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Employing the $\exp(-\varphi(z))$-Expansion Method to Find Analytical Solutions for a (2+1)-dimensional Combined KdV-mKdV Equation

Baixin Chen[1] and Yongyi Gu[1*]

[1] *School of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou 510320, China*

[*] *Corresponding author*

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Abstract

In this paper, we obtain exact solutions of the (2+1)-dimensional combined KdV-mKdV equation by using a symbol calculation approach. First, we give some background on the equation. Second, the $\exp(-\varphi(z))$-expansion method will be introduced to solve the equation. After, using the $\exp(-\varphi(z))$-expansion method to solve the equation, we can get four types of exact solutions, which are hyperbolic, trigonometric, exponential, and rational function solutions. Finally, we can observe the characteristics of the exact solutions via computer simulation more easily.

1. Introduction

Seeking the exact solutions of nonlinear partial differential equations (NLPDEs) is a hotspot in nonlinear science research and the related theory has developed rapidly in recent decades. Because many nonlinear phenomena existing in nature and various fields can be described as NLPDEs. More importantly, the solutions of NLPDEs can account for these complex phenomena as well as applying in these fields [1]-[12], such as atmosphere, optical fiber communications and fluid mechanics. There is a series of NLPDEs, for example, the KdV equation, the KP equation and the Schrödinger equation. Also, there are many effective methods to search exact solutions of NLPDEs, such as Lie symmetry [13], the Hirota bilinear method [14, 15], the extended complex metho d[16], and the $\exp(-\varphi(z))$-expansion method [17]. Particularly, the $\exp(-\varphi(z))$-expansion method first proposed by Zhao and Li [17] can be used to attain analytical traveling wave solutions of numerous NLPDEs, such as the combined KdV-mKdV equation[18], the (1+1)-dimensional classical Boussinesq equations [19] and the Caudrey-Dodd-Gibbon-Sawada-Kotera equation [20].
As we all know, the KdV equation becoming a kind of classical nonlinear partial differential equations can be used to describe small amplitude shallow water waves, stratified internal waves, ion acoustic waves and its model has great practical value in many fields [21]-[23], such as plasma physics, solid state physics and fluid mechanics. With the development of soliton theory and the in-depth research in the KdV equation, we fully understand the properties of it and its abundant solutions. Meanwhile, various extensions of KdV equations are derived. More recently, Wang and Kara [24] built the new (2+1)-dimensional KdV and mKdV equations as

$$u_t - 6uu_x + 6uu_y - u_{xxx} + u_{yyy} + 3u_{xxy} - 3u_{xyy} = 0 \tag{1.1}$$

Email address and ORCID number: **gdguyongyi@163.com, 0000-0002-6651-1714 (Y. Gu), baixinchen2022@163.com, 0000-0002-6064-484X (B. Chen)**

$$u_t - 6u^2u_x + 6u^2u_y - u_{xxx} + u_{yyy} + 3u_{xxy} - 3u_{xyy} = 0 \tag{1.2}$$

Then Malik et al. [25] proposed the (2+1)-dimensional combined KdV-mKdV equation by combining them, which is given by:

$$u_t - a_1uu_x + a_1uu_y - a_2u^2u_x + a_2u^2u_y - a_3u_{xxx} + a_3u_{yyy} + a_4u_{xxy} - a_4u_{xyy} = 0 \tag{1.3}$$

By considering $a_1 = 6, a_2 = 0, a_3 = 1, a_4 = 3$ and $a_1 = 0, a_2 = 6, a_3 = 1, a_4 = 3$, Eq.(1.3) reduces to Eqs.(1.1) and (1.2) respectively. Additionally, although the authors have obtained the analytical solutions to the combined KdV-mKdV equation in [18], the (2+1)-dimensional combined KdV-mKdV equation in [25] has more dimensions and the mixed partial derivatives in contrast to the former, which means a much broader researching space for scholars. Therefore, it makes sense to research the the (2+1)-dimensional combined KdV-mKdV equation deeply. The integrability of the equation and some forms of its solutions are illuminated in Sandeep Malik's paper. In this article, we use the $\exp(-\varphi(z))$-expansion method to attain exact solutions to the (2+1)-dimensional combined KdV-mKdV equation and observe the characteristics of them by computer simulation, which can obtain more abundant solutions to the equation and indicate the validity of the $\exp(-\varphi(z))$-expansion method. The results and simulations are gained by using Maple.

