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Generalizations of The Feng Qi Type Inequality For Pseudo-Integral

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ABSTRACT
In this paper, generalizations of the Feng Qi type integral inequalities for pseudo-integrals are proved. There are considered two cases of the real semiring with pseudo-operations: One discusses pseudo-integrals where pseudo-operations are given by a monotone and continuous function g. The other one focuses on the pseudo-integrals based on a semiring ([a; b]; sup;), where the pseudo-multiplication is generated. Some examples are given to illustrate the validity of these inequalities.

Keywords: *Sugeno integrals, inequality, Feng Qi inequality, Fuzzy integral inequality*

1. INTRODUCTION

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval endowed with pseudo-addition and with pseudo-multiplication (see [18, 22, 25]). Based on this structure there where developed the concepts of -measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform and etc. pseudo-analysis would be an interesting topic to generalize an inequality from the framework of the classical analysis to that of some integrals which contain the classical analysis as special cases [3, 4, 5, 7, 9, 10, 11, 12, 13, 14, 20, 24].

The integral inequalities are good mathematical tools both in theory and application. Different integral inequalities including Chebyshev, Jensen, Holder and Minkowski inequalities are widely used in various fields of mathematics, such as probability theory,

differential equations, decision-making under risk and information sciences.,

The aim of this paper is to study some general Feng Qi type inequality for pseudo-integrals of monotone functions. We think that our results will be useful for those areas in which the classical Feng Qi inequality plays a role whenever the environment is non-deterministic. In [29], Feng Qi studied a very interesting integral inequality and proved the following result:

Theorem 1.1. Let n be a positive integer. Suppose $f(x)$ has continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a)\geq 0$, for $0\leq i\leq n-1$ and $f^{(n)}(x)\geq n!$, then

$$\int_a^b [f(x)]^{(n+2)}dx \geq \left(\int_a^b f(x)dx\right)^{(n+1)}$$

In [1] the Feng Qi type inequality for Sugeno integral is presented with several examples given to illustrate the validity of this inequalities.

Theorem 1.2. Let μ be the Lebesgue measure on $\mathbb{R}$ and let $f:[0,1]\to[0,\infty)$ be a real valued function

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such that $\int_0^1 fd\mu = p$. If f is a continuous and strictly decreasing function, such that $f\left(p^{n+1}\right) \geq p^{\left(\frac{n+1}{n+2}\right)}$, then the inequality

$$(s)\int_0^1 f^{n+2}d \geq \left((s)\int_0^1 fd\mu\right)^{n+1}$$

holds for all $n \geq 0$.

Theorem 1.3. Let μ be the Lebesgue measure on $\mathbb{R}$ and let $f:[0,1] \to [0,\infty)$ be a real valued function such tha t $\int_0^1 fd\mu = p$. If f is a continuous and strictly decreasing function, such that $f\left(1-p^{n+1}\right) \geq p^{\left(\frac{n+1}{n+2}\right)}$, then the inequality

$$(s)\int_0^1 f^{n+2}d \geq \left((s)\int_0^1 fd\mu\right)^{n+1}$$

holds for all $n \geq 0$.

The paper is organized as follows: Section 2 contain some of preliminaries, such as pseudo-operations and pseudo-analysis as well as integrals. In Section 3, we have proved generalizations of the Feng Qi type inequality for pseudo-integrals. Finally, a conclusion is given in Section 4.

2. PRELIMINARIES

In this section, some definitions and basic properties of the Sugeno integrals and pseudo-integrals which will be used in the next sections are presented.

Definition 2.1 Let Σ be a σ-algebra of subsets of X and let $\mu:\Sigma \to [0,\infty)$ be a non-negative, extended real-valued set function, we say that μ is a fuzzy measure iff:

(FM1) $\mu(\varnothing)=0$;

(FM2) $\mathrm{E},\mathrm{F} \in \Sigma$ and $\mathrm{E} \subseteq \mathrm{F}$ imply $\mathrm{E} \leq \mathrm{F}$ (monotonicity);

(FM3) $\left(E_n\right) \subseteq \Sigma, \mathrm{E}_1 \subseteq E_2 \subseteq \dots$ imply $\lim \mu(E_n) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$ (continuity from below);

(FM4) $\left(E_n\right) \subseteq \Sigma,\ \mathrm{E}_1 \supseteq E_2 \supseteq \dots, \mu(\mathrm{E}_1) < \infty$ imply $\lim \mu(E_n) = \mu\left(\bigcap_{i=1}^{\infty} E_i\right)$ (continuity from above).

