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Identification of the Solely Time-Dependent Zero-Order Coefficient in a Linear Bi-Flux Diffusion Equation from an Integral Measurement

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Abstract

Bi-flux diffusion equation, can be easily affected by the existence of external factors, is known as an anomalous diffusion. In this paper, the inverse problem (IP) of determining the solely time-dependent zero-order coefficient in a linear Bi-flux diffusion equation with initial and homogeneous boundary conditions from an integral additional specification of the energy is considered. The unique solvability of the inverse problem is demonstrated by using the contraction principle for sufficiently small times.

1. Introduction

The classical diffusion model is contingent on Fick's law and, this generally describes some practical problems such as heat and mass diffusion, bacterial infection diffusion, and predator-prey models, [1, 2, 3]. However, a number of sensitive particle systems, for example population systems arise in biology, are known as an anomalous diffusion. Because the diffusion motion of particles can be easily affected by the existence of the disturbing exogenous agents. To model this anomalous diffusion, the Bi-flux approach to diffusion problems was introduced in [4, 5, 6, 7] to deal with bizarre evolutionary processes by using a discrete formulation. The theory leads the flux to bifurcate into two separate currents corresponding to two independent potentials. The 1st and 2nd flows derive from the classical Fick potential and a new potential, respectively. From this theory the Bi-flux diffusion equation is derived as:

$$Z_\tau(\chi,\tau)+\beta(1-\beta)RZ_{\chi\chi\chi\chi}(\chi,\tau)-\beta DZ_{\chi\chi}(\chi,\tau)=H(\chi,\tau,Z), \tag{1.1}$$

where $Z(\chi,\tau)$ is the particle concentration, D is the diffusion coefficient, R is the reaction coefficient, $H(\chi,\tau,Z)$ linear or nonlinear reaction function, and β $(0\leq\beta\leq 1)$ is the fraction of particles displaced regarding to the Fick's law.
If $\beta=0$, the homogeneous equation (1.1) $(H(\chi,\tau,Z)=0)$ reduces to a stationary equation and $Z(\chi,\tau)=$constant. If $\beta=1$, the equation (1.1) reduces to the classical diffusion model which arises in many physical processes such as thermodynamics [8], predator-prey problems [2], bio-heat conduction [9], heat exchangers [10], and mass transport in groundwater [11]. If the diffusion coefficient $D=0$, the equation (1.1) is Mullin's equation which arises in thermal grooving by surface diffusion [12]. If the reaction function $H(\chi,\tau,Z)$ is linear and in the form $H(\chi,\tau,Z)=k(\tau)Z(\chi,\tau)+h(\chi,\tau)$, then the eq. (1.1) can be rewritten as

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$$Z_\tau(\chi,\tau)+\beta(1-\beta)RZ_{\chi\chi\chi\chi}(\chi,\tau)-\beta DZ_{\chi\chi}(\chi,\tau)=k(\tau)Z(\chi,\tau)+h(\chi,\tau). \tag{1.2}$$

In this work we consider an IP of recovering the time-dependent zero-order coefficient $k(\tau)$ in a linear Bi-flux diffusion equation (1.2) together with the particle concentration $Z(\chi,\tau)$ in the rectangle domain $\Pi_T=\{(\chi,\tau):(\chi,\tau)\in[0,1]\times[0,T]\}$ for some fixed $T>0$, subject to the initial condition (IC)

$$Z(\chi,0)=z_0(\chi),\ \chi\in[0,1], \tag{1.3}$$

and the boundary conditions (BCs)

$$Z(0,\tau)=Z(1,\tau)=Z_{\chi\chi}(0,\tau)=Z_{\chi\chi}(1,\tau)=0,\ \tau\in[0,T], \tag{1.4}$$

and the additional condition (AC)

$$\int_0^1 Z(\chi,\tau)d\chi=m(\tau),\ \tau\in[0,T], \tag{1.5}$$

where, D and R are the positive diffusion and reaction coefficients respectively, $0<\beta<1$, $h(\chi,\tau)$ is the source term, $z_0(\chi)$ is the initial particle concentration, and $m(\tau)$ is the extra integral measurement, which often refers specification of the energy/mass, to obtain the solution of the IP.

