equations (3.17), (3.20) and (3.21). □

Remark 3.4. *If $a=c=0$ in Theorem 3.2, we obtain the Bertrand curve conditions in the 3D Lie groups in the literature [15] where*

$$\overline{\gamma}(\overline{s})=\overline{\gamma}(\sigma(s))=\gamma(s)+b(s)\mathbf{n}(s)$$

Corollary 3.5. *Let $\gamma:J\subset\mathbb{R}\to\mathbb{G}$ be a Bertrand curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa,\tau\neq 0$. γ is a Bertrand curve where $\overline{\gamma}(\overline{s})=\overline{\gamma}(\sigma(s))=\gamma(s)+b(s)\mathbf{n}(s)$ iff there exist real number b and ℓ satisfying*

$$1-b\kappa=\ell b(\tau-\tau_{\mathbb{G}})\quad \ell\kappa-(\tau-\tau_{\mathbb{G}})\neq 0$$

In the following corollary, we show the existence of Bertrand curves with general helix in Lie group $\mathbb{G}$.

Corollary 3.6. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a general helix in Lie group $\mathbb{G}$ with the curvatures κ, τ satisfying $\ell\kappa - (\tau - \tau_{\mathbb{G}}) \neq 0$ and $\kappa + \ell(\tau - \tau_{\mathbb{G}}) \neq 0$ where ℓ is a real number. Then, $\overline{\gamma}$ is given by*

$$\overline{\gamma}(\sigma(s)) = \gamma(s) + \frac{ks}{\ell - k}\mathbf{t} + \frac{s}{\ell - k}\mathbf{b} \tag{3.23}$$

where $k \neq 0$ is constant.

Proof. Suppose that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a general helix in Lie group $\mathbb{G}$ with the curvatures κ, τ. From Theorem 2.4, we can write

$$\frac{\tau - \tau_{\mathbb{G}}}{\kappa} = k \tag{3.24}$$

where $k \neq 0$ is constant. On the other hand, if we take $b = 0$ in the conditions of (ii) in Theorem 3.2, we have

$$a\kappa = c(\tau - \tau_{\mathbb{G}}), \quad c' \neq 0, \quad 1 + a' = \ell c' \tag{3.25}$$

By using equations (3.24) and (3.25), we get

$$a = kc, \quad c = \frac{s}{\ell - k}.$$

Hence, equation (3.23) is satisfied. □

Corollary 3.7. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve with the curvatures κ, τ and $\overline{\gamma} : \overline{I} \subset \mathbb{R} \to G$ be a Bertrand mate curve of γ with the curvatures $\overline{\kappa}, \overline{\tau}$. Then $\overline{\gamma}$ is a general helix iff γ is a general helix in 3D Lie groups.*

Proof. By using equations (3.19) and (3.22), we get

$$\frac{\overline{\tau} - \overline{\tau}_{\mathbb{G}}}{\overline{\kappa}} = \xi_1\xi_2 \frac{1 + \ell\left(\dfrac{\tau - \tau_{\mathbb{G}}}{\kappa}\right)}{\ell - \left(\dfrac{\tau - \tau_{\mathbb{G}}}{\kappa}\right)}$$

and

$$\frac{\tau - \tau_{\mathbb{G}}}{\kappa} = \frac{\left(\dfrac{\overline{\tau} - \overline{\tau}_{\mathbb{G}}}{\overline{\kappa}}\right)\ell - \xi_1\xi_2}{\left(\dfrac{\overline{\tau} - \overline{\tau}_{\mathbb{G}}}{\overline{\kappa}}\right) + \xi_1\xi_2\ell}.$$

Therefore, $\overline{\gamma}$ is a general helix (i.e $\dfrac{\overline{\tau} - \overline{\tau}_{\mathbb{G}}}{\overline{\kappa}} =$ constant) iff γ is a general helix (i.e $\dfrac{\tau - \tau_{\mathbb{G}}}{\kappa} =$ constant). □

Corollary 3.8. *If $\mathbb{G}$ is Abelian Lie group, the results obtained correspond to the generalized Bertrand curves given in study [10].*

4. Generalized Mannheim curves in $3D$ Lie groups

In this part, we obtain generalized Mannheim curves in $3D$ Lie groups and we obtain some characterizations.

