Gazi University Journal of Science
GU J Sci
25(2):317-321 (2012)

ORIGINAL ARTICLE

Solution of the Klein-Gordon Equation with Position-Dependent Mass for Exponential Scalar and Vector Potentials by an Alternative Approach

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Received: 21.08.2011 Accepted: 13.10.2011

ABSTRACT

The s-wave Klein-Gordon equation, with position-dependent mass, is solved for the exponential vector and scalar potentials by an alternative approach. The asymptotic iteration method is used to obtain the energy eigenvalues. The results are the exact analytical and are in good agreement with the results previously.

Keywords: Position-dependent mass, Asymptotic Iteration Method (AIM), Klein-Gordon equation, Exponential vector and scaler potentials, Eigenvalues.

1. INTRODUCTION

Recently, solution of the Schrödinger equation with position-dependent mass has attracted considerable attention. A lot of studies have been performed to obtain the solutions of the Schrödinger, Klein-Gordon and Dirac equations with position-dependent mass for various potentials by means of different methods [1-8]. It has been seen that the position dependent mass is a useful tool to investigate the electronic properties of semiconductors and quantum dots [9], ^{3}He clusters [10], impurities in crystals [11].

In this study, we have intended to solve the Klein-Gordon equation with position-dependent mass by using an alternative and practical method as called the asymptotic iteration method (AIM). The AIM has used extensively for both non-relativistic and relativistic cases for various potential used in different physical systems [12-16]. In these days, this method has been expanded position-dependent mass cases. With this goal, the bound-state solution of the position-dependent mass Klein–Gordon equation including inversely linear potential and the generalized Hulthén potential have been obtained [17, 18]. The solution of exact pseudospin symmetry solution of the Dirac equation for spatially-dependent mass Coulomb potential including a Coulomb-like tensor interaction via asymptotic iteration method has been given [19]. Therefore, investigation of the solutions of the Klein-Gordon and Dirac equations

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with the position-dependent mass for different potential cases will be very important to define whether the AIM for solution of more complex problems is used or not. For purpose, in the present paper, we aimed to show solution of the Klein-Gordon equation for the exponential vector and scalar potentials with position-dependent mass by using the asymptotic iteration method.

In the next section, we briefly outline AIM. In section 3, we apply AIM to obtain bound state solutions of exponential potential with position-dependent mass. Then, in section 4 we discussion some special cases of the energy eigenvalues. Finally, in section 5, we remark on these results.

2. THE AIM

AIM is proposed to solve the second-order differential equations of the form [20, 21].

$$y'' = \lambda_0(x)y' + s_0(x)y \tag{1}$$

where $\lambda_0(x) \neq 0$. The variables, $s_0(x)$ and $\lambda_0(x)$, are sufficiently differentiable. The differential Eq. 1 has a general solution [20]

$$y(x) = \exp\left(-\int^x \alpha(x')dx'\right)\left[C_2 + C_1\int^x \exp\left(\int^{x'}[\lambda_0(x'') + 2\alpha(x'')]dx''\right)dx'\right] \tag{2}$$

if $k > 0$, for sufficiently large k, we obtain the $\alpha(x)$ values from

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k\text{-}1}(x)}{\lambda_{k\text{-}1}(x)} = \alpha(x), \quad k = 1,\ 2,\ 3,... \tag{3}$$

where

$$\lambda_k(x) = \lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x)$$

$$s_k(x) = s'_{k-1}(x) + s_0(x)\lambda_{k-1}(x), \qquad k = 1,\ 2,\ 3,... \tag{4}$$

It should be noted that one can also start the recurrence relations from $k = 0$ with the initial conditions $\lambda_{-1} = 1$ and $s_{-1} = 0$ [22]. For a given potential, the radial Schrödinger equation is converted to the form of Eq. 1. Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$, parameters are calculated by the recurrence relations given by Eq. 4.

The energy eigenvalues are obtained from the roots of the quantization condition, given by the termination condition of the method in Eq. 3. The quantization condition of the method together with Eq. 4 can also be written as follows

$$\delta_k(x) = \lambda_k(x)s_{k\text{-}1}(x) - \lambda_{k\text{-}1}(x)s_k(x) = 0 \quad k = 1,\ 2,\ 3,... \tag{5}$$

The energy eigenvalues are obtained from this equation if the problem is exactly solvable. If not, for a specific n principal quantum number, we choose a suitable x_0 point, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential, and the approximate energy eigenvalues are obtained from the roots of this equation for sufficiently great values of k with iteration.

