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The Existence and Uniqueness of Periodic Solutions for A Kind of Forced Rayleigh Equation

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ABSTRACT

In this study, the coincidence degree theory has been used to determine new results on the existence and uniqueness of T-periodic solutions for a type of Rayleigh equation as follows

$$u''(t) + f\big(t, u'(t)\big)u'(t) + g\big(t, u(t)\big) = p(t).$$

Keywords: *Rayleigh equation; Periodic solutions; existence; uniqueness; coincidence degree*

1. INTRODUCTION

The dynamic behaviors of Rayleigh-type equations with and without deviating arguments have been extensively studied. It is still being studied because of its applications in many disciplines such as mechanics, physics, engineering and various other technical fields. In the recent past, the Rayleigh equation and Rayleigh type equations have been studied. These studies were focused on existence and uniqueness of periodic solutions with and without deviating arguments. (see [1-10]).

Recently, in 2008, Li and Huang [4] studied the Rayleigh equation of the form

$$u''(t) + f\big(t, u'(t)\big) + g\big(t, u(t)\big) = p(t). \tag{1.1}$$

They established sufficient conditions for the existence and uniqueness of T -periodic solutions of this equation.

The aim of this work is to determine sufficient conditions for existence and uniqueness of T-periodic solutions of the Rayleigh equation of the following form

$$u''(t) + f\big(t, u'(t)\big)u'(t) + g\big(t, u(t)\big) = p(t)$$

or an equivalent system

$$\begin{cases} \dfrac{du}{dt} = v(t) \\ \dfrac{dv}{dt} = -f\big(t, v(t)\big)v(t) - g\big(t, u(t)\big) + p(t) \end{cases} \tag{1.2}$$

where $p: R \to R$ and $f, g: R \times R \to R$ are continuous functions, p is T-periodic, f and g are T-periodic in the first argument with period $T > 0$.

2. PRELIMINARY RESULTS

First, assume an operator equation in a Banach space X as follows

$$Lz = \lambda Nz, \qquad \lambda \in (0,1) \tag{2.1}$$

where $L: DomL \cap X \to X$ is a linear operator and λ is a parameter. P and Q represent two projectors,

$P: DomL \cap X \to X$ and $Q: X \to X/ImL.$

For easy understanding, the continuation theorem [1, p.40] has been explained as follows.

Lemma 2.1. *Let X be a Banach space. Suppose that $L: DomL \subset X \to X$ is a Fredholm operator with index*

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zero and $N:\bar{\Omega} \to X$ is L-compact on $\bar{\Omega}$ with Ω open bounded in X. Moreover, assume that all the following conditions are satisfied.

(i) *$Lz \neq \lambda Nz$, for all $z \in \partial\,\Omega\, \cap DomL, \lambda \in (0,1)$;*

(ii) *$QNz \neq 0$, for all $z \in \partial\,\Omega\, \cap KerL$;*

(iii) *The Brower degree*

$d[QN, \Omega\, \cap KerL, 0] \neq 0$.

Then equation $Lz = Nz$ has at least one solution in $\bar{\Omega}$.

Second, the Brousk theorem has been described as follows:

Lemma 2.2. *([2]). Suppose $\Omega \subset R^n$ is an open bounded set which including the origin and symmetric with respect to the origin, if $A: \bar{\Omega} \to R^n$ is a continuous mapping, and*

$Az = -A(-z) \neq 0, \qquad z \in \partial\,\Omega,$

then $d[A, \Omega\,, 0] \neq 0$.

For ease of exposition, throughout this paper we will adopt the following notations:

$$|u|_k = \left(\int_0^T (|u(t)|^k dt\right)^{\frac{1}{k}},$$

$$|u|_\infty = \max_{t\in[0,T]} |u(t)|.$$

Let us denote

$$X = \left\{z = \left(u(t), v(t)\right)^T \in C(R, R^2): z\ is\ T - \text{periodic}\right\},$$

which is a Banach space endowed with the norm $\|.\|$ defined by

$\|z\| = |u|_\infty + |v|_\infty, \qquad$ for all $z \in X$.

