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Some Fixed Point Theorems of R-Weakly Commuting Mappings in Multiplicative Metric Spaces

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ABSTRACT

In this paper, we present a unique common fixed point theorem for pointwise R-weakly commuting maps in complete multiplicative metric space. Another result of R-weakly commuting of type (P) is also established. Our results generalize the results of the main theorem of Xiaoju He, Meimei Song and Danping Chen (Common fixed points for weak commutative mappings on a multiplicative metric space) by using R-weakly commuting maps.

MSC: Primary 46B20, 47A12

Keywords: R-Weakly commuting maps, R-weakly commuting of type (P), multiplicative metric space, common fixed points.

1. INTRODUCTION AND PRELIMINARIES

Study of the importance of fixed points of mappings satisfying certain contraction condition is important in various research activities. Michael Grossman and Robert Katz [1] introduced multiplicative calculus which is also known as non-newtonian calculus. Regarding multiplicative calculus, Muttalip Ozavsar [2] used multiplicative contraction mappings and proved some fixed point theorems of mappings on complete multiplicative metric space. After this many scholars try to fit the well known mappings which are applicable in different spaces to multiplicative metric space and search its application for other streams. We found some of the

application about the multiplicative metric space. Agamieza E. Bashirov, Emine Misirli Kurpinar and Ali Ozyapici [3] used multiplicative calculus as a mathematical tool for economics, finance etc. Luc Florac, Hans van Assen [4] used multiplicative calculus in Biomedical Image Analysis, Ugur Kadak and Muharrem Ozluk [5] generalized Runge-Kutta method, A. Bashirov, M. Riza [6] used complex multiplicative differentiation and established multiplicative Cauchy-Riemann equation and also complex fourier series which are expressed in terms of exponents are suitable for multiplicative calculus. Misirli and Gurefe [7] used multiplicative calculus in numerical

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methods and many more application which we can't find out presently and a lot of application which are at the door step, which we can see in the near future. Recently, Demet et. al. [16] established some results on fixed points of non-Newtonian contraction mappings on non-Newtonian metric spaces.

In 1999, R. P. Pant [8,9] used the notion of R-weakly commuting maps. In the year 2006, Imdad Mohd. Ali and Javid Ali [10] introduced R-weakly commuting of type (P) in fuzzy metric space. In this paper, we discuss the common fixed points for R-weak commuting and R-weakly commuting maps of type (P) in complete multiplicative metric space.

2. SOME BASIC PROPERTIES

Definition 1. [3] Let (X, d) be a non empty set. A multiplicative metric is a mapping $d : X \times X \rightarrow R^+$ satisfying the following conditions:

(a) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) \cdot d(z, y)$ for all $x, y \in X$ (multiplicative triangle inequality).

Example 1. [2] Let R^+ be the collection of all n -tuples of positive real numbers. Let $d : R_+^n \times R_+^n \rightarrow R$ be defined as follows:

$$d(x, y) = \left| \frac{x_1}{y_1} \right| \left| \frac{x_2}{y_2} \right| \dots \left| \frac{x_n}{y_n} \right|,$$

where

$$x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in R_+^n$$

and $|\cdot| : R_+ \rightarrow R_+$ is defined as follows:

$$|a| = \begin{cases} a, & \text{if } a \geq 1; \\ \frac{1}{a}, & \text{if } a \leq 1. \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric are satisfied.

Definition 2. [2] Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x) = \{y \mid d(x, y) < \varepsilon\}$, $\varepsilon > 1$, there exists a

natural number N such that $n \geq N$, then $x_n \in B_\varepsilon(x)$. Here sequence $\{x_n\}$ is said to be multiplicative converging to x , denoted by $x_n \rightarrow x$ as $(n \rightarrow \infty)$.

Definition 3. [2] Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X . The sequence is called a multiplicative Cauchy sequence if it holds that for all $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.

Definition 4. [2] We call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is multiplicative convergence to $x \in X$.

Definition 5. [2] Let (X, d) be a multiplicative metric space. A mapping $f : X \times X \rightarrow X$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0, 1)$ such that $d(f(x_1), f(x_2)) \leq d(x_1, x_2)^\lambda$ for all $x, y \in X$.

Definition 6. [11] Suppose that S, T are two self-mappings of a multiplicative metric space (X, d) , S, T are called commutative mappings if it holds that for all $x \in X$, $STx = TSx$.

