Gazi University Journal of Science
GU J Sci
29(1):177-185 (2016)

Transmuted Power Function Distribution

Muhammad Ahsan ul HAQ[1], Nadeem Shafique BUTT[2♠], Rana Muhammad USMAN[1], Ahmed A. FATTAH[3]

[1]*College of Statistical and Actuarial Sciences, University of the Punjab, Lahore, Pakistan*
[2]*Department of Family and Community Medicine, King Abdul Aziz Univeristy, Jeddah, KSA*
[3]*Institue of Statistical Studies and Research, Cairo University, Egypt*

Received: 21/10/2015 *Accepted: 11/01/2016*

ABSTRACT

This study provides a three parameter Transmuted Power Function distribution that is the generalization of the Power Function distribution. Structural properties of the proposed distribution are derived including survival function, hazard rate, moments, quantiles, mode, Rényi entropy, smallest and largest densities of ordered statistics. The estimation of the model parameters is performed using maximum likelihood method. Two real data sets are used to demonstrate the flexibility of the new model.

Keywords: *Transmuted, Power Function, Reliability Function, Moment Generating Function, Rényi Entropy, Order Statistics, Maximum Likelihood Estimation.*

1. INTRODUCTION

In the modern era, large number of continuous univariate models exist, among these univariate models some particular occupy a central role because of their demonstrated utility in wide variety of circumstances. Many parametric models are used to model lifetime data. Power function distribution is flexible lifetime

♠Corresponding author, e-mail: nshafique@kau.edu.sa

distribution model which is the special case of beta distribution. Power function distribution was derived from Pareto distribution using the inverse transformation [1]. According to [1], if Y is power function distribution then Y^{-1} is the Pareto distribution model. Meniconi and Barry [2] explore the performance of Power function distribution on electrical components and illustrated that power function distribution is most suitable distribution on electrical component data as compared to log-normal, Weibull and exponential models. Likewise, numerous probability models are used to model income distribution, but these models are mathematically more complicated to manage. The power function distribution on the other hand is very helpful in this regard [3]. The power function distribution can be used to fit the distribution of likelihood ratios in statistical tests. Further the introduction and derivations of statistical properties of the power function distribution discussed by [4-7].

Characterizations of power function distribution using order statistics and record values has been studied by [8], [9] discussed ordered statistics to estimate the scale and location of power function distribution. [10-12] provided detailed discussion on parameter estimation of power function distribution using various estimation procedures like method of moments, maximum likelihood, percentiles, method of least square, and Bayesian estimation with various loss functions. Bayesian analysis of power function distribution was discussed using three single and as well as three double priors and the accuracy of these priors was assessed using simulation studies [13]. An initial test estimator for a scale parameter of the power function distribution was proposed by [14]. Abdulsathar, Renjini [15] estimate the Gini-index and Lorenz curve of power function distribution and the shape parameter using Bayesian approach. The estimators was developed using weighted squared error and squared error loss functions.

Cordeiro and dos Santos Brito [16] derived Beta power function distribution, Tahir, Alizadeh [17] introduced Weibull power function (WPF) distribution, and Oguntunde, Odetunmibi [18] studied the Kumaraswamy Power function distribution.

Cumulative distribution function (cdf) and probability density function (pdf) of power function distribution is given by;

$$F(y;\alpha,\beta) = \left(\frac{y}{\beta}\right)^{\alpha} \quad \text{(1)}$$

$$f(y) = \frac{\alpha y^{\alpha-1}}{\beta^{\alpha}}\ ; \quad 0 < y < \beta, \alpha > 0 \quad \text{(2)}$$

Where β is scale and α is shape parameter.

The purpose of this research is to provide more flexible generalization of power function distribution using the quadratic rank transmutation map introduced by Shaw and Buckley [19]. The generalized distribution is called Transmuted Power Function distribution (T-Ps).

For an arbitrary cdf $F(y)$, [19] defined the transmuted generalized family with cdf and pdf given by

$$F(y) = (1+\theta)G(y) - \theta G^2(y) \quad \text{(3)}$$

and

$$f(y) = g(y)[1+\theta-2\theta G(y)], \quad \text{(4)}$$

respectively and where $|\theta| \leq$. The generalized distribution reduces to parent distribution for $\theta = 0$. Using this approach various generalized distributions have been generated; Transmuted Weibull Lomax, Transmuted Lindley, Transmuted Pareto [20-22]

Furthermore, the study is organized as follows; in section 2 graphical representations of probability density function and the hazard function is given. Section 3 deals with various mathematical properties, random number generation and estimation of proposed distribution.