We define a linear operator $L: DomL \subset X \to X$ by setting

$$DomL = \left\{z = \left(u(t), v(t)\right)^T \in C^1(R, R^2): z\ is\ T - \text{periodic}\right\}$$

and $z \in DomL$

$$Lz = z' = (u'(t), v'(t))^T. \tag{2.2}$$

Also define a nonlinear operator $N: X \to X$ by setting

$$Nz = (v(t), -f\left(t, v(t)\right)v(t) - g\left(t, u(t)\right) + p(t))^T. \tag{2.3}$$

In the context of (2.2) and (2.3), the operator equation

$Lz = \lambda Nz$

is equivalent to the following system

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \lambda \begin{pmatrix} v(t) \\ -f\left(t, v(t)\right)v(t) - g\left(t, u(t)\right) + p(t) \end{pmatrix}, \quad \lambda \in (0,1) \tag{2.4}$$

Again from (2.2) and (2.3), we can get

$$KerL = R^2,$$

and

$$ImL = \left\{z \in X: \int_0^T z(s)ds = 0\right\}.$$

Hence, the linear operator L is Fredholm operator with index zero.

Define the continuous projectors $P: X \to KerL$ and $Q: X \to X/ImL$ by setting

$$Pz(t) = \frac{1}{T}\int_0^T z(s)ds$$

and

$$Qz(t) = \frac{1}{T}\int_0^T z(s)ds.$$

Thus, $ImP = KerL$ and $KerQ = ImL$. Moreover, the generalized inverse (of L) $K_p: ImL \to DomL \cap KerP$ is described as

$$(K_p z)(t) = \begin{pmatrix} \int_0^t u(s)ds - \frac{1}{T}\int_0^T \int_0^t u(s)ds\, dt \\ \int_0^t v(s)ds - \frac{1}{T}\int_0^T \int_0^t v(s)ds\, dt \end{pmatrix}, z(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \epsilon ImL. \tag{2.5}$$

Therefore, from (2.3) and (2.5) it is easy to see that N is L-compact on $\bar{\Omega}$, where Ω is any open bounded set in X.

Lemma 2.3. *([4]).Assume that the following condition holds.*

$$(\mathrm{H}_1)\ \big(g(t,u_1)-g(t,u_2)\big)(u_1-u_2)<0, for\ all\ t\in R, u_1,u_2\in R\ and\ u_1\neq u_2.$$

Then system (1.2) *has at most one T-periodic solution.*

3. MAIN RESULTS

Theorem 3.1. *Let* (H_1) *hold. Furthermore, assume that the following conditions ensure.*

(H_2) *There exists a positive constant* d^**such that*

$$u\big(g(t,u)-p(t)\big)<0\ for\ all\ t\in R, |u|\geq d^*.$$

(H_3) *There exists nonnegative constants r and K such that*

$$r<\frac{1}{T},\qquad |f(t,u)|\leq 2r+\frac{K}{|u|},\qquad for\ all\ t\in R, u\in R\backslash\{0\}$$

Then system (1.2) *has a unique T-periodic solution.*

Proof. By Lemma 2.3, along with (H_1), it can be easily seen that system (1.2) has at most one *T*-periodic solution. Therefore, to prove theorem 3.1, it is enough to show that system (1.2) has at least one *T*-periodic solution. For it, we shall apply Lemma 2.1. Firstly, we shall claim that set of all possible *T*-periodic solutions of (2.4) are bounded.