2. The $\exp(-\varphi(z))$-expansion method

Considering the following nonlinear PDE:

$$F(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, \cdots) = 0, \tag{2.1}$$

in which F is a polynomial of the unknown function $u(x,y,t)$ and its partial derivatives, and it also involves nonlinear terms.

Step 1. Insert traveling wave transform

$$u(x,y,t) = u(z), \quad z = \kappa x + \lambda y + \omega t,$$

into Eq.(2.1) to reduce it into the ODE,

$$P(u, u', u'', u''', \cdots) = 0, \tag{2.2}$$

in which P is a polynomial of u and its derivatives, while $' := \frac{d}{dz}$.

Step 2. Assume that the exact solutions of Eq.(2.2) have the following form:

$$u(z) = \sum_{\upsilon=0}^{n} C_\upsilon (\exp(-\varphi(z)))^\upsilon, \tag{2.3}$$

in which C_υ, $(0 \le \upsilon \le n)$ are constants to be determined later, such that $C_n \ne 0$ and $\varphi = \varphi(z)$ satisfies the ODE as follows:

$$\varphi'(z) = b_1 + \exp(-\varphi(z)) + b_2 \exp(\varphi(z)). \tag{2.4}$$

where b_1 and b_2 are constants and the solutions of Eq.(2.4) are given as below:
When $b_1^2 - 4b_2 > 0$, $b_2 \ne 0$,

$$\varphi(z) = \ln\left(\frac{-\sqrt{(b_1^2-4b_2)}\tanh(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma)) - b_1}{2b_2}\right), \tag{2.5}$$

$$\varphi(z) = \ln\left(\frac{-\sqrt{(b_1^2-4b_2)}\coth(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma)) - b_1}{2b_2}\right). \tag{2.6}$$

When $b_1^2 - 4b_2 < 0$, $b_2 \ne 0$,

$$\varphi(z) = \ln\left(\frac{\sqrt{(4b_2-b_1^2)}\tan(\frac{\sqrt{(4b_2-b_1^2)}}{2}(z+\varsigma)) - b_1}{2b_2}\right), \tag{2.7}$$

$$\varphi(z)=\ln\left(\frac{\sqrt{(4b_2-b_1^2)}\cot(\frac{\sqrt{(4b_2-b_1^2)}}{2}(z+\varsigma))-b_1}{2b_2}\right) \tag{2.8}$$

When $b_1^2-4b_2>0$, $b_1\neq 0$, $b_2=0$,

$$\varphi(z)=-\ln\left(\frac{b_1}{\exp(b_1(z+\varsigma))-1}\right). \tag{2.9}$$

When $b_1^2-4b_2=0$, $b_1\neq 0$, $b_2\neq 0$,

$$\varphi(z)=\ln\left(-\frac{2(b_1(z+\varsigma)+2)}{b_1^2(z+\varsigma)}\right). \tag{2.10}$$

When $b_1^2-4b_2=0$, $b_1=0$, $b_2=0$,

$$\varphi(z)=\ln(z+\varsigma). \tag{2.11}$$

in which ς is an arbitrary constant and $C_n\neq 0, b_1, b_2$ are constants in Eqs.(2.5)-(2.11). Considering the homogeneous balance between the highest order derivatives and nonlinear terms of Eq.(2.2), we define the degree of $u(z)$ as $D(u(z))=n$ and the positive integer n can be ascertained by the following expressions

$$D\left(\frac{d^\alpha u}{dz^\alpha}\right)=n+\alpha, D\left(u^\beta\left(\frac{d^\alpha u}{dz^\alpha}\right)^s\right)=n\beta+s(n+\alpha). \tag{2.12}$$

Step 3. Plugging Eq.(2.3) into Eq.(2.2), we obtain a polynomial of $\exp(-\varphi(z))$. Then collect all terms with the same power about $\exp(-\varphi(z))$ and let the coefficients of them equal zero respectively. After that, we get a set of algebraic equations and by solving them we confirm the values of $C_n\neq 0, b_1, b_2$. Finally, we substitute the obtained values into Eq.(2.3) as well as Eqs.(2.5)-(2.11) to achieve the determination of the exact solutions for the original PDE.

