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Coupled Fixed Point Theorems For Mixed G-Monotone Mappings In Partially Ordered Metric Spaces Via New Functions

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ABSTRACT

In this paper, we prove some coupled coincidence point results for mixed g-monotone mappings in partially ordered metric spaces via new functions. The main results of this paper are generalizations of the main results of Luong and Thuan [N.V. Luong, N.X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. 74 (2011) 983-992].

Keywords: Coupled fixed point, C-class function, Partially Ordered Metric Space.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory play a major role in mathematics. The Banach contraction principle [19] is the simplest one corresponding to fixed point theory. So a large number of mathematicians have extended it and have

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been interested in fixed point theory in some metric spaces. One of this spaces is partially ordered metric space, that is, metric spaces endowed with a partial ordering [2, 3, 7, 8, 9, 15].

The existence of a fixed point for contraction type mappings in partially ordered metric spaces has been considered by many authors [5, 6]. Bhaskar and Lakshmikantham [15] introduced the notion of a coupled fixed point and proved some coupled fixed point theorems for mappings satisfying a mixed monotone property and discussed the existence and uniqueness of a solution for a periodic boundary value problem. Lakshmikantham and ´Ciri´c [16] introduced the concept of a mixed g-monotone mapping and proved coupled coincidence and common fixed point theorems that extend theorems from [15]. Subsequently, many authors obtained several coupled coincidence and coupled fixed point theorems in some ordered metric spaces [1, 12, 14, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31].

Definition 1. ([15]) Let $(X,\leq)$ be a partially ordered set and $F: X\times X\to X$. The mapping F is said to have the mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x,y\in X$,

$x_1,x_2\in X,\quad x_1\leq x_2 \Rightarrow F(x_1,y)\leq F(x_2,y)$

and

$y_1,y_2\in X,\quad y_1\leq y_2 \Rightarrow F(x,y_1)\geq F(x,y_2)$.

Definition 2. ([15]) An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F: X\times X\to X$ if $F(x,y)=x, F(y,x)=y$.

Definition 3. ([16]) An element $(x,y)\in X\times X$ is called a coupled coincidence point of a mapping $F: X\times X\to X$ and $g: X\to X$ if $F(x,y)=gx, F(y,x)=gy$.

Definition 4. ([16]) Let X be non-empty set and $F: X\times X\to X$ and $: X\to X$. We say F and g are commutative if $gF(x,y)=F(gx,gy)$ for any $x,y\in X$.

Definition 5. ([16]) Let $(X,\leq)$ be a partially ordered set and $F: X\times X\to X, g: X\to X$ be mappings. F is said to have the mixed g-monotone property if $F(x,y)$ is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x,y\in X$,

$x_1,x_2\in X,\quad gx_1\leq gx_2 \Rightarrow F(x_1,y)\leq F(x_2,y)$

and

$y_1,y_2\in X,\quad gy_1\leq gy_2 \Rightarrow F(x,y_1)\geq F(x,y_2)$.

Lemma 1. ([29]) Let X be non-empty set and $g: X\to X$ be a mapping . Then, there exists a subset $E\subseteq X$ such that $g(E)=g(X)$ and $g: E\to X$ is one-to-one.

Theorem 1. ([15]) Let $(X,\leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space. Let $F: X\times X\to X$ be a continuous mapping having the mixed monotone property. Assume that there exists a $k\in[0,1)$ with

$d\big(F(x,y),F(u,v)\big)\leq\frac{k}{2}[d(x,u)+d(y,v)]$ for all $x\geq u$ and $y\leq v$.

If there exist two elements $x_0,y_0\in X$ with

$x_0\leq F(x_0,y_0)$ and $y_0\geq F(y_0,x_0)$

then there exist $x,y\in X$ such that

$x=F(x,y)$ and $y=F(y,x)$.