2. THE T-PS DISTRIBUTION

The random variable Y follows the T-Ps distribution with the probability density function,

$$f(y,\alpha,\beta,\theta) = \frac{\alpha y^{\alpha-1}}{\beta^{\alpha}}\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right] \quad \text{(5)}$$

The corresponding cumulative distribution function is

$$F(y,\alpha,\beta,\theta) = \left(\frac{y}{\beta}\right)^{\alpha}\left[1+\theta-\theta\left(\frac{y}{\beta}\right)^{\alpha}\right] \quad \text{(6)}$$

where α , β and θ are shape, scale and transmuted parameters respectively.

2.1. Reliability Analysis

The reliability function S(y), which is the probability of an item not failing prior to sometime y, is defined by S(y) = 1−F(y). The survival function of a T-Ps distribution is given by

$$S(y) = 1 - \left(\frac{y}{\beta}\right)^{\alpha}\left[1+\theta-\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]$$

The hazard function is given by

$$h(y) = \frac{\frac{\alpha y^{\alpha-1}}{\beta^{\alpha}}\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]}{1-\left(\frac{y}{\beta}\right)^{\alpha}\left[1+\theta-\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]}$$

Figure 1 and Figure 2 illustrates various shapes of probability density function and hazard function for various combinations of parameter respectively

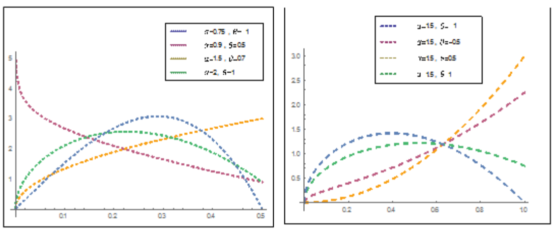


Figure 1: Density Plots of the T-Ps for some parameter values (β=0.5, 1)

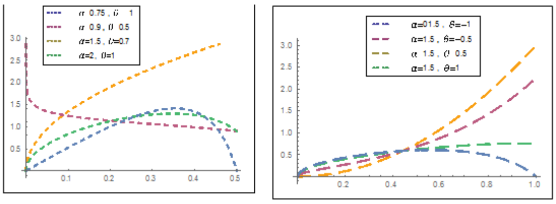


Figure 2: Plots of the T-Ps hazard function for some parameter values (β=0.5, 1)

2.2 Moments

The r[th] moment μ^r of the T-Ps distribution can be derived using relation

$$\mu^r = \int_0^\beta y^r f(y)dy \tag{7}$$

Using the (5) in (7), we get

$$\mu^r = \alpha\beta^r\left[\frac{2\alpha + r(1-\theta)}{(\alpha+r)(2\alpha+r)}\right] \tag{8}$$

Setting r=1 in (8) we can get the mean of the T-Ps distribution

$$Mean = \mu = \alpha\beta\left[\frac{2\alpha + (1-\theta)}{(\alpha+1)(2\alpha+1)}\right]$$

Similarly, we can obtain the variance of T-Ps using the relation $v(Y) = E(Y^2) - E^2(Y)$

2.3 Quantile Function and Random Number Generation

By definition, the quantile function of random variable Y of T-Ps is obtained by inverting the cumulative distribution function as

$$Q = F^{-1}(q)$$

$$y_q = \left[\frac{\beta^\alpha}{\sqrt{\theta}}\left\{\left[\left(\frac{1+\theta}{2\sqrt{\theta}}\right)^2 - q\right]^{\frac{1}{2}} + \frac{1+\theta}{2\sqrt{\theta}}\right\}\right]^{\frac{1}{\alpha}} \tag{9}$$

Using (9) we can generate the random numbers for Transmuted power function distribution, where q is the uniform random variate.

2.4 Mode of T-PS

Taking the first derivative of (4) and equating it to zero, we get

$$Mode = \left[\frac{(\alpha-1)(1+\theta)\beta^\alpha}{2\theta(2\alpha-1)}\right]^{\frac{1}{\alpha}} \tag{10}$$

Using the (10) we can get mode for different values of parameters.