3. Exact solutions of the (2+1)-dimensional combined KdV-mKdV equation

Substituting traveling wave transform

$$u(x,y,t)=u(z), \quad z=\kappa x+\lambda y+\omega t,$$

into Eq.(1.3) and then integrating it, we obtain

$$u+(\frac{1}{2}a_1\lambda-\frac{1}{2}a_1\kappa)u^2+(\frac{1}{3}a_2\lambda-\frac{1}{3}a_2\kappa)u^3+(a_3\lambda^3-a_3\kappa^3+a_4\kappa^2\lambda-a_4\kappa\lambda^2)u''+\delta=0. \tag{3.1}$$

where δ is the integration constant.
Taking the homogeneous balance between u^3 and u'' of Eq.(3.1) according to Eqs.(2.12), we can yield $n=1$ and hence

$$u(z)=C_0+C_1\exp(-\varphi(z)), \tag{3.2}$$

where $C_1\neq 0$, C_0 are constants.
Plugging u, u^2, u^3, u'' into Eq.(3.1) and equating the coefficients with the same order of $\exp(-\varphi(z))$ to zero, we obtain

$$\begin{aligned}
e^{0(-\varphi(z))}:&\omega C_0+1/2a_1\lambda C_0{}^2-1/2a_1\kappa C_0{}^2+1/3a_2\lambda C_0{}^3\\
&-1/3a_2\kappa C_0{}^3+\delta-C_1\kappa^3a_3b_1b_2+C_1\lambda^3a_3b_1b_2\\
&-C_1a_4\kappa\lambda^2b_1b_2+C_1a_4\kappa^2\lambda b_1b_2=0
\end{aligned}$$

$$\begin{aligned}
e^{1(-\varphi(z))}:&-C_1a_3b_1{}^2\kappa^3+C_1a_3b_1{}^2\lambda^3+C_1a_4b_1{}^2\kappa^2\lambda\\
&-C_1a_4b_1{}^2\kappa\lambda^2-2a_3\kappa^3b_2C_1+2a_3\lambda^3b_2C_1\\
&+2C_1\lambda a_4b_2\kappa^2-2C_1\lambda^2a_4b_2\kappa-C_0{}^2C_1a_2\kappa\\
&+C_0{}^2C_1a_2\lambda-C_0C_1a_1\kappa+C_0C_1a_1\lambda\\
&+C_1\omega=0
\end{aligned}$$

$$\begin{aligned}
e^{2(-\varphi(z))}:&-3C_1a_4\kappa\lambda^2b_1+3C_1a_4\kappa^2\lambda b_1+1/2a_1\lambda C_1{}^2\\
&-1/2a_1\kappa C_1{}^2+a_2\lambda C_0C_1{}^2-a_2\kappa C_0C_1{}^2\\
&-3C_1a_3\kappa^3b_1+3C_1a_3\lambda^3b_1=0
\end{aligned}$$

$$\begin{aligned}e^{3(-\varphi(z))}:&-2C_1\,a_3\,\kappa^3+2C_1\,a_3\,\lambda^3+1/3\,a_2\,\lambda\,{C_1}^3\\&-1/3\,a_2\,\kappa{C_1}^3-2C_1\,a_4\,\kappa\,\lambda^2+2C_1\,a_4\,\kappa^2\lambda=0\end{aligned}$$

Having solved the above algebraic equations,we get two different cases:

Case 1.

$$\begin{aligned}C_0&=\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2},\\C_1&=\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2},\\\delta&=\frac{1}{24a_2^2}\Big(-3\,b_1\sqrt{-(a_3\,\lambda^2+\kappa\,(a_3-a_4)\,\lambda+a_3\,\kappa^2)\,a_2}\,\big(\big(2\,\big({b_1}^2-4\,b_2\big)\\&\quad(\kappa-\lambda)\,\big(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda\big)+4\,\omega\big)\,a_2\\&\quad+{a_1}^2\,(\kappa-\lambda)\big)\sqrt{6}+2\,a_1\,\big(6\,\omega\,a_2+{a_1}^2\,(\kappa-\lambda)\big)\Big),\end{aligned}\tag{3.3}$$

where b_1 and b_2 are arbitrary.
Plugging Eqs.(3.3) into Eq.(3.2), we can obtain

$$\begin{aligned}u(z)=&\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2}\\&+\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2}\exp(-\varphi(z)).\end{aligned}\tag{3.4}$$

Employing Eqs.(2.5) to (2.11) into Eq.(3.4) respectively, attains the exact solutions to the (2+1)-dimensional combined KdV-mKdV equation in the following.