Let (X, F, μ) be a fuzzy measure space and f is a non-negative real-valued function on X, by F_+ we denote the set of all non-negative measurable function f with respect to F and F_α denote the set $\{x \in X \mid f(x) \geq \alpha\}$, the α-level of f, for $\alpha \geq 0$ $F_0 = \{x \in X \mid f(x) > 0\} = \mathrm{supp}(f)$ is the support of f. We know that $\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$.

Definition 2.2. Let μ be a fuzzy measure on (X,Σ). If $f \in F$ and $\mathrm{A} \in \Sigma$, then the Sugeno integral (or fuzzy integral) of f f on A, with respect to the fuzzy measure μ, is defined [32] as

$$(s)\int fd\mu = \vee_{\alpha \geq 0}(\alpha \wedge \mu(A \cap F_\alpha)).$$

Where $\vee$, $\wedge$ denotes the operation $\sup$ and inf on $[0,\infty)$ respectively. In particular, if A = X, then

$$(s)\int fd\mu = \vee_{\alpha \geq 0}(\alpha \wedge \mu(F_\alpha)).$$

The following proposition gives most elementary properties of the fuzzy integral and can be found in [32].

Proposition 2.3. Let (X, F, μ) be a fuzzy measure space, with $\mathrm{a},\mathrm{b} \in \Sigma$ and $f, g \in F$. We have

1. $(s)\int_A fd\mu(A)$;
2. $(s)\int_A kd\mu \leq k \wedge \mu(A)$, for k non-negative constant;
3. If $f \leq g$ on A, then $(s)\int_A fd\mu(A) \leq (s)\int_A gd\mu(A)$;
4. If $B \subset A$, then $(s)\int_B fd\mu(A) \leq (s)\int_A fd\mu(A)$;

5. If $\mu(A) \leq \infty$, then $(s)\int_A fd\mu(A) \geq \alpha \Leftrightarrow \mu(A \cap \{f \geq \alpha\}) \geq \alpha$

6. $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow (s)\int_A fd\mu(A) \leq \alpha$

7. $(s)\int_A fd\mu(A) < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$;

8. $(s)\int_A fd\mu(A) > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

Remark 2.4. Let $F(\alpha) = \mu(A \cap F_\alpha)$, from parts (5) and (6) of the above Proposition, it very important to note that

$$F(\alpha) = \alpha \Rightarrow (s)\int_A fd\mu = \alpha.$$

Thus, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$.

Definition 2.5. Let [a, b] be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on [a, b] will be denoted by ≼. The operation $\oplus$ (pseudo-addition) is a function ⊕: [a, b]×[a, b]→[a, b] which is for $x, y, z, 0$ (zero element) ∈[a, b] it satisfies the following requirements:

(i) $x \oplus y = y \oplus x$;

(ii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;

(iii) $x \preceq y \Rightarrow x \oplus z \preceq y \oplus z$;

(iv) $0 \oplus x = x$.

Let $[a,b]_+ = \{x \mid x \in [a,b], 0 \preceq x\}$.

Definition 2.6. A binary operation function $\odot : [a, b] \times [a, b] \to [a, b]$ is called a pseudo-multipication, for $x, y, z, 1$ (unit element)∈[a, b] it satisfies the following requirements:

(i) $x \odot y = y \odot x$;

(ii) $(x \odot y) \odot z = x \odot (y \odot z)$;

(iii) $x \preceq y \Rightarrow x \odot z \preceq y \odot z$ for all $[a,b]_+$;

(iv) $(x \oplus y) \odot z = (x \odot z) \oplus (x \odot y)$;

(v) $1 \oplus x = x$;

(vi) $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist and finite then $\lim_{n\to\infty}(x_n \odot y_n) = \lim_{n\to\infty} x_n \odot \lim_{n\to\infty} y_n$.