2.5 Rényi Entropy

The Rényi entropy represents a measure of variation of the uncertainty. By definition Rényi entropy defined as

$$I_R(\delta) = \frac{1}{1-\delta}\log[I(\delta)],$$

Where

$$I(\delta) = \int f^\delta(y), \qquad \delta > 0 \quad and\ \delta \neq 1.$$

We have

$$I(\delta) = \int_0^\beta \frac{\alpha^\delta y^{\delta(\alpha-1)}}{\beta^{\delta\alpha}}\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^\alpha\right]^\delta dy$$

$$I(\delta) = \int_0^\beta \frac{\alpha^\delta y^{\delta(\alpha-1)}}{\beta^{\delta\alpha}}\sum_{i=0}^{n}\binom{\delta}{i}\theta^i\left(1-2\left(\frac{y}{\beta}\right)^\alpha\right)^i dy$$

$$I(\delta) = \int_0^\beta \frac{\alpha^\delta y^{\delta(\alpha-1)}}{\beta^{\delta\alpha}}\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{\delta}{i}\theta^i\binom{i}{j}(-1)^j(2)^j\left(\frac{y}{\beta}\right)^{\alpha j} dy$$

$$I(\delta) = \alpha^\delta\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\binom{\delta}{i}\theta^i\binom{i}{j}(-1)^j(2)^j}{\beta^{\delta\alpha+\alpha j}}\int_0^\beta y^{\delta(\alpha-1)+\alpha j}dy$$

$$I(\delta) = \alpha^\delta\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\binom{\delta}{i}\theta^i\binom{i}{j}(-1)^j(2)^j}{\beta^{\delta\alpha+\alpha j}}\left[\frac{y^{\delta(\alpha-1)+\alpha j+1}}{\delta(\alpha-1)+\alpha j+1}\right]_0^\beta$$

$$I(\delta) = \alpha^\delta\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{\delta}{i}\theta^i\binom{i}{j}(-1)^j(2)^j\left(\frac{\beta^{-\delta+1}}{\delta(\alpha-1)+\alpha j+1}\right)$$

$$I(\delta) = \alpha^\delta\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\binom{\delta}{i}\theta^i\binom{i}{j}(-1)^j(2)^j}{\beta^{\delta-1}(\delta(\alpha-1)+\alpha j+1)}$$

$$I_R(\delta) = \frac{\delta\log\alpha}{1-\delta} + \log\beta + \frac{1}{1-\delta}\log\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\binom{\delta}{i}\theta^i\binom{i}{j}(-1)^j(2)^j}{(\delta(\alpha-1)+\alpha j+1)}$$

2.6 Order Statistics

Let be random variables and its ordered values is denoted as $Y_{(1)}, Y_{(2)}, Y_{(3)}, \ldots\ldots\ldots, Y_{(n)}$. The probability density function (pdf) of order statistics is obtained using the below function

$$f_{s:n}(y) = \frac{n!}{(s-1)!\,(n-s)!}f(y)[F(y)]^{s-1}[1-F(y)]^{n-s}$$

The density of the n[th] ordered statistics follows the transmuted Power Function distribution is derived as follow

$$f_{s:n}(y)=\frac{n!}{(s-1)!\,(n-s)!}\left(\frac{\alpha y^{\alpha-1}}{\beta^{\alpha}}\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]\right)\left(\left(\frac{y}{\beta}\right)^{\alpha}\left[1+\theta-\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]\right)^{s-1}$$

$$\left(1-\left(\frac{y}{\beta}\right)^{\alpha}\left[1+\theta-\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]\right)^{n-s}$$

The density of the smallest order statistic, is obtained as

$$f_{1:n}(y)=n\left[\left\{\frac{\alpha y^{\alpha-1}}{\beta^{\alpha}}\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]\right\}\left\{1-\left(\frac{y}{\beta}\right)^{\alpha}\left[1+\theta-\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]\right\}^{n-1}\right]$$

The density of the largest order statistic, is obtained as

$$f_{n:n}(y)=n\left[\left\{\frac{\alpha y^{\alpha-1}}{\beta^{\alpha}}\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]\right\}\left\{\left(\frac{y}{\beta}\right)^{\alpha}\left[1+\theta-\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]\right\}^{n-1}\right]$$

$$f_{n:n}(y)=n\alpha\frac{y^{n\alpha-1}}{\beta^{n\alpha}}\left[\left\{1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right\}\left\{1+\theta-\theta\left(\frac{y}{\beta}\right)^{\alpha}\right\}^{n-1}\right]$$