Definition 7. [11] Suppose that S, T are two self-mappings of a multiplicative metric space (X, d) ; S, T are called weak commutative mappings if it holds that for all $x \in X$, $d(STx, TSx) \leq d(Sx, Tx)$.

Remark: Commutative mappings must be weak commutative mappings, but the converse is not true.

Definition 8. [8,9] Let S, T be two self-mappings of multiplicative metric space (X, d) ; S, T are called pointwise R-weak commuting on X if there exists $R > 0$ such that

$$d(STx, TSx) \leq R d(Sx, Tx)$$

for every $x \in X$.

Definition 9.[10] Let S, T are two self-mappings of multiplicative metricspace (X, d) ; S, T are called R-weak commuting mappings of type (P) if there exists some $R > 0$ such that

$$d(SSx, TTx) \leq Rd(Tx, Sx)$$

for every $x \in X$.

Lemma.[9] Let S, T be two self-mappings of a multiplicative metric space (X, d) . If S, T are R-weakly commuting maps of type (P) and $\{x_n\}$ be a sequence in X such that

$$\lim_{n \rightarrow \infty} Sx_n = z = \lim_{n \rightarrow \infty} Tx_n$$

for some $z \in X$, then

$$\lim_{n \rightarrow \infty} STx_n = Tz$$

if T is continuous at z .

Theorem 10. [2] Let (X, d) be a multiplicative metric space and let $f : X \times X \rightarrow X$ be a multiplicative contraction. If (X, d) is complete, then f has a unique fixed point.

3.MAIN RESULTS

Theorem 11. Let A, B, S and T be self mappings from a complete multiplicative metric space into itself satisfying the following conditions:

- $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
- $d(Ax, By) \leq \left\{ \max \{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)\} \right\}^\lambda$,
- $\{A, S\}$ and $\{B, T\}$ are pointwise R-weakly commuting pairs,
- $\{A, S\}$ and $\{B, T\}$ are compatible pairs of reciprocally continuous mappings.

Then A, B, S and T have a unique common fixed point in X .

Proof. Since $A(X) \subset T(X)$ for an arbitrary point x_0 in X , there exists a point x_1 in X such that $Tx_1 = Ax_0$ and for x_1 there exists x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n}$$

and $y_{2n+1} = Tx_{2n} = Bx_{2n-1}$ for $n = 0, 1, 2, \dots$

From (b), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Ax_{2n-1}, Bx_{2n}) \\ &\leq \left\{ \max \left\{ \begin{aligned} &d(Sx_{2n-1}, Tx_{2n}), d(Ax_{2n-1}, Sx_{2n-1}), d(Bx_{2n}, Tx_{2n}), \\ &d(Ax_{2n-1}, Tx_{2n}), d(Bx_{2n}, Sx_{2n-1}) \end{aligned} \right\} \right\}^\lambda \end{aligned}$$

$$\begin{aligned}
&= \left\{ \max \left\{ d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \right. \right. \\
&\quad \left. \left. d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1}) \right\} \right\}^{\lambda} \\
&= \left\{ \max \left\{ d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), 1, d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}) \right\} \right\}^{\lambda} \\
&= d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1}).
\end{aligned}$$

This implies that

$$d(y_{2n}, y_{2n+1}) \leq d^{\frac{\lambda}{1-\lambda}}(y_{2n-1}, y_{2n}).$$

Let $h = \frac{\lambda}{1-\lambda}$, then

$$d(y_{2n}, y_{2n+1}) \leq d^h(y_{2n-1}, y_{2n}). \quad (1)$$

We also obtain

$$d(y_{2n+1}, y_{2n+2}) \leq d^h(y_{2n}, y_{2n+1}). \quad (2)$$

From (1) and (2), we have

$$d(y_{2n}, y_{2n+1}) \leq d^h(y_{2n-1}, y_{2n}) \leq \dots \leq d^{h^n}(y_1, y_0) \text{ for all } n \geq 2.$$

Let m, n be positive integers such that $m \geq n$, then we have

$$\begin{aligned}
d(y_m, y_n) &\leq d(y_m, y_{m-1}) \cdot d(y_{m-1}, y_{m-2}) \dots d(y_{n+1}, y_n) \\
&\leq d^{h^{m-1}}(y_1, y_0) \cdot d^{h^{m-2}}(y_1, y_0) \dots d^{h^n}(y_1, y_0) \\
&= d^{h^{m-1} + h^{m-2} + \dots + h^n}(y_1, y_0) \\
&\leq d^{\frac{h^n}{1-h}}(y_1, y_0).
\end{aligned}$$

This implies that $d(y_m, y_n) \rightarrow 1$ as $m, n \rightarrow \infty$. Hence $\{y_n\}$ is a multiplicative Cauchy. By the completeness of X , there exists $z \in X$ such that $y_n \rightarrow z$ as $n \rightarrow \infty$. Consequently, $Ax_{2n}, Bx_{2n-1}, Sx_{2n}, Tx_{2n+1}$ converges to z as $n \rightarrow \infty$.