Case 1.1. When $b_1^2-4b_2>0$, $b_2\neq 0$,

$$\begin{aligned}u_{11}(z)=&\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2}\\&-\frac{2b_2\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2(\sqrt{(b_1^2-4b_2)}\tanh(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma))+b_1)},\\u_{12}(z)=&\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2}\\&-\frac{2b_2\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2(\sqrt{(b_1^2-4b_2)}\coth(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma))+b_1)}.\end{aligned}$$

Case 1.2. When $b_1^2-4b_2<0$, $b_2\neq 0$,

$$\begin{aligned}u_{13}(z)=&\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2}\\&+\frac{2b_2\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2(\sqrt{(4b_2-b_1^2)}\tan(\frac{\sqrt{4b_2-b_1^2}}{2}(z+\varsigma))-b_1)},\\u_{14}(z)=&\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2}\\&+\frac{2b_2\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2(\sqrt{(4b_2-b_1^2)}\cot(\frac{\sqrt{4b_2-b_1^2}}{2}(z+\varsigma))-b_1)}.\end{aligned}$$

Case 1.3. When $b_1^2-4b_2>0$, $b_1\neq 0$, $b_2=0$,

$$\begin{aligned}u_{15}(z)=&\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2}\\&+\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1}{a_2(\exp(b_1(z+\varsigma))-1)}.\end{aligned}$$

Case 1.4. When $b_1^2-4b_2=0$, $b_1\neq 0$, $b_2\neq 0$,

$$u_{16}(z)=\frac{\sqrt{-6a_2\left(a_3\kappa^2+a_3\kappa\lambda+a_3\lambda^2-a_4\kappa\lambda\right)}b_1-a_1}{2a_2}-\frac{b_1^2(z+\varsigma)\sqrt{-6a_2\left(a_3\kappa^2+a_3\kappa\lambda+a_3\lambda^2-a_4\kappa\lambda\right)}}{2a_2(b_1(z+\varsigma)+2)}.$$

Case 1.5. When $b_1^2-4b_2=0$, $b_1=0$, $b_2=0$,

$$u_{17}(z)=\frac{\sqrt{-6a_2\left(a_3\kappa^2+a_3\kappa\lambda+a_3\lambda^2-a_4\kappa\lambda\right)}b_1-a_1}{2a_2}+\frac{\sqrt{-6a_2\left(a_3\kappa^2+a_3\kappa\lambda+a_3\lambda^2-a_4\kappa\lambda\right)}}{a_2(z+\varsigma)}.$$

Case 2.

$$C_0=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}b_1+a_1}{2a_2},$$

$$C_1=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}}{a_2},$$

$$\delta=\frac{1}{24a_2^2}\Bigg(3b_1\sqrt{-\left(a_3\lambda^2+\kappa\left(a_3-a_4\right)\lambda+a_3\kappa^2\right)a_2}\left(\left(2\left(b_1{}^2-4b_2\right)\right.\right.$$
$$\left(\kappa-\lambda\right)\left(a_3\kappa^2+a_3\kappa\lambda+a_3\lambda^2-a_4\kappa\lambda\right)+4\omega\right)a_2$$
$$\left.+a_1{}^2\left(\kappa-\lambda\right)\right)\sqrt{6}+2a_1\left(6\omega a_2+a_1{}^2\left(\kappa-\lambda\right)\right)\Bigg), \tag{3.5}$$

where b_1 and b_2 are arbitrary constants.
Plugging Eqs.(3.5) into Eq.(3.2), we can obtain

$$u(z)=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}b_1+a_1}{2a_2}-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}}{a_2}\exp(-\varphi(z)). \tag{3.6}$$

Employing Eqs.(2.5) to (2.11) into Eq.(3.6) respectively, attains the exact solutions to the (2+1)-dimensional combined KdV-mKdV equation in the following.