2.7 Maximum-likelihood Estimation

In this section, the maximum likelihood estimates (MLEs), of the parameters that are inherent within the transmuted Power Function distribution function is given by the following: Let be random variables of transmuted Power Function distribution of size n. Then sample likelihood and Log-Likelihood functions of T-Ps is obtained as

$$\prod_{i=1}^{n} f(y)=\frac{\alpha^{n}}{\beta^{n\alpha}}\prod_{i=1}^{n} y^{\alpha-1}\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]$$

Log-likelihood function is

$$L=n\log\alpha-n\alpha\log\beta+\sum_{i=1}^{n}(\alpha-1)\log y+\log\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]$$

Therefore, The MLE's of parameters (α, β, and θ) which maximize the above log-likelihood function must satisfy the normal equations. We take the first derivative of the above log-likelihood equation with respect to parameters and equate to zero respectively.

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}-n\log\beta+\sum_{i=1}^{n}\log y+\frac{\left[-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\log\alpha\right]}{\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]}=0$$

$$\frac{\partial L}{\partial\beta}=-\frac{n\alpha}{\beta}+2\theta\alpha\left(\frac{1}{\beta}\right)^{\alpha+1}\sum_{i=1}^{n}\frac{y^{\alpha}}{\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]}=0$$

$$\frac{\partial L}{\partial\theta}=\sum_{i=1}^{n}\frac{\left[1-2\left(\frac{y}{\beta}\right)^{\alpha}\right]}{\left[1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right]}=0$$

The exact solution of above derived ML estimator for unknown parameters is not possible. So it is more convenient to use non-linear optimization algorithms such as Newton Raphson algorithm to numerically maximize the above likelihood function. After application of large sample property of ML Estimates, MLE $\hat{\theta}$ can be treated as being approximately normal with mean θ and variance-covariance matrix equal to the inverse of the expected information matrix, i.e. $\sqrt{n(\hat{\theta}-\theta)}\rightarrow N(0,nI^{-1}(\theta))$. $I(\theta)$ is the information matrix then its inverse of matrix is $I^{-1}(\theta)$ provide the variances and covariance's. The observed information matrix given by

$$I(\theta)=n\begin{bmatrix}J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\theta}\\ J_{\beta\alpha} & J_{\beta\beta} & J_{\beta\theta}\\ J_{\theta\alpha} & J_{\theta\beta} & J_{\theta\theta}\end{bmatrix}$$

Approximate two sided $100(1-\alpha)\%$ confidence intervals for α, β, and θ are, respectively, given by

$$\hat{\alpha}\pm Z_{\alpha/2}\sqrt{I_{\hat{\alpha}\hat{\alpha}}^{-1}(\hat{\theta})},\qquad \hat{\beta}\pm Z_{\alpha/2}\sqrt{I_{\hat{\beta}\hat{\beta}}^{-1}(\hat{\theta})}\quad\text{and}\quad \hat{\theta}\pm Z_{\alpha/2}\sqrt{I_{\hat{\theta}\hat{\theta}}^{-1}(\hat{\theta})}$$

numDeriv package of R language can be used to compute the Hessian matrix and its inverse, standard errors and asymptotic confidence intervals.

3. APPLICATION

In this section, we use censored and uncensored real data sets to compare the fits of the new model and illustrate the usefulness of the new model.

3.1 Smith and Naylor Data – Uncensored

The data set is obtained from Smith and Naylor [23] represents lifetimes of 50 industrial devices put on life test at time zero. The first set consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. Table 1 presents basic descriptive statistics for the Smith and Naylor data

Table 1. Descriptive statistic of the Smith and Naylor data

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.550	1.375	1.590	1.507	1.685	2.240

In order to determine the shape of the most appropriate hazard function for modeling, graphical analysis data may be used. In this context, the total time in test (TTT) plot is very useful (for more details see Aarset [24]).

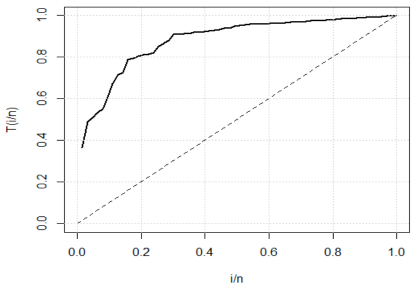


Figure 3. The TTT plot of the Smith and Naylor data.