If A and S are compatible, then

$$\lim_{n \rightarrow \infty} d(ASx_n, SAx_n) = 1$$

that is $Az = Sz$. Also, by reciprocal continuity of A and S , we have

$$\lim_{n \rightarrow \infty} ASx_{2n} = Az \text{ and } \lim_{n \rightarrow \infty} SAx_{2n} = Sz.$$

Since $A(X) \subset T(X)$, so there exists a point z in X such that $Az = Tw$. Using (b), we have

$$\begin{aligned} d(Az, Bw) &\leq \left\{ \max \left\{ d(Sz, Tw), d(Az, Sz), d(Bw, Tw), d(Az, Tw), d(Bw, Sz) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ d(Az, Az), d(Az, Az), d(Bw, Az), d(Az, Az), d(Bw, Az) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ 1, 1, d(Bw, Az) \right\} \right\}^\lambda \\ &= d^\lambda(Bw, Az) \\ \Rightarrow d^{1-\lambda}(Bw, Az) &= 1. \end{aligned}$$

Therefore, $Az = Bw$.

Thus, $Az = Sz = Bw = Tw$.

As pointwise R-weak commutativity of A and S implies that there exists $R > 0$ such that

$$d(ASz, SAz) \leq Rd(Az, Sz)$$

implies $ASz = SAz$ and $SSz = SAz = ASz = AAz$.

Similarly, pointwise R-weak commutativity of B and T implies that

$$BBw = BTw = TBw = TTW.$$

Again from (b), we have

$$\begin{aligned} d(Az, AAz) &= d(Bw, AAz) \\ &= d(AAz, Bw) \\ &\leq \left\{ \max \left\{ d(SAz, Tw), d(AAz, SAz), d(Bw, Tw), d(AAz, Tw), d(Bw, SAz) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ d(AAz, Az), d(AAz, AAz), d(Bw, Bw), d(AAz, Az), d(Az, AAz) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ 1, 1, d(AAz, Az) \right\} \right\}^\lambda. \end{aligned}$$

Therefore, $AAz = Az$. Thus $Az = AAz = SAz$.

Thus, Az is a common fixed point of A and S . Again from (b), we have

$$\begin{aligned} d(Bw, BBw) &= d(Az, BBw) \\ &\leq \left\{ \max \left\{ d(Sz, TBw), d(Az, Sz), d(BBw, TBw), d(Az, TBw), d(BBw, Sz) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ d(Bw, BBw), d(Bw, Bw), d(BBw, BBw), d(Bw, BBw), d(BBw, Bw) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ 1, 1, d(BBz, Bw) \right\} \right\}^\lambda. \end{aligned}$$

Therefore, $BBw = Bw$. Thus $Bw = BBw = TBw$.

Therefore, Bw is common fixed point of B and T .

If $Bw = Az = u$ then $Au = Su = Bu = Tu = u$. Hence u is the common fixed point of A, B, S and T .

In order to prove the uniqueness of fixed point, let v be another common fixed point of A, B, S and T . Then from (b), we have

$$\begin{aligned} d(u, v) &= d(Au, Bv) \\ &\leq \left\{ \max \left\{ d(Su, Tv), d(Au, Su), d(Bv, Tv), d(Au, Tv), d(Bv, Su) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ d(u, v), d(u, u), d(v, v), d(u, v), d(v, u) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ 1, 1, d(u, v) \right\} \right\}^\lambda. \end{aligned}$$

Therefore, $d(u, v) \rightarrow 1$. Thus, $u = v$.

This shows that fixed point is unique and hence completes the proof.