We assume also $0 \odot x = x$ that ⊙ is a distributive pseudo-multiplication with respect to ⊕ , i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a,b], \oplus, \odot)$ is a semiring ([16]). In this paper, we will consider semirings with the following continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x, y)$, ⊙ is arbitrary not idempotent pseudo-multiplication on the interval [a, b]. We have $0 = a$ and the idempotent operation $\sup$ induces a full order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$.

(b) $x \oplus y = \inf(x, y)$, ⊙ is arbitrary not idempotent pseudo-multiplication on the interval [a, b]. We have $0 = b$ and the idempotent operation inf induces a full order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$.

Case II: The pseudo-operations are defined by a monotone and continuous function $g : [a,b] \to [0,\infty]$, i.e., pseudo-operations are given with $x \oplus y = g^{-1}(g(x) + g(x))$ and $x \odot y = g^{-1}(g(x)g(x))$. If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then g(b)=0 and $g(a) = 1$. If the generator g is increasing (respectively decreasing), then the operation induces the usual order (respectively opposite to the usual order) on the interval [a, b] in the following way: $x \preceq y$ if and only if $g(x) \le g(y)$

Case III: Both operations are idempotent. We have.

(a) $x \oplus y = \sup(x, y)$, $x \odot y = \inf(x, y)$, on the interval [a, b]. We have $0 = a$ and $1 = b$. The idempotent operation $\sup$ induces the usual order($x \preceq y$ if and only if $\sup(x, y) = y$).

(b) $x \oplus y = \inf(x, y)$, $x \odot y = \sup(x, y)$ on the interval [a, b]. We have $0 = b$ and $1 = a$. The idempotent operation inf induces an order opposite to the usual order($x \preceq y$ if and only if $\inf(x, y) = y$).

Definition 2.7. A set function $m : \Sigma \to [a, b]_+$ (or semiclosed interval) is a $\sigma - \oplus -$ measure if there holds:

(i) $m(\emptyset) = 0$;

(ii) $m\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} m(A_i)$ for any sequence A_i of pairwise disjoint sets from Σ, where $\bigoplus_{i=1}^{\infty}(x_i) = \lim_{n\to\infty} \bigoplus_{i=1}^{n}(x_i)$.

Let X be a non-empty set. Let A be a σ-algebra of subsets of a set X.

We shall consider the semiring ([a, b], ⊕, ⊙), when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)),$$

$$x \odot y = g^{-1}(g(x)g(y)).$$

For $x \in [a, b]_+$ and $p \in]0, \infty[$, we will introduce the pseudo-power $x_{\odot}^{(p)}$ as follows: if $p = n$ is a natural number, then

$$x_{\odot}^{(n)} = \overbrace{x \odot x \odot \ldots \odot x}^{n}$$

Moreover, $x_{\odot}^{(\frac{1}{n})} = \sup\{y \mid y_{\odot}^{(n)} \le x\}$. Then $x_{\odot}^{(\frac{m}{n})}$ is well defined for any rational $r \in]0,1[$, independently of representation $r = \frac{m}{n} = \frac{m_1}{n_1}; m; n; m_1; n_1$ being positive integers (the result follows from the continuity and monotonicity of ⊙). Due to continuity of ⊙, if p is not rational, then

$$x_{\odot}^{(p)} = \sup\{x_{\odot}^{(r)} \mid r \in]0, p[, r \in \mathbb{Q}\}.$$

Evidently, if $x \odot y = g^{-1}(g(x)g(y))$, then $x_{\odot}^{(p)} = g^{-1}(g^{p}(x))$. On the other hand, if ⊙ is idempotent, then $x_{\odot}^{(p)} = x$ for any $x \in [a, b]$ and $p \in]0, \infty[$.

Let m be a ⊕-measure, where ⊕ has a monotone and cotinuous generator g, then $g \circ m$ is a σ-addetive measure in the following two important case of integral based on semiring ([a, b], ⊕, ⊙) are discussed. Thus, the pseudo-integral of function $f : X \to [a, b]$ is defined by

$$\int_X^{\oplus} f \odot dm = g^{-1}\left(\int_X (g \circ f) d(g \circ m)\right),$$

where the integral applied on the right side is the standard Lebesgue integral. In fact, let $m = g^{-1} \circ \mu$, μ is the standard Lebesgue measure on X, then we obtain

$$\int_X^{\oplus} f dx = g^{-1}\left(\int_X (g(f(x)) dx\right).$$

More on this structure as well as corresponding measures and integrals can be found in [8, 15].