Theorem 2. ([15]) Let $(X,\leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space. Assume that X has the following property,

(1) if a non-decreasing sequence $\{x_n\}\to x$, then $x_n\leq x$ for all $n\in\mathbb{N}$,

(2) if a non-increasing sequence $\{y_n\}\to y$, then $y\leq y_n$ for all $n\in\mathbb{N}$,

Let $F: X\times X\to X$ be a continuous mapping having the mixed monotone property. Assume that there exists a $k\in[0,1)$ with

$d(F(x,y),F(u,v))\leq\frac{k}{2}[d(x,u)+d(y,v)]$ for all $x\geq u$ and $y\leq v$.

If there exist two elements $x_0,y_0\in X$ with

$x_0\leq F(x_0,y_0)$ and $y_0\geq F(y_0,x_0)$

then there exist $x,y\in X$ such that

$x=F(x,y)$ and $y=F(y,x)$.

Theorem 3. ([10]) Let $(X,\leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space. Let $F: X\times X\to X$ be a mapping having the mixed monotone property and there exist two elements $x_0,y_0\in X$ with $x_0\leq F(x_0,y_0)$ and $y_0\geq F(y_0,x_0)$.

Suppose that F, g satisfy

$$\varphi(d\big(F(x,y),F(u,v)\big))\leq\frac{1}{2}\varphi(d(x,u)+d(y,v))-\psi(\tfrac{d(x,u)+d(y,v)}{2})$$

for all $x,y,u,v\in X$ with $x\geq u$ and $y\leq v$. Suppose either

(1) F is continuous or

(2) X has the following property :

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

(a) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$,

then there exist $x, y \in X$ such that

$$x = F(x, y) \quad \text{and} \quad y = F(y, x)$$

that is, F has a coupled fixed point in X.

2. The main results

In this paper, we prove coupled coincidence and common fixed point theorems for mixed g-monotone mappings satisfying more general contractive conditions in partially ordered metric spaces. We prove fixed point theorems via new functions. We also present results on existence and uniqueness of coupled common fixed points.

Let Φ denote all functions $\varphi: [0, \infty) \to [0, \infty)$ which satisfy

(1) φ is continuous and non-decreasing,

(2) $\varphi(t) = 0$ and only if $t = 0$,

(3) $\varphi(t + s) \leq \varphi(t) + \varphi(s),\ \forall t, s \in [0, \infty)$

and Ψ denote all functions $\psi: [0, \infty) \to [0, \infty)$ which satisfy $\lim \psi(t) > 0$ for all $r > 0$ and $\lim_t \psi(t) = 0$ and Ψ_1 denote all functions $\psi: [0, \infty) \to [0, \infty)$ which satisfy $\psi(0) \geq 0, \psi(t) > 0$ for all $t > 0$.

Definition6. A function $\phi: [0, \infty) \to [0, \infty)$ is called an ultra-altering distance function if ϕ is continuous, and $\phi(0) \geqq 0, \phi(t) > 0, t \neq 0$,

In 2014 the concept of C -class functions was introduced by Ansari [4] which

cover a large class of contractive conditions.

Definition 7. [4] A mapping $f: [0, \infty)^2 \to R$ is called C-class function if it is continuous and satisfies following axioms:

(1) $f(s, t) \leq s$,

(2) $f(s, t)$ =s implies that either s=0 or t=0 for all s,t∈[0,∞).

Note that for some f we have $f(0,0) = 0$.

We denote C-class functions as C.

Example 1. [4] The following functions $f: [0, \infty)^2 \to R$ are elements of C:

(1) $f(s, t) = s - t, f(s, t) = s \Rightarrow t = 0$.

(2) $f(s, t) = ms, 0 < m < 1, f(s, t) = s \Rightarrow s = 0$.

(3) $f(s, t) = \frac{s}{(1+t)^r};\ r \in (0, \infty), f(s, t) = s \ \Rightarrow\ s = 0\ or\ t = 0$.

(4) $f(s;\ t) = log_a\left(\frac{t + a^s}{1 + t}\right), a > 1, f(s;\ t) = s \implies s = 0\ or\ t = 0$.