Case 2.1. When $b_1^2-4b_2>0$, $b_2\neq 0$,

$$u_{21}(z)=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}b_1+a_1}{2a_2}+\frac{2b_2\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}}{a_2\left(\sqrt{(b_1^2-4b_2)}\tanh\left(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma)\right)+b_1\right)},$$

$$u_{22}(z)=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}b_1+a_1}{2a_2}+\frac{2b_2\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}}{a_2\left(\sqrt{(b_1^2-4b_2)}\coth\left(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma)\right)+b_1\right)}.$$

Case 2.2. When $b_1^2-4b_2<0$, $b_2\neq 0$,

$$u_{23}(z)=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}b_1+a_1}{2a_2}-\frac{2b_2\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}}{a_2\left(\sqrt{(4b_2-b_1^2)}\tan\left(\frac{\sqrt{4b_2-b_1^2}}{2}(z+\varsigma)\right)-b_1\right)},$$

$$u_{24}(z) = -\frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}b_1 + a_1}{2a_2} - \frac{2b_2\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}}{a_2(\sqrt{(4b_2 - b_1^2)}\cot(\frac{\sqrt{4b_2-b_1^2}}{2}(z+\varsigma)) - b_1)}.$$

Case 2.3. When $b_1^2 - 4b_2 > 0$, $b_1 \neq 0$, $b_2 = 0$,

$$u_{25}(z) = -\frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}b_1 + a_1}{2a_2} - \frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}b_1}{a_2(\exp(b_1(z+\varsigma)) - 1)}.$$

Case 2.4. When $b_1^2 - 4b_2 = 0$, $b_1 \neq 0$, $b_2 \neq 0$,

$$u_{26}(z) = -\frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}b_1 + a_1}{2a_2} + \frac{b_1^2(z+\varsigma)\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}}{2a_2(b_1(z+\varsigma)+2)}.$$

Case 2.5. When $b_1^2 - 4b_2 = 0$, $b_1 = 0$, $b_2 = 0$,

$$u_{27}(z) = -\frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}b_1 + a_1}{2a_2} - \frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}}{a_2(z+\varsigma)}.$$

4. Computer simulations

In this section, the results are illustrated by computer simulations respectively.

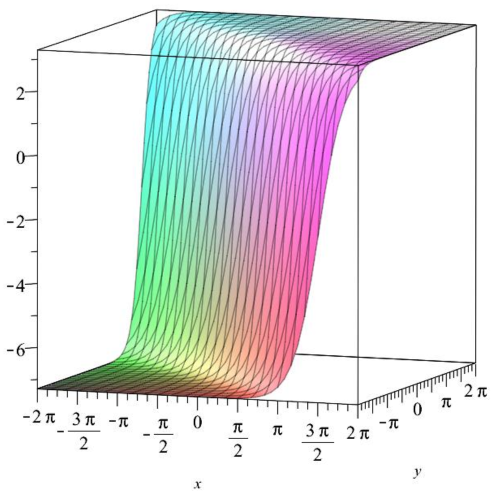

Figure 4.1: 3D profile of $u_{11}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = 1$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, $b_1 = 4$, and $b_2 = 3$.

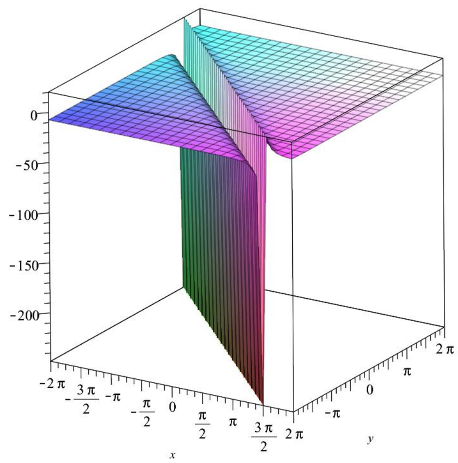

Figure 4.2: 3D profile of $u_{12}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = 1$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, $b_1 = 4$, and $b_2 = 3$.