Definition 4.1. *A curve $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ is a Mannheim curve if there exists a special curve $\gamma^* : \overline{J} \subset \mathbb{R} \to \mathbb{G}$ and a bijection $\zeta : \gamma \to \overline{\gamma}$ where $\mathbf{n}$ and $\mathbf{b}^*$ at $s \in J$, $s^* \in J^*$ coincide. Also, $\gamma^*(s^*)$ is called the Mannheim mate of $\gamma(s)$ in Lie group $\mathbb{G}$.*

Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the Frenet apparatus $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and the curvatures $\kappa, \tau \neq 0$ and $\gamma^*(s^*)$ be a Mannheim mate curve of γ with the Frenet apparatus $\{\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*\}$ and the curvatures $\kappa^*, \tau^* \neq 0$. Then, we have

$$\gamma^*(s^*) = \gamma^*(\varphi(s)) = \gamma(s) + e(s)\mathbf{t}(s) + f(s)\mathbf{n}(s) + g(s)\mathbf{b}(s) \tag{4.1}$$

where $e(s), f(s)$ and $g(s)$ are differentiable functions on J.

Theorem 4.2. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$. γ is a Mannheim curve with Mannheim mate γ^* iff there exist differentiable functions e, f, g satisfying*

$$e\kappa + f' - g(\tau - \tau_{\mathbb{G}}) = 0, \quad g' + f(\tau - \tau_{\mathbb{G}}) \neq 0 \tag{4.2}$$

$$(1 + e' - f\kappa)\kappa = (g' + f(\tau - \tau_{\mathbb{G}}))(\tau - \tau_{\mathbb{G}})$$

Proof. Suppose that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$. By differentiating equation (4.1), we get

$$\frac{d\gamma^*(s^*)}{ds^*}\varphi' = \frac{d\gamma(s)}{ds} + e'(s)\mathbf{t}(s) + e(s)\dot{\mathbf{t}}(s) + f'(s)\mathbf{n}(s) + f(s)\dot{\mathbf{n}}(s) + g'(s)\mathbf{b}(s) + g(s)\dot{\mathbf{b}}(s)$$

Then, we have

$$\mathbf{t}^*\varphi' = (1 + e' - f\kappa)\mathbf{t} + (e\kappa + f' - g(\tau - \tau_{\mathbb{G}}))\mathbf{n} + (g' + f(\tau - \tau_{\mathbb{G}}))\mathbf{b} \tag{4.3}$$

By taking the scalar product of equation (4.3) with $\mathbf{n}$, we find

$$e\kappa + f' - g(\tau - \tau_{\mathbb{G}}) = 0$$

Then, we have

$$\mathbf{t}^*\varphi' = (1 + e' - f\kappa)\mathbf{t} + (g' + f(\tau - \tau_{\mathbb{G}}))\mathbf{b} \tag{4.4}$$

It is seen that

$$(\varphi')^2 = (1 + e' - f\kappa)^2 + (g' + f(\tau - \tau_{\mathbb{G}}))^2 \tag{4.5}$$

Then, we can denote as

$$\mathbf{t}^* = \delta_1\mathbf{t} + \delta_2\mathbf{b} \tag{4.6}$$

for

$$\delta_1 = \frac{1 + e' - f\kappa}{\varphi'} \quad and \quad \delta_2 = \frac{g' + f(\tau - \tau_{\mathbb{G}})}{\varphi'} \tag{4.7}$$