(5) $f(s, t) = log_a(\frac{1 + a^s}{2}), f(s, t) = s \ \Rightarrow\ s = 0$.

(6) $f(s, t) = (s + l)^{\frac{1}{(1+t)^r}} - l, l > 1, l > 1, r \in (0, \infty), f(s, t) = s \ \Rightarrow\ t = 0$.

(7) $f(s, t) = s \log_{a+t} a, a > 1, f(s, t) = s \Rightarrow\ s = 0\ or\ t = 0$.

(8) $f(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})), f(s, t) = s \Rightarrow t = 0$.

(9) $f(s, t) = s\beta(s), \beta: [0, \infty) \to [0,1)$ and is continuous, $f(s, t) = s \Rightarrow s = 0$;

(10) $f(s, t) = s - \frac{t}{k+t}, f(s, t) = s \Rightarrow t = 0$.

(11) $f(s, t) = s - \phi(s), f(s, t) = s \Rightarrow s = 0$, here $\phi: [0, \infty) \to [0, \infty)$ is a continuous function such that $\phi(t) = 0 \Leftrightarrow t = 0$.

(12) $f(s, t) = sh(s, t), f(s, t) = s \Rightarrow s = 0$,here $h: [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0$,

(13) $f(s,t) = s - (\frac{2+t}{1+t})t, f(s,t) = s \Rightarrow t = 0.$

(14) $f(s,t) = \sqrt[n]{ln(1+s^n)}, f(s,t) = s \Rightarrow s = 0.$

(15) $f(s,t) = \varphi(s), f(s,t) = s \Rightarrow s = 0$, here $\varphi\colon [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

(16) $f(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty), f(s,t) = s \Rightarrow s = 0.$

Theorem 4. Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F\colon X \times X \to X$ be a mapping having the mixed monotone property and there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose that F, g satisfy

$$\phi(d(F(x,y), F(u,v))) \leq f(\frac{1}{2}\phi(d(gx,gu) + d(gy,gv)), \psi(\frac{d(gx,gu)+d(gy,gv)}{2})) \quad (2.1)$$

for all $x, y, u, v \in X$ with $gx \leq gu$ and $gy \geq gv$, $F\ (X \times X) \subseteq g(X)$, $g(X)$ is complete and g is continuous and f is element of C .

Suppose that either

(1) F is continuous or

(2) X has the following property:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.

Then there exist $x, y \in X$ such that

$$gx = F(x,y) \text{ and } gy = F(y,x)$$

that is, F and g have a coupled coincidence point in $X \times X$.

Proof. Using Lemma1, there exists $E \subseteq X$ such that $g(E) = g(X)$ and $g\colon E \to X$ is one-to-one. We define a mapping $A\colon g(E) \times g(E) \to X$ by

$$A(gx, gy) = F(x,y), \forall gx, gy \in g(E). \quad (2.2)$$

As g is one to one on $g(E)$, so A is well-defined. Thus, it follows from (2.1) and (2.2) that

$$\phi(d(\mathrm{A(x,y)}, \mathrm{A(u,v)})) \leq f(\frac{1}{2}\phi(d(gx,gu) + d(gy,gv)), \psi(\frac{d(gx,gu)+d(gy,gv)}{2})) \quad (2.3)$$

for all $gx, gy, gu, gv \in g(E)$ with $gx \leq gu$ and $gy \geq gv$. Since F has the mixed g-monotone property, for all $x, y \in X$, we have

$$x_1, x_2 \in X, \quad gx_1 \leq gx_2 \Rightarrow F(x_1, y) \leq F(x_2, y) \quad (2.4)$$

and

$$y_1, y_2 \in X, \quad gy_1 \geq gy_2 \Rightarrow F(x, y_1) \leq F(x, y_2). \quad (2.5)$$

Thus, it follows from (2.2), (2.4) and (2.5) that, for all $gx, gy \in g(E)$,

$$gx_1, gx_2 \in g(X), \quad gx_1 \leq gx_2 \Rightarrow A(gx_1, gy) \leq A(gx_2, gy)$$

and

$$gy_1, gy_2 \in g(X), \quad gy_1 \geq gy_2 \Rightarrow A(gx, gy_1) \leq A(gx, gy_2),$$

which implies that A has the mixed monotone property.