The TTT plot is concave and provides evidence that the monotonic hazard rate is adequate. We compare the fitting of the T-Ps model with 5 models. The cdf of the other fitted models are:

1. The Transmuted Log-Loistic (T-LL) distribution introduced by Aryal [25]. The cdf of T-LL distribution (with three parameters α, β and θ) is

$$F(x) = \frac{(1+\theta)\alpha^{\beta}x^{\beta} + x^{2\beta}}{(\alpha^{\beta}+x^{\beta})^{2}}, \qquad x > 0$$

where $\beta > 0$ is a shape parameter, $\alpha > 0$ is a scale parameter and $\theta > 0$ is a transmuted parameter.

2. The Complementary Burr III Poisson (C-BIII-P) distribution introduced by Hassan et al. (2015). The cdf of C-BIII-P distribution (with three parameters λ, α and β) is

$$F(x) = \frac{e^{\lambda}(1+x^{-\alpha})^{-\beta} - 1}{e^{\lambda} - 1}, \qquad x > 0$$

where α, β are shape parameters and λ is scale parameter.

3. The Poisson-Lomax (P-L) distribution introduced by Al-Zahrani and Sagor [26]. The cdf of P-L distribution (with three parameters α, β and λ) is

$$F(x) = 1 - \frac{1 - e^{-\lambda(1+\beta x)^{-\alpha}}}{e^{-\lambda}}, \qquad x > 0$$

$\alpha > 0$ is shape parameter and $\lambda, \beta > 0$ are scale parameters.

4. Transmuted-Rayligh (T-R) distribution introduced by Merovci [27]. The cdf of T-R distribution (with two parameters α and λ) is

$$F(x) = \left(1 - e^{\left(\frac{x^2}{2\alpha^2}\right)}\right)\left(1 + \theta e^{\left(\frac{x^2}{2\alpha^2}\right)}\right), \qquad x > 0$$

$\alpha > 0$ is scale parameters and $|\theta| > 0$ is transmuted parameter.

In each case, the parameters are estimated by maximum likelihood and also model selection is carried out using log-likelihood function evaluated at the MLEs ($\hat{\ell}$), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC), Anderson-Darling (A^*) and Cram'er–von Mises (W^*) to compare the fitted models. In general, the smaller the values of these statistics, the better the fit to the data. The estimates of the parameters and the standard error values of this estimates are listed in Table 2 while Table 3, gives the rest of the statistics as $-\hat{\ell}$, AIC, CAIC, BIC, HQIC, W^*, A^* and K-S values.

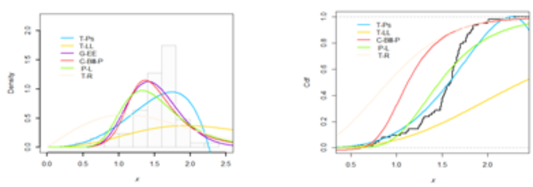
Table 2. MLEs (standard errors in parentheses) to Smith and Naylor data.

Model	Estimates			
	α	β	θ	λ
T-Ps	3.407995 (0.392135)	2.25 (-)	0.9695379 (2.0740642)	
T-LL	2.373395 (2.1e-16)	3.168769 (0.183472)	0.0000000001 (0.230196)	
C-BIII-P	5.274659 (0.43783)	1.56372 (0.44984)		4.10423 (1.12647)
P-L	92.206099 (53.61923)	0.029833 (0.017893)		36.251247 (10.666323)
T-R	1.089476 (1.1e-08)		0.0000000001 (1.7e-12)	

Table 3. The measures $-\hat{\ell}$, AIC, CAIC, BIC and HQIC to Smith and Naylor data.