By differentiating equation (4.6) in $\mathbb{G}$, we have

$$\begin{aligned} \dot{\mathbf{t}}^*\varphi' &= \delta_1'\mathbf{t} + \delta_1\dot{\mathbf{t}} + \delta_2'\mathbf{b} + \delta_2\dot{\mathbf{b}} \\ \varphi'\kappa^*\mathbf{n}^* &= \delta_1'\mathbf{t} + (\delta_1\kappa - \delta_2(\tau - \tau_{\mathbb{G}}))\mathbf{n} + \delta_2'\mathbf{b} \end{aligned} \tag{4.8}$$

By taking the scalar product of (4.8) with $\mathbf{n}$, we have $\delta_1\kappa - \delta_2(\tau - \tau_{\mathbb{G}}) = 0$. From equation (4.7), we get

$$(1 + e' - f\kappa)\kappa = (g' + f(\tau - \tau_{\mathbb{G}}))(\tau - \tau_{\mathbb{G}}) \tag{4.9}$$

where $g' + f(\tau - \tau_{\mathbb{G}}) \neq 0$.
Conversely, suppose that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve in G such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$ and the conditions of (4.2) hold for differentiable functions e, f, g. Then, we can write

$$\frac{d\gamma^*}{ds} = (1 + e' - f\kappa)\mathbf{t} + (g' + f(\tau - \tau_{\mathbb{G}}))\mathbf{b} \tag{4.10}$$

where

$$\varphi' = \sqrt{\langle \frac{d\gamma^*}{ds}, \frac{d\gamma^*}{ds} \rangle} = \frac{\xi_1(g' + f(\tau - \tau_{\mathbb{G}}))\sqrt{\kappa^2 + (\tau - \tau_{\mathbb{G}})^2}}{\kappa}$$

with $\xi_1 = sgn(g' + f(\tau - \tau_{\mathbb{G}}))$. From equation (4.10), we get

$$\mathbf{t}^* = \frac{\xi_1}{\sqrt{\kappa^2 + (\tau - \tau_{\mathbb{G}})^2}}((\tau - \tau_{\mathbb{G}})\mathbf{t} + \kappa\mathbf{b}), \quad \langle \mathbf{t}^*, \mathbf{t}^* \rangle = 1 \tag{4.11}$$

Then, we can denote

$$\mathbf{t}^* = \gamma_1\mathbf{t} + \gamma_2\mathbf{b} \tag{4.12}$$

where

$$\gamma_1 = \frac{\xi_1(\tau - \tau_{\mathbb{G}})}{\sqrt{\kappa^2 + (\tau - \tau_{\mathbb{G}})^2}}, \quad \gamma_2 = \frac{\xi_1\kappa}{\sqrt{\kappa^2 + (\tau - \tau_{\mathbb{G}})^2}}.$$

By differentiating (4.12) with respect to s, we get

$$\dot{\mathbf{t}}^* = \frac{\gamma_1' \mathbf{t} + \gamma_2' \mathbf{b}}{\varphi'} \tag{4.13}$$

Then, from equation (4.13), we get

$$\kappa^* = \|\dot{\mathbf{t}}^*\| = \frac{\xi_2((\tau-\tau_{\mathbb{G}})\kappa' - (\tau-\tau_{\mathbb{G}})'\kappa}{\varphi'(\kappa^2+(\tau-\tau_{\mathbb{G}})^2)} = \frac{-\xi_2\kappa^2\left(\dfrac{\tau-\tau_{\mathbb{G}}}{\kappa}\right)'}{\varphi'(\kappa^2+(\tau-\tau_{\mathbb{G}}))^2} \tag{4.14}$$

and

$$\mathbf{n}^* = \frac{\xi_1\xi_2}{\sqrt{\kappa^2+(\tau-\tau_{\mathbb{G}})^2)}}(-\kappa\mathbf{t}+(\tau-\tau_{\mathbb{G}})\mathbf{b}), \quad \langle \mathbf{n}^*, \mathbf{n}^*\rangle = 1 \tag{4.15}$$

where $\xi_2 = sgn((\tau-\tau_{\mathbb{G}})\kappa' - (\tau-\tau_{\mathbb{G}}))'\kappa)$. Moreover, we can obtain

$$\mathbf{b}^* = \mathbf{t}^* \wedge \mathbf{n}^* = -\xi_2 \mathbf{n}, \quad \langle \mathbf{b}^*, \mathbf{b}^*\rangle = 1 \tag{4.16}$$