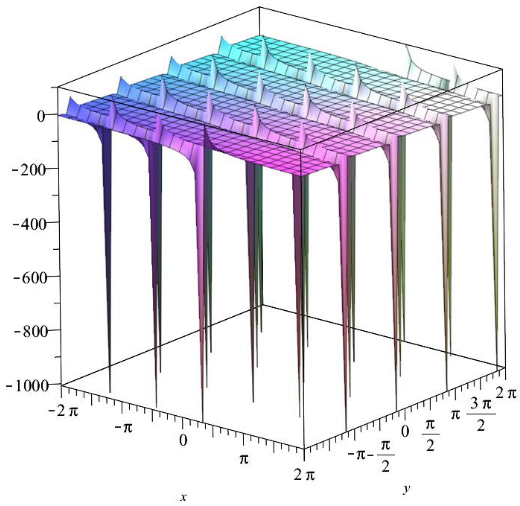

Figure 4.3: 3D profile of $u_{13}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = \frac{1}{10}$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, $b_1 = 4$, and $b_2 = 5$.

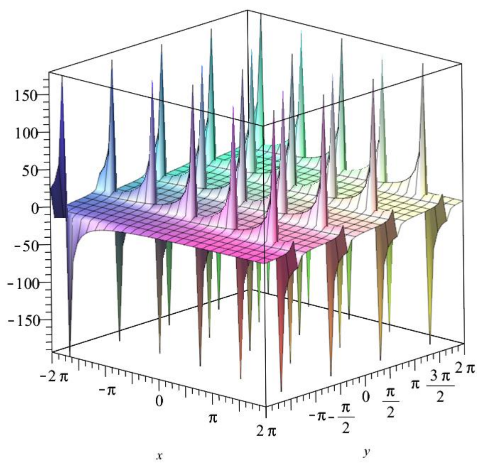

Figure 4.4: 3D profile of $u_{14}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = \frac{1}{10}$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, $b_1 = 4$, and $b_2 = 5$.

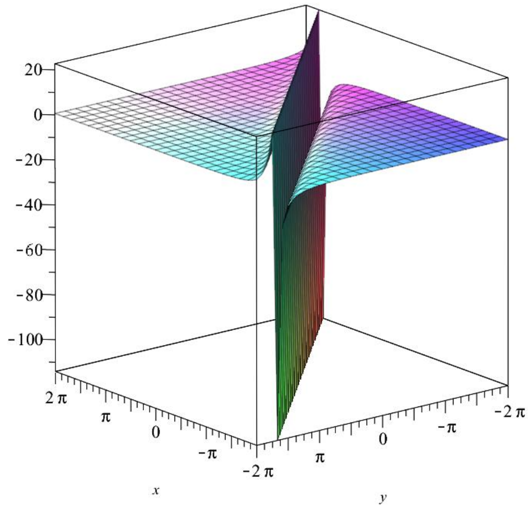

Figure 4.5: 3D profile of $u_{15}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = 1$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, and $b_1 = 1$.

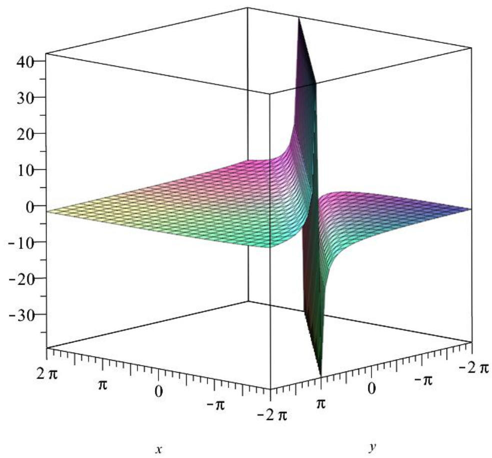

Figure 4.6: 3D profile of $u_{16}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = 1$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, and $b_1 = 1$.

5. Conclusion

In this study, we use the $\exp(-\varphi(z))$-expansion method to obtain abundant new exact solutions to the (2+1)-dimensional combined KdV-mKdV equation. Except the types of hyperbolic and exponential function solutions which are the same as those of Sandeep Malik's paper [25], we also get new types of function solutions including trigonometric and rational solutions. Additionally, the results indicate that utilizing the $\exp(-\varphi(z))$-expansion method to the combined KdV-mKdV equation and the (2+1)-dimensional combined KdV-mKdV equation can get the same forms of solutions, while the solutions to the (2+1)-dimensional combined KdV-mKdV equation have one more case. These solutions can widely stimulate mathematicians and physicists' interest and have potential value to be applied in mathematics and physics. Meanwhile, the effectiveness of the $\exp(-\varphi(z))$-expansion method to seek exact solutions for nonlinear differential equations can be seen from the obtained results.

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Competing interests

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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