Let $z=\big(u(t),v(t)\big)^T\in X$ be an arbitrary *T*-periodic solution of (2.4). From (2.4), we get

$$u''+\lambda f\left(t,\frac{1}{\lambda}u'(t)\right)u'(t)+\lambda^2\big[g\big(t,u(t)\big)-p(t)\big]=0\,,\qquad \lambda\in(0,1),\tag{3.1}$$

Set

$$u(t_1)=\max_{t\in R}u(t)\,,\qquad u(t_2)=\min_{t\in R}u(t)\,,\text{where } t_1,t_2\in R.$$

Then, we get

$$u'(t_1)=u'(t_2)=0,\qquad u''(t_1)\leq 0,\qquad \text{and}\qquad u''(t_2)\geq 0.$$

It is follows from (3.1) that

$$g(t_1,u(t_1))-p(t_1)\geq 0\ \text{ and } g(t_2,u(t_2))-p(t_2)\leq 0.$$

In the context of (H_2), we obtain

$$u(t_1)<d^*\text{ and }u(t_2)>-d^*.$$

Since $u(t)$ is continuous function on R, for the following inequality there exists a constant $\xi\in R$

$$|u(\xi)|\leq d^*.\tag{3.2}$$

Let $\xi=mT+\bar{\xi}$, where $\bar{\xi}\in[0,T]$, and m is an integer. Then, we have

$$|u(t)|=\left|u(\bar{\xi})+\int_{\bar{\xi}}^{t}u'(s)ds\right|\leq d^*+\int_{\bar{\xi}}^{t}|u'(s)|ds\,, t\in[0,T],$$

and

$$|u(t)|=|u(t-T)|=\left|u(\bar{\xi})-\int_{t-T}^{\bar{\xi}}|u'(s)|ds\right|\leq d^*+\int_{t-T}^{\bar{\xi}}|u'(s)|ds\,, t\in\big[\bar{\xi},\bar{\xi}+T\big].$$

Bringing together the above two inequalities we ascertain

$$\begin{aligned}|u|_\infty&=\max_{t\in[0,T]}|u(t)|=\max_{t\in[\bar{\xi},\bar{\xi}+T]}|u(t)|\\&\leq\max_{t\in[\bar{\xi},\bar{\xi}+T]}\left\{d^*+\frac{1}{2}\left(\int_{\bar{\xi}}^{t}|u'(s)|ds+\int_{t-T}^{\bar{\xi}}|u'(s)|ds\right)\right\}\\&\leq d^*+\frac{1}{2}\int_0^T|u'(s)|ds\end{aligned}$$

$$\leq d^* + \frac{1}{2}\sqrt{T}|u'|_2. \tag{3.3}$$

Set

$$\Omega_1 = \{t: t \in [0,T], |u(t)| > d^*\}, \Omega_2 = \{t: t \in [0,T], |u(t)| \leq d^*\}.$$

Multiplying $u(t)$ and Eq. (3.1) and then integrating it from 0 to T, by (H_2), (H_3), (3.3) and schwarz inequality, we get

$$|u'|_2^2 = -\int_0^T u''(t)u(t)dt$$

$$= \int_0^T \left\{\lambda f\left(t, \frac{1}{\lambda}u'(t)\right)u'(t) + \lambda^2\big[g\big(t,u(t)\big) - p(t)\big]\right\}u(t)dt$$

$$= \int_0^T \lambda f\left(t, \frac{1}{\lambda}u'(t)\right)u'(t)u(t)dt + \lambda^2\int_{\Omega_1}[g\big(t,u(t)\big) - p(t)]u(t)dt$$

$$+\lambda^2\int_{\Omega_2}\big[g\big(t,u(t)\big) - p(t)\big]u(t)dt$$

$$\leq \int_0^T \lambda\left[2r + \frac{\lambda K}{|u'(t)|}\right]|u'(t)||u(t)|dt$$

$$+\lambda^2\int_{\Omega_2}\big|g\big(t,u(t)\big) - p(t)\big||u(t)|dt$$

$$\leq 2r|u|_\infty\int_0^T|u'(t)|dt + |u|_\infty T(max\{|g(t,u) - p(t)|: t \in R, |u| \leq d^*\} + K)$$

$$\leq r\big(\sqrt{T}|u'|_2 + 2d^*\big)\sqrt{T}|u'|_2$$

$$+ T(max\{|g(t,u) - p(t)|: t \in R, |u| \leq d^*\} + K)\left(\frac{1}{2}\sqrt{T}|u'|_2 + d^*\right) \tag{3.4}$$