Finally, we get

$$\tau^* - \tau_{\mathbb{G}}^* = -\langle \dot{\mathbf{b}}^*, \mathbf{n}^*\rangle = \frac{\xi_1\sqrt{\kappa^2+(\tau-\tau_{\mathbb{G}})^2}}{\varphi'} \neq 0 \tag{4.17}$$

Then, γ is a Mannheim curve in Lie group $\mathbb{G}$. □

Proposition 4.3. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ and $\gamma^* : J^* \subset \mathbb{R} \to \mathbb{G}$ be Mannheim curve pair with the Frenet vectors $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ and $\{\mathbf{t}^*,\mathbf{n}^*,\mathbf{b}^*\}$, respectively. Then $\tau_{\mathbb{G}} = \tau_{\mathbb{G}}^*$ for $\tau_{\mathbb{G}} = \frac{1}{2}\langle[\mathbf{t},\mathbf{n}],\mathbf{b}\rangle$ and $\tau_{\mathbb{G}}^* = \frac{1}{2}\langle[\mathbf{t}^*,\mathbf{n}^*],\mathbf{b}^*\rangle$.*

Proof. The proof is easily seen from equations (4.11), (4.15) and (4.16). □

Remark 4.4. *If $e = g = 0$ in Theorem 4.2, we satisfy the Mannheim curve conditions in the 3D Lie groups in the literature [14] where*

$$\gamma^*(s^*) = \gamma^*(\varphi(s)) = \gamma(s) + f(s)\mathbf{n}(s)$$

Corollary 4.5. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$. γ is a Bertrand curve where $\gamma^*(s^*) = \gamma^*(\varphi(s)) = \gamma(s) + f(s)\mathbf{n}(s)$ iff there exist real number f satisfying*

$$\kappa = f(\kappa^2 + (\tau-\tau_{\mathbb{G}})^2).$$

Corollary 4.6. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a general helix with the curvatures $\kappa, \tau \neq 0$. Then, the Mannheim mate γ^* is a straight line in Lie group $\mathbb{G}$.*

Proof. Suppose that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a general helix with the curvatures $\kappa, \tau \neq 0$ in Lie group $\mathbb{G}$. Since the ratio $\dfrac{\tau-\tau_{\mathbb{G}}}{\kappa}$ is constant, we get $\kappa^* = 0$. Then, the Mannheim mate γ^* is a straight line. □

Corollary 4.7. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve with the curvatures κ, τ and $\gamma^* : J^* \subset \mathbb{R} \to G$ be a Mannheim mate of γ with the curvatures κ^*, τ^*. Then γ^* is a general helix iff γ is a slant helix in Lie group G.*

Proof. From equations (4.14) and (4.17), we get

$$\frac{\tau^* - \tau_{\mathbb{G}}}{\kappa^*} = -\xi_1\xi_2\kappa \frac{\left(1+\left(\dfrac{\tau-\tau_{\mathbb{G}}}{\kappa}\right)^2\right)^{\frac{3}{2}}}{\left(\dfrac{\tau-\tau_{\mathbb{G}}}{\kappa}\right)'}$$

Hence, the desired is achieved. □

Corollary 4.8. *If $\mathbb{G}$ is Abelian Lie group, the results obtained correspond to the generalized Mannheim curves given in study [11].*

5. Conclusion

In this study, we examined generalized Bertrand and Mannheim curves in $3D$ Lie groups inspired by [10] and [11] studies. We have shown that we obtain the results in studies [10], [11], [14] and [15], especially considering the Abelian Lie groups. In connection with this study, special curve types can be studied in Lie groups with different metric structures in the future.

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Author's contributions

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