The second class is when $x \oplus y = \max(x, y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_X^{\oplus} f \odot dm = \sup\left(f(x) \odot \psi(x)\right),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-additive [19]. For any continuous function

$f:[0,\infty]\to[0,\infty]$ the integral $\int^{\oplus} f\odot dm$ can be obtained as a limit of g-integrals, [19]. We denoted by μ the usual Lebesgue measure on ℝ . We have

$m(A)=ess\sup(x\mid x\in A)=\sup\{a\mid \mu(x\mid x\in A, x>a)>0\}$

Theorem 2.8. ([21]). Let m be a sup-measure on $([0,\infty]$,**B**$[0,\infty])$, where **B(**$[0,\infty])$ is the Borel σ-algebra on $[0,\infty]$, $m(A)=ess\sup_{\mu}(\psi(x)\mid x\in A)$,and $\psi:[0,\infty]\to[0,\infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family (m_λ) of $\oplus_\lambda$ -measure on ($[0,\infty]$, **B**), where $\oplus_\lambda$ is a generated by g^λ (the function g of the power λ), $\lambda\in(0,\infty)$, such that $\lim_{\lambda\to\infty} m_\lambda = m$.

Theorem 2.9. ([21]). Let ([0, ∞], sup, $\odot$) be a semiring, when $\odot$ is a generated with g, i.e., we have $x\odot y=g^{-1}(g(x)g(\mathrm{y}))$ for every $x,y\in(0,\infty)$. Let m be the same as in Theorem 2.8. Then there exists a family {m_λ} of $\oplus_\lambda$ -measures, where $\oplus_\lambda$ is a generated by $g^\lambda, \lambda\in(0,\infty)$ such that for every continuous function $f:[0,\infty]\to[0,\infty]$,

$$\int^{\sup} f\odot dm=\lim_{\lambda\to\infty}\int^{\oplus_\lambda} f\odot dm_\lambda=\lim_{\lambda\to\infty}(g^\lambda)^{-1}\left(\int g^\lambda\left(f(x)\right)dx\right)$$

3. FENG QI INEQUALITY FOR PSEUDO-INTEGRALS

The aim of this section is to show that Feng Qi type inequality is deriving from [1] for the pseudo-integral.

Now we peresent generalizations of two above mentioned theorems for pseudo-integral.

Theorem 3.1. For a given measurable space (X, A), let $f:[0,1]\to[0,1]$ be a real-valued function such that $(s)\int_0^1 f d\mu = p$ .If f is a continuous and strictly decreasing function, such that $f\left(p^{n+1}\right)\geq p^{(\frac{n+1}{n+2})}$ and let a generator $\mathrm{g}:[0,1]\to[0,\infty)$ of pseudo-addition $\oplus$ and pseudo-multipication $\odot$ be decreasing function, then the inequality

$$\int_{[0,1]}^{\oplus} f_{\odot}^{n+2}\odot dm \geq \left(\int_{[0,1]}^{\oplus} f_{\odot}\odot dm\right)_{\odot}^{n+1}$$

holds for all $n\geq 0$ and $\sigma-\oplus$ -measure m.

Proof. We apply the classical Feng Qi inequality and obtion:

$$\int_0^1 (g\circ f)^{n+2} d(g\circ m) \geq \left(\int_0^1 (g\circ f) d(g\circ m)\right)^{n+1}$$

Since function g is decreasing function, so g^{-1} is also decreasing function and we obtion

$$g^{-1}\left(\int_0^1 (g\circ f)^{n+2} d(g\circ m)\right) \geq g^{-1}\left(\int_0^1 (g\circ f) d(g\circ m)\right)^{n+1}.$$