Since $0 \leq r < \frac{1}{T}$, (3.4) signifies that there exists a positive constant D_1 such that

$$|u'|_2 \leq D_1 \quad \text{and} \quad |u|_\infty \leq D_1\,. \tag{3.5}$$

Set $t_1 \in [0,T]$ such that $|u(t_1)| = \max_{t\in[0,T]}|u(t)|$, then $u'(t_1) = 0$. In the context of the first equation of (2.4), we have

$$v(t_1) = 0.$$

Then we can choose a positive constant D_2 such that

$$|v(t)| = \left|v(t_1) + \int_{t_1}^t v'(s)ds\right|$$

$$\leq |v(t_1)| + \int_{t_1}^t|v'(s)|ds$$

$$\leq \int_0^T|v'(s)|ds$$

$$\leq \int_0^T\left|\lambda f\left(t, \frac{1}{\lambda}u'(t)\right)u'(t) + \lambda^2\big[g\big(t,u(t)\big) - p(t)\big]\right|dt$$

$$\leq \int_0^T \lambda\left[2r + \frac{\lambda K}{|u'(t)|}\right]|u'(t)|dt + \int_0^T\big|\lambda^2\big[g\big(t,u(t)\big) - p(t)\big]\big|dt$$

$$\leq 2r\int_0^T|u'(t)|dt + T[K + max\{|g(t,u) - p(t)|: t \in R, |u| \leq D_1\}]$$

$$\leq 2r\sqrt{T}|u'(t)|_2 + T[K + max\{|g(t,u) - p(t)|: t \in R, |x| \leq D_1\}]$$

$$\leq D_2 \tag{3.6}$$

where $t \in [t_1, t_1 + T]$.

Set

$$\Omega = \{z = (u,v)^T \in X: |u|_\infty + |v|_\infty < D_1 + D_2 + d^* + 1 = D\},$$

It is known that the system (2.4) has no solution on $\partial\Omega$ as $\lambda \in (0,1)$. Let $z = (u,v)^T \in \partial\Omega \cap KerL = \partial\Omega \cap R^2$. z is a constant vector in R^2 with $\|z\| = D$. From (H_2), if $v = 0$, we get

$$|u|_\infty = D > d^* + 1$$

and

$$-\frac{1}{T}\int_0^T (f(t,v)v + g(t,u) - p(t))dt = -\frac{1}{T}\int_0^T (g(t,u) - p(t))dt \neq 0.$$

Thus, in any case

$$QNz = (v, -\frac{1}{T}\int_0^T (f(t,v)v + g(t,u) - p(t))dt)^T \neq 0, \qquad z \in \partial\Omega \cap KerL. \tag{3.7}$$

Define a continuous mapping $A: \bar{\Omega} \to R^2$ by

$$Az = (v,u)^T, \qquad for\ all\ z = (u,v)^T \in \bar{\Omega}.$$

Clearly, Ω is symmetric with regard to the origin and

$$Az = -A(-z) \neq 0, \qquad \text{for all } z \in \partial\Omega \cap KerL$$

by applying Lemma 2.2, we have

$$\mathrm{d}[A, \Omega \cap KerL, 0] \neq 0. \tag{3.8}$$

Like the proof of (3.7), it is easy to prove that

$$\phi(z,\lambda) = \lambda Az + (1-\lambda)QNz$$
$$= \left(\lambda v + (1-\lambda)v, \lambda u - (1-\lambda)\frac{1}{T}\int_0^T \big(f(t,v)v + g(t,u) - p(t)\big)dt\right)^T$$

is homotopy mapping such that $\phi(z,\lambda) \neq 0$ on $(\partial\Omega \cap KerL) \times [0,1]$.