Suppose that assumption (1) holds. Since F is continuous, A is also continuous. Using the Theorem 3 with the mapping A, it follows that A has a coupled fixed point $(u, v) \in g(E) \times g(E)$.

Suppose that assumption (2) holds. We can conclude similarly in the proof of Theorem 3 that the mapping A has a coupled fixed point $(u, v) \in g(X) \times g(X)$.

Finally, we prove that F and g have a coupled fixed point in X. Since (u, v) is a coupled fixed point of A, we get

$$u = A(u,v), \;\; v = A(v,u). \quad (2.6)$$

Since $(u, v) \in g(X) \times g(X)$, there exists a point $(u', v') \in X \times X$ such that

$$u = gu', \ \ v = gv'. \tag{2.7}$$

Thus, it follows from (2.6) and (2.7) that

$$gu' = A(gu', gv'), \ \ gv' = A(gv', gu'). \tag{2.8}$$

Also, from (2.2) and (2.8), we get

$$\mathrm{gu}' = F(u', v'), \ \ gv' = F(v', u').$$

Therefore, (u', v') is a coupled coincidence point of F and g. This completes the proof. ■

Now with choice f we have let $f(s,t) = ks, 0 < k < 1$ then

Corollary 1. Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property and there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose that F, g satisfy

$$\phi(d(F(x,y), F(u,v))) \leq \frac{k}{2}\phi(d(gx, gu) + d(gy, gv))$$

for all $x, y, u, v \in X$ with $gx \leq gu$ and $gy \geq gv$, $F\ (X \times X) \subseteq g(X)$, $g(X)$ is complete and g is continuous .

Suppose that either

(1) F is continuous or

(2) X has the following property:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.

Then there exist $x, y \in X$ such that

$$gx = F(x, y) \text{ and } gy = F(y, x)$$

that is, F and g have a coupled coincidence point in $X \times X$.

Proof In Theorem 4, taking $f(s;t) = ks, 0 < k < 1.$ ■

Corollary 2.([31]) Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X and there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose that F, g satisfy

$$\phi\left(d\big(F(x,y), F(u,v)\big)\right) \leq \frac{1}{2}\phi\big(d(gx, gu) + d(gy, gv)\big) - \psi(\frac{d(gx, gu) + d(gy, gv)}{2}))$$

for all $x, y, u, v \in X$ with $gx \leq gu$ and $gy \geq gv$, $F\ (X \times X) \subseteq g(X)$, $g(X)$ is complete and g is continuous .

Suppose that either

(1) F is continuous or

(2) X has the following property:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.

Then there exist $x, y \in X$ such that

$$gx = F(x, y) \text{ and } gy = F(y, x)$$

that is, F and g have a coupled coincidence point in $X \times X$.

Proof. In Theorem 4, taking $f(s;t) = s - t$. ■

Corollary 3. Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X and there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose that F, g satisfy

$$\phi\left(d\big(F(x,y), F(u,v)\big)\right) \leq \frac{\frac{1}{2}\phi\big(d(gx, gu) + d(gy, gv)\big)}{1 + \psi(\frac{d(gx, gu) + d(gy, gv)}{2})}$$

for all $x, y, u, v \in X$ with $gx \leq gu$ and $gy \geq gv$, $F\ (X \times X) \subseteq g(X)$, $g(X)$ is complete and g is continuous .

Suppose that either

(1) F is continuous or

(2) X has the following property:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.

Then there exist $x, y \in X$ such that

$$gx = F(x,y) \text{ and } gy = F(y,x)$$

that is, F and g have a coupled coincidence point in $X \times X$.