Theorem 12. Let S, T, A and B be self mappings from a complete multiplicative metric space X into itself satisfying the following conditions:

- a) $S(X) \subset B(X)$ and $T(X) \subset A(X)$,
- b) $\{A, S\}$ and $\{B, T\}$ are R-weakly commuting of type (P),
- c) One of S, T, A and B is continuous,

$$d) \quad d(Sx, Ty) \leq \left\{ \max \left\{ \begin{array}{l} d(Ax, By), d(Ax, Sx), d(By, Ty) \\ d(Sx, By), d(Ax, Ty) \end{array} \right\} \right\}^\lambda, \lambda \in \left(0, \frac{1}{2}\right), \forall x, y \in X.$$

Then S, T, A and B have a unique common fixed point in X .

Proof. Since $S(X) \subset B(X)$, consider a point $x_0 \in X$, $\exists x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$, $\exists x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$, $\exists x_{2n+1} \in X$ such that $Sx_{2n} = Bx_{2n+1} = y_{2n}$, $\exists x_{2n+2} \in X$ such that $Tx_{2n+1} = Ax_{2n+2} = y_{2n+1}, \dots$

Now this we can define a sequence $\{y_n\}$ in X , we obtain

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \left\{ \max \left\{ \begin{array}{l} d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}) \\ d(Sx_{2n}, Bx_{2n+1}), d(Ax_{2n}, Tx_{2n+1}) \end{array} \right\} \right\}^\lambda \end{aligned}$$

$$\begin{aligned} &\leq \left\{ \max \left\{ d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n+1}) \right\} \right\}^\lambda \\ &\leq \left\{ \max \left\{ d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), 1, d(y_{2n-1}, y_{2n}).d(y_{2n}, y_{2n+1}) \right\} \right\}^\lambda \\ &= d^\lambda(y_{2n-1}, y_{2n}).d^\lambda(y_{2n}, y_{2n+1}). \end{aligned}$$

This implies that

$$d(y_{2n}, y_{2n+1}) \leq d^{\frac{\lambda}{1-\lambda}}(y_{2n-1}, y_{2n}).$$

Let $h = \frac{\lambda}{1-\lambda}$, then

$$d(y_{2n}, y_{2n+1}) \leq d^h(y_{2n-1}, y_{2n}). \quad (3)$$

We also obtain

$$d(y_{2n+1}, y_{2n+2}) \leq d^h(y_{2n}, y_{2n+1}). \quad (4)$$

From (3) and (4), we know

$$d(y_n, y_{n+1}) \leq d^h(y_{n-1}, y_n) \leq \dots \leq d^{h^n}(y_1, y_0) \text{ for all } n \geq 2.$$

Let $m, n \in \mathbb{N}$, such that $m \geq n$, then we get

$$\begin{aligned} d(y_m, y_n) &\leq d(y_m, y_{m-1}).d(y_{m-1}, y_{m-2}) \dots d(y_{n+1}, y_n) \\ &\leq d^{h^{m-1}}(y_1, y_0).d^{h^{m-2}}(y_1, y_0) \dots d^{h^n}(y_1, y_0) \\ &\leq d^{\frac{h^n}{1-h}}(y_1, y_0). \end{aligned}$$

This implies that $d(y_m, y_n) \rightarrow 1$ as $m, n \rightarrow \infty$. Hence $\{y_n\}$ is a multiplicative Cauchy. By the completeness of X , there exists $z \in X$ such that $y_n \rightarrow z$ as $n \rightarrow \infty$.

Moreover, since $\{Sx_{2n-1}\} = \{Bx_{2n-2}\} = \{y_{2n-1}\}$ and $\{Tx_{2n}\} = \{Ax_{2n-1}\} = \{y_{2n}\}$ are subsequences of $\{y_n\}$, so we obtain

$$\lim_{n \rightarrow \infty} Sx_{2n-1} = \lim_{n \rightarrow \infty} Bx_{2n-2} = \lim_{n \rightarrow \infty} Tx_{2n} = \lim_{n \rightarrow \infty} Ax_{2n-1} = z.$$

Case 1: Suppose A is continuous, then

$$\lim_{n \rightarrow \infty} ASx_{2n} = \lim_{n \rightarrow \infty} AAx_{2n} = Az.$$

Since $\{A, S\}$ is R-weakly commuting maps of type (P), we have

$$d(SSx_{2n}, AAx_{2n}) \leq Rd(Ax_{2n}, Sx_{2n}).$$

Let $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} SAx_{2n} = Az.$$

Now, we obtain

$$d(SAx_{2n}, Tx_{2n+1}) \leq \left\{ \max \left\{ d(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \right. \right. \\ \left. \left. d(SAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, Tx_{2n+1}) \right\} \right\}^\lambda.$$

Letting $n \rightarrow \infty$, we obtain

$$\begin{aligned} d(Az, z) &\leq \left\{ \max \left\{ d(Az, z), d(Az, Az), d(z, z), d(Az, z), d(Az, z) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ d(Az, z), 1 \right\} \right\}^\lambda \\ &= d^\lambda(Az, z). \end{aligned}$$

This implies that $d(Az, z) = 1$ i.e. $Az = z$.