For left side of inequality we have

$$g^{-1}\left(\int_0^1 (g\circ f)^{n+2} d(g\circ m)\right) = g^{-1}\left(\int_0^1 \mathrm{g}(g^{-1}(g\circ f)^{n+2}) d(g\circ m)\right)$$

$$= g^{-1}\left(\int_0^1 \mathrm{g}(f_{\odot}^{\,n+2}) d(g\circ m)\right)$$

$$= g^{-1}\left(g\left(g^{-1}\left(\int_0^1 \mathrm{g}(f_{\odot}^{\,n+2}) d(g\circ m)\right)\right)\right)$$

$$= g^{-1}\left(g\left(\int_0^1 f_{\odot}^{\,n+2}\odot dm\right)\right)$$

$$= \int_{[0,1]}^{\oplus} f_{\odot}^{n+2}\odot dm.$$

For right side of inequality we have

$$g^{-1}\left(\int_0^1 (g\circ f) d(g\circ m)\right)^{n+1} = g^{-1}\left(\int_0^1 \mathrm{g}(g^{-1}(g\circ f)) d(g\circ m)\right)^{n+1}$$

$$= g^{-1}\left(\int_0^1 \mathrm{g}(f_{\odot}) d(g\circ m)\right)^{n+1}$$

$$= g^{-1}\left(g\left(g^{-1}\left(\int_0^1 \mathrm{g}(f_{\odot}) d(g\circ m)\right)^{n+1}\right)\right)$$

$$= g^{-1}\left(g\left(\int_{[0,1]}^{\oplus} f_{\odot}\odot dm\right)^{n+1}\right).$$

Hence we have

$$\int_{[0,1]}^{\oplus} f_{\odot}^{n+2} \odot dm \geq \left(\int_{[0,1]}^{\oplus} f_{\odot} \odot dm \right)_{\odot}^{n+1}.$$

Theorem 3.2. For a given measurable space (X, A), let $f:[0,1]\to[0,1]$ be a real-valued function such that $(s)\int_0^1 fd\mu = p$ .If f is a continuous and strictly decreasing function, such that $f\left(1-p^{n+1}\right) \geq p^{(\frac{n+1}{n+2})}$ and let a generator $g:[0,1]\to[0,\infty)$ of pseudo-addition $\oplus$ and pseudo-multipication $\odot$ be decreasing function, then the inequality

$$\int_{[0,1]}^{\oplus} f_{\odot}^{n+2} \odot dm \geq \left(\int_{[0,1]}^{\oplus} f_{\odot} \odot dm \right)_{\odot}^{n+1}$$

holds for all $n \geq 0$ and $\sigma-\oplus$-measure m.

Proof. The proof is similar with the Theorem 3.1.

Example 3.3. Let $g(x)=\ln(x)$, then

$$x \oplus y = xy, \qquad x \odot y = e^{\ln(x).\ln(y)}.$$

By Theorem 3.1, the following inequality holds:

$$\ln \int_0^1 e^{(\ln f(x))^{n+2}} \geq \left(\ln \int_0^1 e^{(\ln f(x))} \right)^{n+1}.$$

In the sequel, we generalize the Feng Qi inequality by the semiring ([a, b], max, $\odot$), where $\odot$ is generated .

Theorem 3.4. Let $f:[0,1]\to[0,1]$ be a real-valued, continuous and strictly increasing function such that $(s)\int_0^1 fd\mu = p$. If $\odot$ is represented by a increasing generator g and m is a complete sup-measure same as in Theorem 2.9, then with condition $f\left(1-p^{n+1}\right) \geq p^{(\frac{n+1}{n+2})}$, the inequality

$$\int_{[0,1]}^{\sup} f_{\odot}^{n+2} \odot dm \geq \left(\int_{[0,1]}^{\sup} f_{\odot} \odot dm \right)_{\odot}^{n+1}$$

holds for all $n \geq 0$ and $\sigma-\oplus-$measure m.