Proof. In Theorem 4, taking $(s;t) = \frac{s}{1+t}$. ■

Theorem 5. Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property and there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose that F, g satisfy

$$\phi(d(\ F(x,y), F(u,v))) \leq f(\tfrac{1}{2}\phi(d(gx,gu) + d(gy,gv)), \psi(\tfrac{d(gx,gu)+d(gy,gv)}{2})) \tag{2.1}$$

for all $x, y, u, v \in X$ with $gx \leq gu$ and $gy \geq gv$, $F\ (X \times X) \subseteq g(X)$, $g(X)$ is complete, g is continuous and f is element of C .

Suppose that either

(1) F is continuous or

(2) X has the following property:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.

Then there exist $x, y \in X$ such that

$$gx = F(x,y) \text{ and } gy = F(y,x)$$

and

$$x = gx = F(x,y), y = gy = F(y,x)$$

that is, F and g have a coupled coincidence point in $X \times X$.

Proof. Following the proof of Theorem 4, F and g have a coupled coincidence point. We only have to show that $x = gx$ and $y = gy$.

Now, x_0 and y_0 be two points in the statement of the Theorem 4. Since $F\ (X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. In the same way we construct $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing in this way we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n) \ \forall n \geq 0. \tag{2.9}$$

Since $gx \geq gx_{n+1}$ and $gy \leq gy_{n+1}$, from (2.1) and (2.9), we have

$$\begin{aligned} \phi(\ d(gx_{n+1}, gx)) &= \phi(d((F(x_n, y_n), F(x,y))) \\ &\leq f(\tfrac{1}{2}\phi(d(gx_n, gx) + d(gy_n, gy)), \psi(\tfrac{d(gx_n,gx)+d(gy_n,gy)}{2})) \end{aligned} \tag{2.10}$$

Similarly, since $gy_{n+1} \geq gy$ and $gx_{n+1} \leq gx$, from (2.1) and (2.9), we have

$$\begin{aligned} \phi(\ d(gy, gy_{n+1})) &= \phi(d((F(y,x), F(y_n, x_n))) \\ &\leq f(\tfrac{1}{2}\phi(d(gy, gy_n) + d(gx, gx_n)), \psi(\tfrac{d(gy,gy_n)+d(gx,gx_n)}{2})) \end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we have

$\phi\big(d(gx_{n+1}, gx)\big) + \phi(d(gy, gy_{n+1}))$

$$\leq 2f(\tfrac{1}{2}\phi(d(gy, gy_n) + d(gx, gx_n)), \psi(\tfrac{d(gy,gy_n)+d(gx,gx_n)}{2})) \tag{2.12}$$

By property (3) of ϕ, we have

$$\phi(d(gx_{n+1}, gx) + d(gy, gy_{n+1})) \leq \phi\big(d(gx_{n+1}, gx)\big) + \phi(d(gy, gy_{n+1})) \tag{2.13}$$

From (2.12) and (2.13), we have

$\phi(d(gx_{n+1}, gx) + d(gy, gy_{n+1})) \leq 2f(\tfrac{1}{2}\phi(d(gy, gy_n) + d(gx, gx_n)), \psi(\tfrac{d(gy,gy_n)+d(gx,gx_n)}{2}))$

which implies

$$\phi(d(gx_{n+1}, gx) + d(gy, gy_{n+1})) \leq \phi(d(gy, gy_n) + d(gx, gx_n))$$

Using the fact that ϕ is non-decreasing, we get

$$d(gx_{n+1}, gx) + d(gy, gy_{n+1}) \leq d(gy, gy_n) + d(gx, gx_n) \tag{2.14}$$

Set $\delta_n = d(gx_{n+1}, gx) + d(gy, gy_{n+1})$ then sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty}[d(gx_{n+1}, gx) + d(gy, gy_{n+1})] = \delta$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then taking the limit as $n \to \infty$ (equivalently, $\delta_n \to \delta$) of both sides of (2.13) and have in mind that we suppose $lim\psi(t) > 0$ for all $r > 0$ and ϕ is continuous, we have

$\phi(\delta) = \lim_{n\to\infty}\phi(\delta_n) \leq 2\lim_{n\to\infty} f(\tfrac{1}{2}\phi(\delta_{n-1}), \psi(\tfrac{\delta_{n-1}}{2}))$