The IPs for the classical diffusion equation ($\beta=1$) are satisfactorily studied theoretically and numerically. For instance, IPs of determining the time-dependent heat source are studied in [13, 14, 15, 16], and IPs of reconstructing the time-dependent reaction coefficient are investigated in [17, 18, 19, 20, 21] with different boundary conditions (local, non-local or non-classical conditions).

On contrary to the IPs for the classical diffusion equation, our aim is to study the Bi-flux diffusion equation to identify the time-dependent zero-order coefficient $k(\tau)$ along with the particle concentration $Z(\chi,\tau)$ theoretically, for the first time, in the rectangular domain, using the IC (1.3), homogeneous BCs (1.4) and the AC (1.5).

The paper is organized as follows. In Section 2, the equivalent IP formulation is derived. In Section 3, the equivalent IP reduced to a system of Volterra integral equations with respect to the unknown functions $Z(\chi,\tau)$, and $k(\tau)$. Then the theorem of the existence and uniqueness of the solution of the IP is phrased and proved by means of the contraction principle.

2. Mathematical formulation of the equivalent IP

In this section, first we will give the classical solution of the IP (1.2)-(1.5), and then reduce the considered IP to the equivalent problem.

Definition 2.1. *The classical solution of the IP (1.2)-(1.5) is the pair $\{Z(\chi,\tau),k(\tau)\}$ subject to the following properties:*

(i) $Z(\chi,\tau)\in C^{4,1}(D_T)$,
(ii) $k(\tau)\in C[0,T]$,
(iii) *the pair $\{Z(\chi,\tau),k(\tau)\}$ satisfies the eq. (1.2) and the conditions (1.3)-(1.5) in ordinary sense.*

$Z(\chi,\tau)\in C^{4,1}(\Pi_T)$ means that $Z(\chi,\tau)$ and its partial derivatives with respect to χ upto fourth and τ upto first order are continuous on Π_T.

Let us reduce the IP (1.2)-(1.5) to the equivalent inverse problem to prove the existence and uniqueness theorem for the IP.

Lemma 2.2. *Let $z_0(\chi)\in C^2[0,1]$, $m(\tau)\in C^1[0,T]$, $m(\tau)\neq 0$ for all $\tau\in[0,T]$ and*

$$z_0(0)=z_0(1)=z_0''(0)=z_0''(1)=0,$$

$$\int_0^1 z_0(\chi)d\chi=m(0),$$

are satisfied. Then the problem of finding the classical solution of the IP (1.2)-(1.5) is equivalent to the problem of finding the classical solution of the IP from the eq. (1.2) with the IC (1.3), the BCs (1.4) and the AC

$$m'(\tau)+\beta(1-\beta)R\int_0^1 Z_{\chi\chi\chi\chi}(\chi,\tau)d\chi-\beta D\int_0^1 Z_{\chi\chi}(\chi,\tau)d\chi=k(\tau)m(\tau)+\int_0^1 h(\chi,\tau)d\chi. \tag{2.1}$$

Proof. It is obvious that if the pair $\{Z(\chi,\tau),k(\tau)\}$ is the classical solution of the IP (1.2)-(1.5), then the AC (2.1) is also satisfied.

Let us show that the condition (1.5) is fulfilled if the pair $\{Z(\chi,\tau),k(\tau)\}$ is a classical solution of the IP (1.2)-(1.4) with the AC (2.1).