Model	$-\hat{\ell}$	AIC	CAIC	BIC	HQIC	W^*	A^*	K-S
T-Ps	**21.9802**	**49.960**	**50.367**	**56.389**	**52.48918**	**0.2933592**	**1.678026**	**0.1903 (0.02083)**
T-LL	77.25721	160.514	60.921	66.944	163.0431	0.5351413	2.940356	0.6279 (2.2e-16)
C-BIII-P	29.4495	64.8989	65.305	71.328	67.42763	0.435267	2.745621	0.5415 (2.22e-16)
P-L	31.0027	68.0053	68.412	74.437	70.53403	0.767541	4.195883	0.2236 (0.00368)
T-R	49.7909	103.5818	103.782	107.868	105.2676	0.4382479	2.404961	0.5834 (2.2e-16)

Table 3 shows that T-Ps distribution fitted the data better than the other models. In order to assess if the model is appropriate, we plot in Figure 2 (a) and (b) the histogram of the data and the T-Ps T-LL, C-BIII-P, P-L and T-R distributions and the empirical and their estimated cdf functions, respectively. These plots indicate that the T-Ps distribution provides a better fit to these data than all its sub-models.

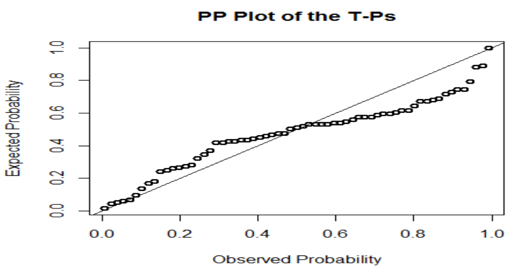


(a) (b)

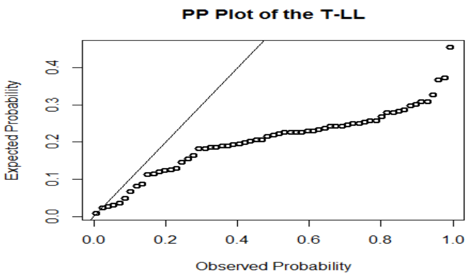
Figure 4. (a) Estimated densities of the T-Ps T-LL, C-BIII-P, P-L and T-R distributions for the data.

(b) Estimated cdf function from the fitted T-Ps T-LL, G-EE, C-BIII-P, P-L and T-R distributions and the empirical cdf for the data.

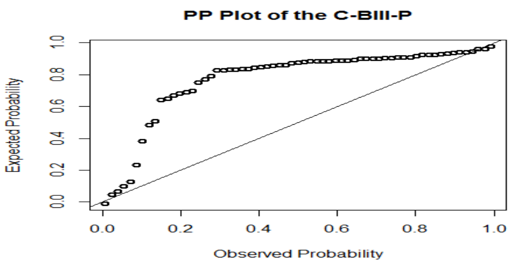
In addition, Figure 5 (a), (b), (c), (d) and (e) present the probability-probability (P-P) plot for the fitted distributions which specified used to determine how well a specific distribution fits to the observed data. This plot will be approximately linear if the specified distribution is the correct model.



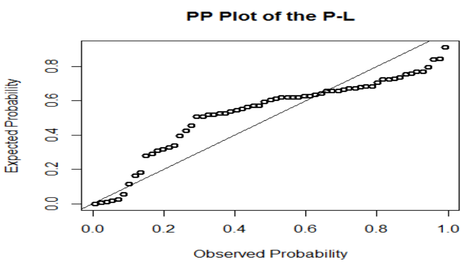
(a)



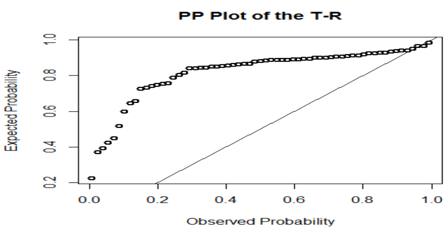
(b)



(c)



(d)



(e)

Figure 5. (a), (b), (c), (d) and (e) are the P-P plot for the T-Ps T-LL, C-BIII-P, P-L and T-R distributions respectively.

3.2 MURTHY DATA- UNCENSORED

The second data set has been obtained from Murthy, Xie [28] page 180 represents 50 items. Table4 presents basic descriptive statistics for data set.

Table 4. Descriptive statistic of Murthy et al. data

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.013	1.390	5.320	7.821	10.040	48.100

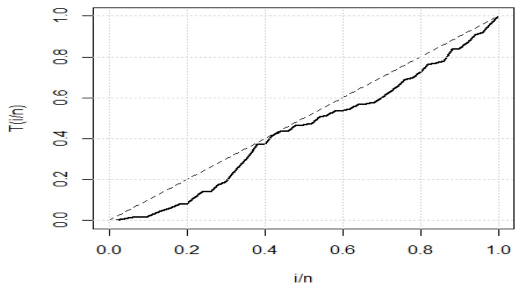


Figure 6. The TTT plot of the Petroleum rock samples.