Hence, by using the homotopy invariance theorem, we have

$$d[QN, \Omega \cap KerL, 0] = d[A, \Omega \cap KerL, 0] \neq 0.$$

It is now known that Ω satisfies all the requirement in Lemma2.1, and then $Lz = Nz$ has at least one solution in the Banach space X, therefore, it is proved that system (1.2) has a unique T-periodic solution. Hence, the proof is completed. □

4. AN EXAMPLE

In this section, an example is provided to demonstrate the above ascertained outcomes.

Example 4.1. Let us consider the following forced Rayleigh equations:

$$u''(t) + \frac{1}{2014\pi}\left[\sin\big(u'(t)\big)\, e^{cos(u'(t))} + e^{\sin(u'(t))}\cos(\frac{1}{4}t)\right]u'(t) - \left(13 + cos^2\left(\frac{1}{4}t\right)\right)u^{2013}(t)$$
$$= sin^2\left(\frac{1}{4}t\right). \tag{4.1}$$

The equivalent system of (4.1) can be constructed as follows

$$\begin{cases} \dfrac{du}{dt} = v(t) \\ \dfrac{dv}{dt} = -\dfrac{1}{2014\pi}\left[\sin\big(v(t)\big)\, e^{cos(v(t))} + e^{\sin(v(t))}\cos(\frac{1}{4}t)\right]v(t) \\ \qquad\qquad + \left(13 + cos^2\left(\frac{1}{4}t\right)\right)u^{2013}(t) + sin^2\left(\frac{1}{4}t\right). \end{cases} \tag{4.2}$$

Since

$$f(t,u) = \frac{1}{2014\pi}\left(\sin(u)\, e^{cos(u)} + e^{\sin(u)}\cos(\frac{1}{4}t)\right),$$

$$g(t,u) = -\left(13 + cos^2\left(\frac{1}{4}t\right)\right)u^{2013},$$

$$p(t) = sin^2\left(\frac{1}{4}t\right),$$

One can easily check that the conditions (H_1) and (H_3) are provided. Now let show that (H_2) holds. Choose $d^* = 1$. For all $t \in R$ and $|u| \geq d^* = 1$, we have

$$u\big(g(t,u) - p(t)\big) = u\left(-\left(13 + cos^2\left(\frac{1}{4}t\right)\right)u^{2013} - sin^2\left(\frac{1}{4}t\right)\right)$$

$$= -13u^{2014} - cos^2\left(\frac{1}{4}t\right)u^{2014} - sin^2\left(\frac{1}{4}t\right)u$$

$$= \left(-12 - cos^2\left(\frac{1}{4}t\right)\right)u^{2014} - \left(u^{2014} + sin^2\left(\frac{1}{4}t\right)u\right)$$

Since $|u| \geq 1$ and $sin^2\left(\frac{1}{4}t\right) \leq 1$, automatically $\left(u^{2014} + sin^2\left(\frac{1}{4}t\right)u\right) \geq 0$. As a result

$$u\big(g(t,u) - p(t)\big) < 0.$$

So (H_2) holds. Since (H_1), (H_2) and (H_3) holds, by Theorem 3.1, system (4.2) has a unique 8π-periodic solution. Therefore, Rayleigh Eq. (4.1) has a unique 8π-periodic solution.

5. CONCLUSION

Since $f(t,u) = \frac{1}{2014\pi}\left(\sin(u)\, e^{cos(u)} + e^{\sin(u)}\cos(\frac{1}{4}t)\right)$, it can be easily seen that all the results in [1-10] and the references therein cannot be applicable to Eq. (4.1) to obtain the existence and uniqueness of 8π-periodic solution. This implies that the results of this paper are essentially new.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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