Integrating the eq. (1.2) with respect to χ from 0 to 1 yields

$$\int_0^1 Z_\tau(\chi,\tau)d\chi+\beta(1-\beta)R\int_0^1 Z_{\chi\chi\chi\chi}(\chi,\tau)d\chi-\beta D\int_0^1 Z_{\chi\chi}(\chi,\tau)d\chi=k(\tau)\int_0^1 Z(\chi,\tau)d\chi+\int_0^1 h(\chi,\tau)d\chi. \tag{2.2}$$

Considering the difference of the equations (2.2) and (2.1) gives the following initial value problem

$$\begin{cases} Y'(\tau)-k(\tau)Y(\tau)=0,\ \tau\in[0,T], \\ Y(0)=0, \end{cases} \tag{2.3}$$

where $Y(\tau)=\int_0^1 Z(\chi,\tau)d\chi-m(\tau)$. The Cauchy problem (2.3) has only a trivial solution $Y(\tau)=0$. i.e.

$$\int_0^1 Z(\chi,\tau)d\chi=m(\tau),\ \tau\in[0,T].$$

Thus the condition (1.5) is fulfilled. □

3. Unique Solvability of the IP

The elements of the system $\{y_n(\chi)\}_{n=1}^{\infty}=\left\{\sqrt{2}\sin(\mu_n\chi)\right\}_{n=1}^{\infty}$ are bi-orthonormal on $[0,1]$ with $\mu_n=n\pi$ for each $n=1,2,...,$ i.e.:

$$(y_n(\cdot),y_m(\cdot))=\int_0^1 y_n(\chi)y_m(\chi)d\chi=\begin{cases} 1 & ,m=n \\ 0 & ,m\neq n \end{cases}.$$

Also the system $\{y_n(\chi)\}_{n=1}^{\infty}$ is complete and forms a Riesz basis in $L_2[0,1]$.
The following lemma is useful to prove the unique solvability of the IP:

Lemma 3.1. *Suppose that the assumptions*

A$_1$ $z_0(\chi)\in C^3[0,1],\ z_0^{(4)}(\chi)\in L_2[0,1],\ z_0(0)=z_0(1)=z_0''(0)=z_0''(1)=0,$
A$_2$ $h(\chi,\tau)\in C^{3,0}(\Pi_T),\ h_{\chi\chi\chi\chi}(\cdot,\tau)\in L_2[0,1],\ h(0,\tau)=h(1,\tau)=h_{\chi\chi}(0,\tau)=h_{\chi\chi}(1,\tau)=0,$

are satisfied. Then

$$\sum_{n=1}^{\infty}\mu_n^3|z_{0n}|\le c_1\left\|z_0^{(4)}\right\|_{L_2[0,1]},\ \sum_{n=1}^{\infty}\mu_n|z_{0n}|\le c_2\left\|z_0^{(4)}\right\|_{L_2[0,1]},$$

and

$$\sum_{n=1}^{\infty}\mu_n^3|h_n(\tau)|\le c_1\left\|h_{\chi\chi\chi\chi}(\cdot,\tau)\right\|_{L_2[0,1]},\ \sum_{n=1}^{\infty}\mu_n|h_n(\tau)|\le c_2\left\|h_{\chi\chi\chi\chi}(\cdot,\tau)\right\|_{L_2[0,1]},$$

hold. Here $z_{0n}=(z_0(\cdot),y_n(\cdot))$, $h_n(\tau)=(h(\cdot,\tau),y_n(\cdot))$, *and*

$$c_1=\left(\sum_{n=1}^{\infty}\frac{1}{\mu_n^2}\right)^{1/2}=\frac{1}{\sqrt{6}},\ c_2=\left(\sum_{n=1}^{\infty}\frac{1}{\mu_n^6}\right)^{1/2}=\frac{1}{\sqrt{945}}.$$

Proof. Let us prove first one of these estimates. The others can be proved analogously. By applying integration by parts four times we get

$$\mu_n^3 z_{0n}=\mu_n^3(z_0(\cdot),y_n(\cdot))=\mu_n^3\int_0^1 z_0(\chi)y_n(\chi)d\chi=\mu_n^3\frac{1}{\mu_n^4}\int_0^1 z_0^{(4)}(\chi)y_n(\chi)d\chi=\frac{1}{\mu_n}\int_0^1 z_0^{(4)}(\chi)y_n(\chi)d\chi=\frac{1}{\mu_n}\left(z_0^{(4)}(\cdot),y_n(\cdot)\right).$$