The wave functions are determined by using the following wave function generator

$$y_n(x) = C_2 \exp\left(-\int^x \frac{s_k(x')}{\lambda_k(x')}dx'\right) \tag{6}$$

where $k \geq n$, n represents the radial quantum number and k shows the iteration number. For exactly solvable potentials, the radial quantum number n is equal to the iteration number k and the eigenfunctions are obtained directly from Eq. 6. For nontrivial potentials that have no exact solutions, k is always greater than n in these numerical solutions and the approximate energy eigenvalues are obtained from the roots of Eq. 5 for sufficiently great values of k by iteration.

3. BOUND STATE SOLUTIONS OF EXPONENTIAL POTENTIAL WITH POSITION-DEPENDENT MASS

Let us consider radial s-wave Klein-Gordon equation of a spinless paricle with position-dependent mass. In the relativistic atomic numbers ($\hbar = c = 1$), the equation is written as follows

$$\frac{d^2}{dr^2}u(r) + \left[\left(E - V(r)\right)^2 - \left(m(r) + S(r)\right)^2\right]u(r) = 0 \tag{7}$$

where the radial wave function is $\psi(r) = u(r)/r$ and $V(r)$ and $S(r)$ are vector and scalar potentials, respectively. As mentioned above, in the present paper, we search the solution of the Klein-Gordon equation for exponential-type scalar and vector potentials. These potentials can be considered as following forms

$$S(r) = -S_0 e^{-\alpha r}, \qquad V(r) = -V_0 e^{-\alpha r} \tag{8}$$

where S_0, V_0, and α are constants. If we accept a specific form of the position-dependent mass as

$$m(r) = m_0(1 - qe^{-\alpha r}) \tag{9}$$

and insert into Eq. 7 together Eq. 8, it becomes

$$\frac{d^2}{dr^2}u(r) - (K_1 e^{-2\alpha r} + K_2 e^{-\alpha r} - E^2 + m_0^2)u(r) = 0 \tag{10}$$

where

$$K_1 = (S_0 + m_0 q)^2 - V_0^2, \quad K_2 = -2m_0^2 q - 2m_0 S_0 - 2EV_0 \tag{11}$$

Using a new variable such as $\rho = e^{-\alpha r}$, we can obtain

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \eta^2 - \frac{\xi}{\rho} - \frac{\varepsilon^2}{\rho^2}\right]u(\rho) = 0 \tag{12}$$

where

$$\eta = \frac{-\sqrt{K_1}}{\alpha}, \qquad \xi = \frac{K_2}{\alpha^2}, \qquad \varepsilon = \frac{\sqrt{m_0^2 - E^2}}{\alpha} \tag{13}$$

In order to solve Eq. 12 by using the AIM procedure, we can use the following wave function similar to reference [23, 24]

$$u(\rho) = \rho^{\varepsilon} e^{\eta\rho} f(\rho) \tag{14}$$

If we insert this wave function into Eq. 12, we obtain following equation

$$\frac{d^2}{d\rho^2}f(\rho) = -\frac{(2\varepsilon + 2\eta\rho + 1)}{\rho}\frac{d}{d\rho}f(\rho) - \frac{(2\varepsilon\eta + \eta - \xi)}{\rho}f(\rho) \tag{15}$$

The Eq. 15 has the same form as Eq. 1. Thus, we can use the AIM to get general solution of this equation. Obtaining $\lambda_0(\rho)$ and $s_0(\rho)$ with the recursion relation and calculating $\lambda_k(\rho)$ and $s_k(\rho)$, we combine these results with the condition given by Eq. 5

$$s_0\lambda_1 - s_1\lambda_0 = 0 \Rightarrow \varepsilon_0 = \frac{\xi - \eta}{2\eta} \tag{16}$$

$$s_1\lambda_2 - s_2\lambda_1 = 0 \Rightarrow \varepsilon_1 = \frac{\xi - 3\eta}{2\eta} \tag{17}$$

$$s_2\lambda_3 - s_3\lambda_2 = 0 \Rightarrow \varepsilon_2 = \frac{\xi - 5\eta}{2\eta}, \quad etc. \tag{18}$$

If we want to obtain a general expression for Eqs. 16, 17, 18, we can write the eigenvalues as the following form

$$\varepsilon_n = \frac{\xi - (2n+1)\eta}{2\eta}, \quad n = 0,\ 1,\ 2,... \tag{19}$$