$= 2f\left(\tfrac{1}{2}\phi(\delta), \lim_{\delta_{n-1}\to\delta}\psi\left(\tfrac{\delta_{n-1}}{2}\right)\right) \leq \phi(\delta)$.

a contradiction. Thus $\delta = 0$, that is

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty}[d(gx_{n+1}, gx) + d(gy, gy_{n+1})] = 0$$

Hence $d(gx_{n+1}, gx) = 0$ and $d(gy, gy_{n+1}) = 0$, that is $x = gx$ and $y = gy$. ■

Corollary 4. ([31]) Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property and there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose that F, g satisfy

$$\phi(d(F(x,y), F(u,v))) \leq f(\tfrac{1}{2}\phi(d(gx, gu) + d(gy, gv)), \psi(\tfrac{d(gx,gu)+d(gy,gv)}{2})) \tag{2.1}$$

for all $x, y, u, v \in X$ with $gx \leq gu$ and $gy \geq gv$, $F\ (X \times X) \subseteq g(X)$, $g(X)$ is complete and g is continuous .

Suppose that either

(1) F is continuous or

(2) X has the following property:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.

Then there exist $x, y \in X$ such that

$$gx = F(x,y) \text{ and } gy = F(y,x)$$

and

$$x = gx = F(x,y), y = gy = F(y,x)$$

that is, F and g have a coupled coincidence point in $X \times X$.

Theorem 6. In addition to hypotheses of Theorem 4, suppose that for every $(x,y),(z,t)$ in $X\times X$, there exists a (u,v) in $X\times X$ that is comparable to (x,y) and (z,t), then F and g have a unique coupled fixed point.

Proof. From Theorem 4, the set of coupled fixed points of F is non-empty. Suppose (x,y) and (z,t) are coupled coincidence points of F, that is $gx=F(x,y), gy=F(y,x), gz=F(z,t)$ and $gt=F(t,z)$. We will prove that

$$gx=gz \quad \text{and} \quad gy=gt.$$

By assumption, there exists (u,v) in $X\times X$ such that $(F(u,v),F(v,u))$ is comparable with $(F(x,y),F(y,x))$ and $(F(z,t),F(t,z))$. Put $u_0=u$ and $v_0=v$ and choose $u_1,v_1\in X$ so that $gu_1=F(u_0,v_0)$ and $gv_1=F(v_0,u_0)$. Then, similarly as in the proof of Theorem 3, we can inductively define sequences $\{gu_n\},\{gv_n\}$ with

$$gx_{n+1}=F(u_n,v_n) \text{ and } gy_{n+1}=F(v_n,u_n) \quad \text{for all } n.$$

Further set $x_0=x, y_0=y, z_0=z$ and $t_0=t$, in a similar way, define the sequences $\{gx_n\},\{gy_n\}$ and $\{gz_n\},\{gt_n\}$. Then it is easy to show that

$$gx_n\to F(x,y), gy_n\to F(y,x) \text{ and } gz_n\to F(z,t), gt_n\to F(t,z)$$

as $n\to\infty$. Since

$$\big(F(x,y),F(y,x)\big)=(gx_1,gy_1)=(gx,gy) \text{ and } (F(u,v),F(v,u))=(gu_1,gv_1)$$

Since $gx\le gu_1$ and $gy\ge gv_1$, or vice versa. It is easy to show that, similarly, (gx,gy) and (gu_n,gv_n) are comparable for all $n\ge 1$, that is, $gx\le gu_n$ and $gy\ge gv_n$, or vice versa. Thus from (2.1), we have

$$\begin{aligned}\phi(\,d(gu_{n+1},gx))&=\phi(d((F(u_n,v_n),F(x,y)))\\&\le f(\tfrac{1}{2}\phi(d(gu_n,gx)+d(gv_n,gy)),\psi(\tfrac{d(gu_n,gx)+d(gv_n,gy)}{2})) \qquad (2.16)\end{aligned}$$