By using Cauchy-Schwartz and Bessel inequalities

$$\sum_{n=1}^{\infty}\mu_n^3|z_{0n}|=\sum_{n=1}^{\infty}\frac{1}{\mu_n}\left|\left(z_0^{(4)}(\cdot),y_n(\cdot)\right)\right|\le\left(\sum_{n=1}^{\infty}\frac{1}{\mu_n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left|\left(z_0^{(4)}(\cdot),y_n(\cdot)\right)\right|^2\right)^{1/2}\le c_1\left\|z_0^{(4)}\right\|_{L_2[0,1]}$$

with $c_1=\left(\sum_{n=1}^{\infty}\frac{1}{\mu_n^2}\right)^{1/2}=\frac{1}{\sqrt{6}}$. □

Since the system $\{y_n(\chi)\}_{n=1}^{\infty}$ forms a Riesz basis, we can seek the solution of the IP (1.2)-(1.4) and (2.1) as

$$Z(\chi,\tau)=\sum_{n=1}^{\infty} Z_n(\tau)y_n(\chi), \tag{3.1}$$

where

$$Z_n(\tau)=\int_0^1 Z(\chi,\tau)y_n(\chi)d\chi.$$

Applying Fourier method yields $Z_n(\tau)$, $n=1,2,...$ satisfy the following initial-value problems (IVPs)

$$\begin{cases} Z_n'(\tau)+K_nZ_n(\tau)=H_n(\tau;Z,k), \\ Z_n(0)=z_{0n},\ n=1,2,..., \end{cases} \tag{3.2}$$

where $K_n=\mu_n^4\beta(1-\beta)R+\mu_n^2\beta D$, $H_n(\tau;Z,k)=k(\tau)Z_n(\tau)+h_n(\tau)$.
The solution of the IVPs (3.2) are

$$Z_n(\tau)=z_{0n}\exp(-K_n\tau)+\int_0^{\tau} H_n(s;Z,k)\exp(-K_n(\tau-s))ds. \tag{3.3}$$

Substituting (3.3) into (3.1) we get the classical solution of the IP (1.2)-(1.4) (or the first component of the pair $\{Z(\chi,\tau),k(\tau)\}$) as

$$Z(\chi,\tau)=\sum_{n=1}^{\infty}\left[z_{0n}\exp(-K_n\tau)+\int_0^{\tau} H_n(s;Z,k)\exp(-K_n(\tau-s))ds\right]y_n(\chi). \tag{3.4}$$

To derive the equations for the unknown coefficient $k(\tau)$ consider

$$\int_0^1 Z_{\chi\chi\chi\chi}(\chi,\tau)dx=\int_0^1\sum_{n=1}^{\infty}\mu_n^4 Z_n(\tau)y_n(\chi)d\chi,$$

and

$$\int_0^1 Z_{\chi\chi}(\chi,\tau)dx=-\int_0^1\sum_{n=1}^{\infty}\mu_n^2 Z_n(\tau)y_n(\chi)d\chi,$$

into the AC (2.1). Then we get

$$k(\tau)=\frac{1}{m(\tau)}\Bigg[m'(\tau)-h_{int}(\tau)$$

$$+\sqrt{2}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)K_n}{\mu_n}\left\{z_{0n}\exp(-K_n\tau)+\int_0^{\tau} H_n(s;Z,a)\exp(-K_n(\tau-s))ds\right\}\Bigg], \tag{3.5}$$

where $h_{int}(\tau)=\int_0^1 h(\chi,\tau)d\chi$.
The equations (3.4) and (3.5) are Volterra type integral equations with regard to $Z(\chi,\tau)$, and $k(\tau)$ and form the system of integral equations. The inverse problem (1.2)-(1.4) and (2.1) reduced to the system of equations (3.4) and (3.5). Thus, we can conclude that solving the system of integral equations (3.4) and (3.5) and the inverse problem (1.2)-(1.4) and (2.1) are equivalent.
Before setting and proving the existence and uniqueness theorem of the solution of the system (3.4) and (3.5), let us give the following Banach spaces which will be used in the proof of the main theorem:

I

$$B_T = \Bigg\{Z(\chi,\tau)=\sum_{n=1}^{\infty} Z_n(\tau)y_n(\chi):\ Z_n(\tau)\in C[0,T],$$
$$J_T(Z)=\left(\sum_{n=1}^{\infty}(\mu_n^4\|Z_n\|_{C[0,T]})^2\right)^{1/2}<+\infty\Bigg\},$$

and the norm of $Z(\chi,\tau)$ defined as $\|Z\|_{B_T}\equiv\left(\sum_{n=1}^{\infty}(\mu_n^4\|Z_n\|_{C[0,T]})^2\right)^{1/2}$.

II $E_T = B_T \times C[0,T]$ is the space of the all pairs $\kappa(\chi,\tau) = \{Z(\chi,\tau), k(\tau)\}$ and the norm is

$$\|\kappa\|_{E_T} = \|Z\|_{B_T} + \|k\|_{C[0,T]}.$$

Theorem 3.2. *Let the assumptions of the Lemma 2.2 and 3.1 are satisfied. Then, the system (3.4) and (3.5) (or the IP (1.2)-(1.4) and (2.1)) has a unique solution for small T.*

Proof. Let $\kappa(\chi,\tau) = \{Z(\chi,\tau), k(\tau)\}$ is an arbitrary element belongs to $E_T = B_T \times C[0,T]$. Then the system of equations (3.4) and (3.5) we can be rewritten into the operator equation form as

$$\kappa = \Phi(\kappa)$$

where $\Phi(\kappa) \equiv \{\phi_1, \phi_2\}$ and

$$\phi_1(\kappa) = \sum_{n=1}^{\infty}\left[z_{0n}\exp(-K_n\tau) + \int_0^{\tau} H_n(s;Z,k)\exp(-K_n(\tau-s))ds\right] y_n(\chi),$$

and

$$\phi_2(\kappa) = \frac{1}{m(\tau)}\left[m'(\tau) - h_{int}(\tau)\right.$$

$$\left. +\sqrt{2}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)K_n}{\mu_n}\left\{z_{0n}\exp(-K_n\tau) + \int_0^{t} H_n(s;Z,k)\exp(-K_n(\tau-s))ds\right\}\right].$$

It is obvious that

$$K_n = \mu_n^4\beta(1-\beta)R + \mu_n^2\beta D > 0,\ n = 1,2,....$$

Thus

$$\exp(-K_n\tau) \le 1, \text{ and } \exp(-K_n(\tau-s)) \le 1 \text{ for } 0 \le s \le \tau,\ 0 \le \tau \le T.$$

Let us prove that Φ is a contraction operator in two steps by considering these estimates.
I) Firstly prove that Φ is a continuous onto map on $E_T \to E_T$. It means that we require to prove $\phi_1(\kappa) \in B_T$ and $\phi_2(\kappa) \in C[0,T]$ for an arbitrary element $\kappa(x,\tau) = \{Z(\chi,\tau), k(\tau)\} \in E_T$ with $Z(\chi,\tau) \in B_T$, $k(\tau) \in C[0,T]$.
Let us verify that

$$J_T(\phi_1) = \left(\sum_{n=1}^{\infty}(\mu_n^4\|(\phi_1)_n\|_{C[0,T]})^2\right)^{1/2} < +\infty,$$

where $(\phi_1)_n(\tau) = RHS(Z_n(\tau))$. In other words let us show that $\phi_1(\kappa) \in B_T$. Under the assumptions of the Lemma 3.1 and Theorem 3.2, we obtain

$$J_T(\phi_1) \le 2\left\|z_0^{(4)}\right\|_{L_2[0,1]} + 2T\left\|h_{\chi\chi\chi\chi}\right\|_{L_2[0,1]} + 2T\|k\|_{C[0,T]}\|Z\|_{B_T}. \tag{3.6}$$