The TTT plot for the current data is displayed in Figure 4, which is convex and according to Aarset [24] provides evidence that the hazard rate is decreasing. We compare the fitting of the Kw-Ps model with 4 non-nested models. The cdf of the fitted models are:

1. The Lindley-Poisson (L-P) distribution introduced by Gui, Zhang [29]. The cdf of L-P distribution (with two parameters θ and λ) is

$$F(x)=\frac{e^{\lambda}-e^{\frac{\lambda\, e^{-\theta x}(\theta+\theta x+1)}{\theta+1}}}{e^{\lambda}-1},\qquad x>0$$

where $\theta,\lambda>0$ are scale parameters.

2. The transmuted Pareto (T-P) distribution introduced by Merovcia and Pukab [22]. The cdf of T-P distribution (with three parameters α,β and θ) is

$$F(x)=\left(1-\left(\frac{\beta}{x}\right)^{\alpha}\right)\left(1+\theta\left(\frac{\beta}{x}\right)^{\alpha}\right),\qquad x>\beta$$

where $\alpha>0$ is shape parameter, $\beta>0$ is scale parameter and $\theta>0$ is transmuted parameter.

3. The Burr XII Negative Binomial (B-N-B) distribution introduced by Ramos, Percontini [30]. The cdf of B-N-B distribution (with five parameters $\alpha,\theta,\alpha,\beta$ and λ) is

$$F(x)=\frac{(1-\beta)^{-\theta}-\left(1-\beta\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\alpha}\right)^{-\theta}}{(1-\beta)^{-\theta}-1},\qquad x>0$$

where are scale parameters and $\lambda, \alpha, \theta > 0$ are shape parameters.

4. The exponentiated Generalized Frechet (E-GF) introduced by Cordeiro, Ortega [31]. The cdf of E-GF distribution (with three parameters α, β, θ and λ) is

$$F(x) = \left(1 - \left(1 - e^{-\left(\frac{\theta}{x}\right)^{\lambda}}\right)^{\alpha}\right)^{\beta}, \qquad x > \beta$$

where $\lambda, \alpha, \beta > 0$ are shape parameters and $\theta > 0$ is scale parameter.

In each case, the parameters are estimated by maximum likelihood and also model selection is carried out using ($\hat{\ell}$), (AIC), (CAIC), (HQIC), (BIC), (A^*) and (W^*) to compare the fitted models. In general, the smaller the values of these statistics, the better the fit to the data. The estimates of the parameters and the numerical values of the statistics are listed in Table 5 while Table 6. gives the rest of the statistics as $-2\hat{\ell}$, AIC, CAIC, BIC, HQIC, W^*, A^* and K-S values.

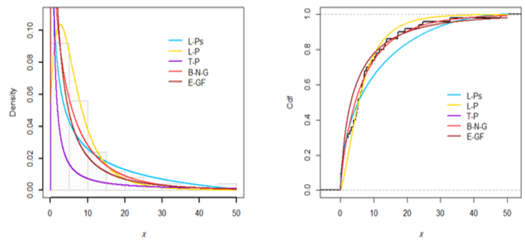
Table 5. MLEs (standard errors in parentheses) of Murthy et al. data.

Model	MLE estimates				
	α	β	θ	a	λ
T-Ps	**0.5341457** **(2.448862)**	**48.2** ()	**0.9021547** **(1.470397)**		
L-P			0.1556775 (0.03555861)		2.3552723 (1.12232986)
T-P	0.2513567 (5.384748)	0.0˙ ($2.3e-03$)	-1.000 (1.026425)		
B-N-W	$9.340596e+01$ (3416.115339)	$1.000000e-10$ (0.1483753)	$1.355848e+01$ 0.6996486	$1.795449e+03$ 7248.1744800	0.8156424 0.2123960
E-GF	32.1573966 ()	0.2889853 ()	1049.6397226 (43.99935)		0.3027114 ()

Table 6. The measures $-2\hat{\ell}$, AIC, CAIC, BIC, HQIC, W^*, A^* and KS to Murthy data.