Proof. Since $\left(g^{\lambda}(x)\right)^{-1} = g^{-1}\left(x^{\frac{1}{\lambda}}\right)$, we have

$$x \odot y = g^{-1}(g(x)g(y)) = \left(g^{\lambda}\right)^{-1}\left(g^{\lambda}(x)g^{\lambda}(y)\right) = x \odot_{\lambda} y$$

In other words, g^{λ} is a generator of $\odot$. By Theorem 2.9 we have

$$\int^{\sup} f \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_{\lambda}} f \odot dm_{\lambda} = \lim_{\lambda\to\infty} (g^{\lambda})^{-1}\left(\int g^{\lambda}\left(f(x)\right)dx\right)$$

Since g is a decreasing function, so $g^{-1}, g^{\lambda}, \left(g^{\lambda}\right)^{-1}$ are also decreasing function. Hence we have

$$\left(\int_{[0,1]}^{\sup} f_{\odot} \odot dm \right)_{\odot}^{n+1} =$$

$$\left(\int_{[0,1]}^{\sup} \left(g^{\lambda}\right)^{-1}\left(g^{\lambda}(f(x)\right) \odot dm \right)_{\odot}^{n+1}$$

$$=$$

$$\left(\lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \int_0^1 g^{\lambda}\left(\left(g^{\lambda}\right)^{-1}\left(g^{\lambda}(f(x)\right)\right) dx \right)_{\odot}^{n+1}$$

$$=$$

$$\left(g^{\lambda}\right)^{-1}\left(g^{\lambda}\left(\lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \int_0^1 g^{\lambda}(f(x))dx \right)\right)^{n+1}$$

$$=$$

$$\lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \left(\int_0^1 g^{\lambda}(f(x))dx \right)^{n+1}.$$

By classical Feng Qi inequality, we have

$$\lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \left(\int_0^1 g^{\lambda}(f(x))dx \right)^{n+1}$$

$$\leq \lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \left(\int_0^1 \left(g^{\lambda}(f(x))\right)^{n+2} dx \right)$$

$$= \lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \left(\int_0^1 g^{\lambda}\left(\left(g^{\lambda}\right)^{-1}\left(g^{\lambda}(f(x))\right)^{n+2} \right) dx \right)$$

$$=$$

$$\lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \left(\int_0^1 g^{\lambda}(f_{\odot}^{n+2}) dx \right)$$

$$=$$

$$\int_{[0,1]}^{\sup} f_{\odot}^{n+2} \odot dm$$

Example 3.5. Let $g^{\lambda}(x) = e^{\lambda x}$, the corresponding pseudo-operations are:

$$x \oplus y = \lim_{\lambda\to\infty} \frac{1}{\lambda}\ln\left(e^{\lambda x} + e^{\lambda y}\right) = \max(x, y),$$

$$x \odot y = \lim_{\lambda\to\infty} \frac{1}{\lambda}\ln\left(e^{\lambda x} e^{\lambda y}\right) = x + y$$

By Theorem 3.1, equation (2,1) and definition $x_{\odot}^{p}$ the following inequality holds:

$$\sup((n+2) f(x) + \psi(x)) \geq (n+1)(\sup(f(x) + \psi(x)).$$

Theorem 3.6. Let $f:[0,1]\to[0,1]$ be a real-valued, continuous and strictly decreasing function such that $(s)\int_0^1 f d\mu = p$. If $\odot$ is represented by a decreasing generator g and m is a complete sup-measure same as in Theorem 2.9, then with condition $f\left(p^{n+1}\right) \geq p^{(\frac{n+1}{n+2})}$, the inequality

$$\int_{[0,1]}^{\sup} f_{\odot}^{n+2} \odot dm \geq \left(\int_{[0,1]}^{\sup} f_{\odot} \odot dm\right)_{\odot}^{n+1}$$

holds for all $n \geq 0$ and $\sigma - \oplus$-measure m.

Proof. The proof is similar with the Theorem 3.4.

Remark 3.7. Typical example for two above case are operation $\oplus = \vee$ and $\odot = \wedge$ that already proved in fuzzy case [1].

4. CONCLUSION

This paper proposed a Feng Qi type inequality for pseudo-integrals. The first class is including the pseudo-integral based on a function reduces on the g-integral, where pseudo-addition and pseudo-multiplication are defined by a monotone and continuous function g. The second class is including the pseudo-integral based on the semiring $([a,\ b],\ \sup,\odot)$ is given by sup-measure, where $x \odot y$ is generated by $g^{-1}\left(g(x)g(y)\right)$. For further investigation, we will investigate other integral inequalities for Pseudo-integral.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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