Since $Z(\chi,\tau) \in B_T$, and $k(\tau) \in C[0,T]$, the norms $\|k\|_{C[0,T]}$, $\|Z\|_{B_T}$ are finite. Therefore, $J_T(\phi_1)$ is also finite. Thus $\phi_1(\kappa)$ belongs to the space B_T.
Now let's demonstrate that $\phi_2(\kappa) \in C[0,T]$. From the eq. (3.5), we obtain

$$\|\phi_2(\kappa)\|_{C[0,T]} \le \frac{1}{\alpha}\left[\|m'\|_{C[0,T]} + \|h_{int}\|_{C[0,T]}\right.$$

$$\left. +M\left(\left\|z_0^{(4)}\right\|_{L_2[0,1]} + T\left\|h_{\chi\chi\chi\chi}\right\|_{L_2[0,1]} + T\|k\|_{C[0,T]}\|Z\|_{B_T}\right)\right] \tag{3.7}$$

where $\min_{0\le\tau\le T}|m(\tau)| \ge \alpha > 0$, $M = 2\sqrt{2}(c_1\beta(1-\beta)R + c_2\beta D)$.
As in the previous part we can conclude that $\|\phi_2(\kappa)\|_{C[0,T]}$ is bounded. So $\phi_2(\kappa) \in C[0,T]$.
From the eqs. (3.6) and (3.7), we can obtain

$$\|\Phi(\kappa)\|_{E_T} \le L_1(T) + L_2(T)\|k\|_{C[0,T]}\|Z\|_{B_T},$$

where

$$L_1(T)=\left(2+\frac{M}{\alpha}\right)\left\|z_0^{(4)}\right\|_{L_2[0,1]}+T\left(2+\frac{M}{\alpha}\right)\|h_{\chi\chi\chi\chi}\|_{L_2[0,1]}+\frac{1}{\alpha}\left[\|m'\|_{C[0,T]}+\|h_{int}\|_{C[0,T]}\right],$$

$$L_2(T)=T\left(2+\frac{M}{\alpha}\right).$$

Since $L_2(T)$ tends to zero as $T\to 0$, and $L_1(T)$ is a continuous function of T, there exists a sufficiently small $T>0$ such that

$$(L_1(T)+1)^2 L_2(T)<1.$$

Let us define a ball $A:=\left\{\kappa\in E_T:\|\kappa\|_{E_T}\le L_1(T)+1\right\}$ for the fixed T. Then, for every $\kappa\in A$, we get

$$\begin{aligned}\|\Phi(\kappa)\|_{E_T} &\le L_1(T)+L_2(T)\|k\|_{C[0,T]}\|Z\|_{B_T}\\ &\le L_1(T)+L_2(T)(L_1(T)+1)^2<L_1(T)+1.\end{aligned}$$

Thus, Φ is an onto continuous map on E_T.

II) This step our aim is to prove that the operator Φ is a contraction mapping operator. Assume that let z_1 and z_2 be any two elements of E_T. According to the definition of the space E_T yields $\|\Phi(\kappa_1)-\Phi(\kappa_2)\|_{E_T}=\|\phi_1(\kappa_1)-\phi_1(\kappa_2)\|_{B_T}+\|\phi_2(\kappa_1)-\phi_2(\kappa_2)\|_{C[0,T]}$. Here $\kappa_i=\left\{Z^i(\chi,\tau),k^i(\tau)\right\}$, $i=1,2$. consider the following differences

$$\phi_1(\kappa_1)-\phi_1(\kappa_2)=\sum_{n=1}^{\infty}\left[\int_0^{\tau}\left[H_n(s;Z^1,k^1)-H_n(s;Z^2,k^2)\right]\exp(-K_n(\tau-s))ds\right]y_n(\chi),$$

and

$$\phi_2(\kappa_1)-\phi_2(\kappa_2)=\frac{\sqrt{2}}{m(\tau)}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)K_n}{\mu_n}\int_0^{\tau}\left[H_n(s;Z^1,k^1)-H_n(s;Z^2,k^2)\right]\exp(-K_n(\tau-s))ds.$$