Similarly,

$$\begin{aligned}\phi(\,d(gy,gv_{n+1}))&=\phi(d((F(y,x),F(v_n,xu_n)))\\&\le f(\tfrac{1}{2}\phi(d(gy,gv_n)+d(gx,gu_n)),\psi(\tfrac{d(gy,gv_n)+d(gx,gu_n)}{2})) \qquad (2.17)\end{aligned}$$

From (2.16) and (2.17) and the property of ϕ, we have

$$\begin{aligned}\phi(d(gu_{n+1},gx)+d(gy,gv_{n+1}))&\le\phi\big(\,d(gu_{n+1},gx)\big)+\phi(\,d(gy,gv_{n+1}))\\&\le 2f(\tfrac{1}{2}\phi(d(gy,gv_n)+d(gx,gu_n)),\psi(\tfrac{d(gy,gv_n)+d(gx,gu_n)}{2})) \qquad (2.18)\end{aligned}$$

which implies

$$\phi(d(gu_{n+1},gx)+d(gy,gv_{n+1}))\le\phi(d(gx,gu_n)+d(gy,gv_n))$$

Thus,

$$d(gx_{n+1},gx)+d(gy,gy_{n+1})\le d(gy,gy_n)+d(gx,gx_n)$$

That is, the sequence $\{d(gu_n,gx)+d(gy,gv_n)\}$ is decreasing. Therefore, there is some $\alpha\ge 0$ such that

$$\lim_{n\to\infty}[d(gu_n,gx)+d(gy,gv_n)]=\alpha$$

We shall show that $\alpha=0$. Suppose, to the contrary, that $\alpha>0$. Then taking the limit as $n\to\infty$ in (2.18), we have

$$\phi(\alpha)\le 2f\left(\frac{1}{2}\phi(\alpha),\lim_{\delta_{n-1}\to\delta}\psi\left(\frac{d(gu_n,gx)+d(gy,gv_n)}{2}\right)\right)<\phi(\alpha)$$

a contradiction. Thus $\alpha=0$, that is

$$\lim_{n\to\infty}[d(gu_n,gx)+d(gy,gv_n)]=0$$

It implies

$$\lim_{n\to\infty}d(gu_n,gx)=\lim_{n\to\infty}d(gy,gv_n)=0 \qquad (2.19)$$

Similarly, we show that

$$\lim_{n\to\infty} d(gu_n, gz) = \lim_{n\to\infty} d(gt, gv_n) = 0 \quad (2.20)$$

From (2.19), (2.20) and by the uniqueness of the limit, it follows that, we have $gx = gz$ and $gy = gt$. Hence (gx, gy) is the unique coupled point of coincidence of F and g. ■

Example 2. Let $f(s,t) = \frac{s}{1+t}$ and $X = [0, +\infty)$ endowed with standart metric $d(x,y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete metric space. Define the mapping $F: X \times X \to X$ by

$$F(x,y) = \begin{cases} y & if \quad x \geq y, \\ x & if \quad x < y. \end{cases}$$

Suppose $g: X \to X$ is such that $gx = x^2$ for all $x \in X$ and $\phi(t): [0, +\infty) \to [0, +\infty)$ is such that $\phi(t) = \frac{1}{t}$. Assume that $\psi(t) = \frac{t}{1+t},\ t \neq 0$ and $\psi(0) = \frac{1}{1000}$.

It is easy to show that for all $x, y, u, v \in X$ with $gx \leq gu$ and $gy \geq gv$, we have

$$\phi(\ d((F(x,y), F(u,v))) \leq \frac{\frac{1}{2}\phi(d(gx,gu) + d(gy,gv))}{1 + \psi(\frac{d(gx,gu) + d(gy,gv)}{2})}$$

Thus, it satisfies all conditions of Theorem 4. So we deduce the existence of $F$and g have a coupled coincidence point $(x, y) \in X \times X$. Here, (0,0) is a coupled coincidence point of $F$and g.

Authors.contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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