By means of Eq. 13, we can obtain the energy eigenvalues for the exponential-type scalar and vector potentials with position-dependent mass

$$E_n^2 = m^2 - \left(\alpha n + \frac{\alpha}{2} + \frac{1}{2}\frac{K_2}{\sqrt{K_1}}\right)^2 \tag{20}$$

If we substitute Eq. 11 into Eq. 20, we get as following equation

$$E_n(q) = \frac{-\beta_n(q) \pm \sqrt{\beta_n^2(q) - 4(S_0 + m_0 q)^2 \chi_n(q)}}{2(S_0 + m_0 q)^2} \tag{21}$$

where

$$\beta_n(q) = -2\alpha(n + \frac{1}{2})V_0\sqrt{(S_0 + m_0 q)^2 - V_0^2} + 2V_0 m_0(S_0 + m_0 q), \tag{22}$$

$$\chi_n(q) = \alpha^2(n + \frac{1}{2})^2((S_0 + m_0 q)^2 - V_0^2) - 2\alpha(n + \frac{1}{2})m_0(S_0 + m_0 q)\sqrt{(S_0 + m_0 q)^2 - V_0^2} + m_0^2 V_0^2 \tag{23}$$

This result is the same as Ref. [25]. If we calculate the corresponding unnormalized eigenfunctions by using the wave function generator given by Eq. 6, it becomes

$$f_n(\rho) = -(1)^n C_2 \frac{\Gamma(n+2\varepsilon_n+1)}{\Gamma(2\varepsilon_n+1)}\, {}_1F_1(-n, 2\varepsilon_n+1; -2\eta\rho) \tag{24}$$

where Γ and ${}_1F_1$ are denoted to the gamma and the confluent hypergeometric functions, respectively. Finally, we can write the total radial wave function by using Eqs. 14 and 24

$$u_n(\rho) = N\rho^{\varepsilon_n} e^{\eta\rho}\, {}_1F_1(-n, 2\varepsilon_n+1; -2\eta\rho) \tag{25}$$

where N is normalization constant.

4. DISCUSSION

In this section we investigate some special cases of the energy eigenvalues given by Eq. 21. According to this,

(i) If we consider constant mass case q=0, the energy eigenvalue becomes as

$$E_n(0) = \frac{-\beta_n(0) \pm \sqrt{\beta_n^2(0) - 4S_0^2\chi_n(0)}}{2S_0^2} \tag{26}$$

where

$$\beta_n(0) = -2\alpha(n+\frac{1}{2})V_0\sqrt{S_0^2 - V_0^2} + 2m_0S_0V_0, \tag{27}$$

$$\chi_n(0) = \alpha^2(n+\frac{1}{2})^2(S_0^2 - V_0^2) - 2\alpha(n+\frac{1}{2})m_0S_0\sqrt{S_0^2 - V_0^2} + m_0^2V_0^2 \tag{28}$$

The Eq. 26 is the same as Refs. [24, 25]. Therefore, if vector potential is stronger than the scalar potential ($V_0 > S_0$), there is no bound state.

(ii) When $q \neq 0$, the energy eigenvalue is shown by the Eq. 21. If it is taken case S_0 instead of case $S_0 + m_0q$ for Eq. 21, it is obtained Eq. 26. This result means that the mass m_0q has only an additional scalar potential effect.

(iii) If we investigate the pure vector potential case ($S_0 = 0,\ V_0 \neq 0$), there are always a bound states in case $m_0q > V_0$. If we take the pure scalar potential ($S_0 \neq 0,\ V_0 = 0$), the energy eigenvalues are given as the following equation

$$E_n(q) = \pm\sqrt{\alpha(n+\frac{1}{2})(2m_0 - \alpha(n+\frac{1}{2}))} \tag{29}$$

5. CONCLUSION

In the present article, we have sought the solution of the radial s-wave Klein-Gordon equation for exponential-type scalar and vector potentials with position-dependent mass by using the asymptotic iteration method. We have shown exactly analytical solutions of the energy eigenvalues. We have compared with the studies previously. We have seen that our results are in good agreement. Also, we have investigated some special cases of scalar and vector potentials. We have shown that our energy eigenvalues for constant-mass case q=0 are in good agreement with the results previously. We can say that the asymptotic iteration method gives sufficiently accurate results for practical purposes. Thus, it is worth extending this method to obtain solution of the Klein-Gordon equation for other potentials with position-dependent mass.

ACKNOWLEDGEMENT

The authors thank Dr. O. Bayrak for useful comments on the manuscript.

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