We can obtain from the last equations

$$\|\phi_1(\kappa_1)-\phi_1(\kappa_2)\|_{B_T}\le 2T\left(\left\|Z^2\right\|_{B_T}\left\|k^1-k^2\right\|_{C[0,T]}+\left\|k^1\right\|_{C[0,T]}\left\|Z^1-Z^2\right\|_{B_T}\right),$$

and

$$\|\phi_2(\kappa_1)-\phi_2(\kappa_2)\|_{C[0,T]}\le\frac{TM}{\alpha}\left(\left\|Z^2\right\|_{B_T}\left\|k^1-k^2\right\|_{C[0,T]}+\left\|k^1\right\|_{C[0,T]}\left\|Z^1-Z^2\right\|_{B_T}\right).$$

From the these inequalities it follows that

$$\|\Phi(\kappa_1)-\Phi(\kappa_2)\|_{E_T}\le L_2(T)C(k^1,Z^2)\|\kappa_1-\kappa_2\|_{E_T},$$

where $L_2(T)=T\left(2+\frac{M}{\alpha}\right)$, and $C(k^1,Z^2)=\max\left\{\left\|k^1\right\|_{C[0,T]},\left\|Z^2\right\|_{B_T}\right\}=L_1(T)+1$.

Since $L_1(T)+1\le(L_1(T)+1)^2$,

$$0<L_2(T)C(k^1,Z^2)=L_2(T)(L_1(T)+1)\le L_2(T)(L_1(T)+1)^2<1,$$

i.e. $0<L_2(T)C(k^1,Z^2)<1$. Therefore, the operator Φ is contraction mapping operator.

Thus we can conclude that the operator Φ is contraction mapping and it is a continuous onto map on E_T. Then the solution of $\kappa=\Phi(\kappa)$ exists and unique regarding to Banach fixed point theorem. □

Example 3.3. *Consider the IP (1.2)-(1.5) with unknown smooth $k(\tau)$ and inputs:*

$$z_0(\chi)=\sin(\pi\chi),\ m(\tau)=\frac{2}{\pi}e^{\tau},\ h(\chi,\tau)=\left[\left(1+\pi^4\beta(1-\beta)R+\pi^2\beta D\right)e^{\tau}-1\right]\sin(\pi\chi).$$

These inputs satisfy the conditions of the Lemma 2.2 and Lemma 3.1, i.e.

$$z_0(\chi)\in C^3[0,1],\ z_0^{(4)}(\chi)\in L_2[0,1],\ z_0(0)=z_0(1)=z_0''(0)=z_0''(1)=0,$$

$$h(\chi,\tau)\in C^{3,0}(\Pi_T),\ h_{\chi\chi\chi\chi}(\cdot,\tau)\in L_2[0,1],\ h(0,\tau)=h(1,\tau)=h_{\chi\chi}(0,\tau)=h_{\chi\chi}(1,\tau)=0,$$

and

$$m(\tau)\in C^1[0,T],m(\tau)\neq 0\ \textit{for all}\ \tau\in[0,T]\ \textit{and}\ \int_0^1 z_0(\chi)d\chi=m(0)=\frac{2}{\pi}.$$

According to the Theorem 3.2, the solution of the inverse problem (1.2)-(1.5) exists and unique and the solution is

$$\{Z(\chi,\tau),k(\tau)\}=\left\{e^{\tau}\sin(\pi\chi),e^{-\tau}\right\}.$$

4. Conclusion

The manuscript studies the IP of obtaining the solely time-dependent zero-order coefficient in a linear Bi-flux diffusion equation from an extra integral observation. The theorem of existence and uniqueness of the solution of the IP is proved for an adequately small time interval by applying the contraction mapping principle. This work is new and has never been studied theoretical or numerical before. The numerical method of the IP will be considered with a suitable scheme as a future work.

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