Model	$-\hat{\ell}$	AIC	AIC_C	BIC	HQIC	W^*	A^*	K-S
T-Ps	**152.248**	**310.497**	**311.018**	**316.233**	**312.681**	**0.11336**	**0.71690**	**0.1211 (0.4559)**
L-P	157.97	319.942	320.197	323.766	321.398	0.06008	0.31416	0.1599 (0.1552)
T-P	183.832	373.665	374.187	379.401	375.849	0.69895	3.80678	0.2769 (0.0009)
B-N-W	150.718	311.436	312.799	320.996	315.077	0.08542	0.42609	0.1108 (0.5717)
E-GF	154.681	317.362	318.250	325.009	320.274	0.22195	1.17341	0.2235 (0.03657)

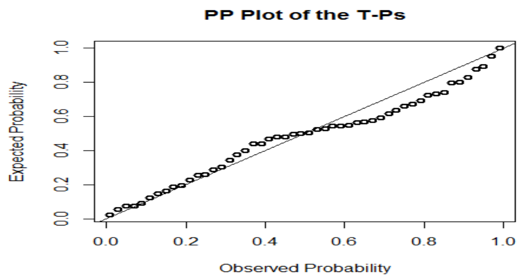
In order to assess if the model is appropriate, we plot in Figure 5 (a) and (b) the histogram of the data and the T-Ps, L-P, T-P, B-N-W E-GF distributions and the empirical and their estimated cdf functions, respectively. These plots indicate that the T-Ps distribution provides a better fit to these data than all its sub-models.



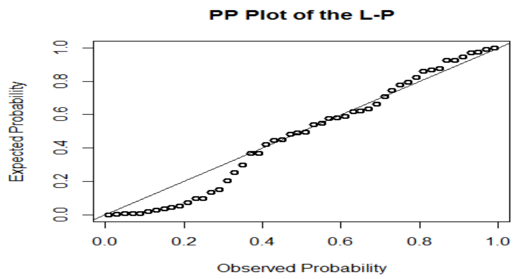
(a) (b)

Figure 7. (a) Estimated densities of the T-Ps, L-P, T-P, B-N-B and E-GF distributions for the data. (b) Estimated cdf function from the fitted T-Ps, L-P, T-P, B-N-B and E-GF distributions and the empirical cdf for the data.

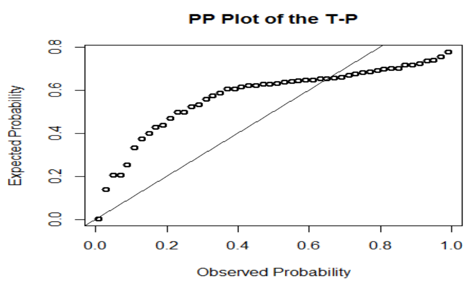
In addition, Figure 7 (a), (b), (c), (d) and (e) present the probability-probability (P-P) plot for the T-Ps, L-P, T-P, B-N-B and E-GF distributions.



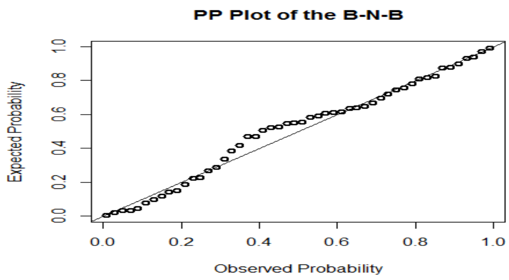
(a)



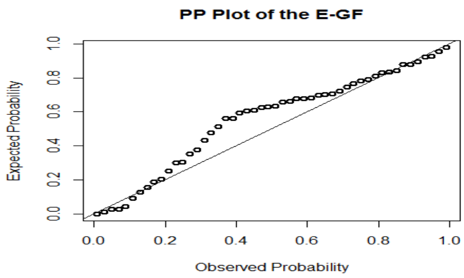
(b)



(c)



(d)



(e)

Figure 8: (a), (b), (c), (d) and (e) are the P-P plot for the T-Ps, L-P, T-P, B-N-B and E-GF distributions respectively.

4. CONCLUSION

In this paper, a new generalization of the power function distribution introduced by [2] is proposed, which extends the power function distribution. For new generalization we derived its mathematical properties, explicit expressions for the moments, quantile function, generating functions and obtain the density function of order statistics. We also discussed the maximum likelihood estimation for estimating the model parameters and compare it with other distributions to show its flexibility. The proposed distribution is applied to two real data sets and provides better fit than other distributions.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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