Again, we can obtain

$$d(Sz, Tx_{2n+1}) \leq \left\{ \max \left\{ d(Az, Bx_{2n+1}), d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1}), \right. \right. \\ \left. \left. d(Sz, Bx_{2n+1}), d(Az, Tx_{2n+1}) \right\} \right\}^\lambda.$$

Letting $n \rightarrow \infty$, we obtain

$$\begin{aligned} d(Sz, z) &\leq \left\{ \max \left\{ d(Az, z), d(z, Sz), d(z, z), d(Sz, z), d(z, z) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ d(Sz, z), 1 \right\} \right\}^\lambda \\ &= d^\lambda(Sz, z) \end{aligned}$$

which implies that $d(Sz, z) = 1$ i.e. $Sz = z$.

Now, $z = Sz \in S(X) \subseteq B(X)$, so $\exists z^* \in X$ such that $z = Bz^*$. Then

$$\begin{aligned} d(z, Tz^*) &= d(Sz, Tz^*) \\ &\leq \left\{ \max \left\{ d(Az, Bz^*), d(Az, Sz), d(Bz^*, Tz^*), \right. \right. \\ &\quad \left. \left. d(Sz, Bz^*), d(Az, Tz^*) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ d(z, Tz^*), 1 \right\} \right\}^\lambda \\ &= d^\lambda(z, Tz^*) \end{aligned}$$

which implies that $d(z, Tz^*) = 1$ i.e. $Tz^* = z$.

Since $\{B, T\}$ is R-weakly commuting maps of type (P), we have

$$d(Bz, Tz) = d(BBz^*, TTz^*) \leq Rd(Tz^*, Bz^*) = Rd(z, z) = R.$$

But $R > 0$ and $d(x, y) \geq 1$. Therefore, $d(Bz, Tz) = 1$, so $Bz = Tz$. Lastly, we have

$$\begin{aligned} d(z, Tz) &= d(Sz, Tz) \\ &\leq \left\{ \max \left\{ d(Az, Bz), d(Az, Sz), d(Bz, Tz), d(Sz, Bz), d(Az, Tz) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ d(z, Tz), 1 \right\} \right\}^\lambda \\ &= d^\lambda(z, Tz) \end{aligned}$$

which implies that $d(z, Tz) = 1$ i.e. $Tz = z$.

Case 2: Suppose that B is continuous, we can obtain the same result by the way of case 1.

Case 3: Suppose that S is continuous, then

$$\lim_{n \rightarrow \infty} SAx_{2n} = \lim_{n \rightarrow \infty} SSx_{2n} = Sz.$$

Since $\{A, S\}$ is R-weakly commuting maps of type (P), then

$$d(AAx_{2n}, SSx_{2n}) \leq Rd(Sx_{2n}, Ax_{2n}).$$

Let $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} ASx_{2n} = Sz.$$

Now, we obtain

$$d(SSx_{2n}, Tx_{2n+1}) \leq \left\{ \max \left\{ \begin{aligned} &d(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, SSx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\ &d(SSx_{2n}, Bx_{2n+1}), d(ASx_{2n}, Tx_{2n+1}) \end{aligned} \right\} \right\}^\lambda.$$

Letting $n \rightarrow \infty$, we obtain

$$\begin{aligned} d(Sz, z) &\leq \left\{ \max \left\{ d(Sz, z), d(Sz, Sz), d(z, z), d(Sz, z), d(Sz, z) \right\} \right\}^\lambda \\ &= \left\{ \max \left\{ d(Sz, z), 1 \right\} \right\}^\lambda \\ &= d^\lambda(Sz, z). \end{aligned}$$

This implies that $d(Sz, z) = 1$ i.e. $Sz = z$.

Since, $z = Sz \in S(X) \subseteq B(X)$, so $\exists z^* \in X$ such that $z = Bz^*$. Then

$$d(SSx_n, Tz^*) \leq \left\{ \max \left\{ d(ASx_n, Bz^*), d(ASx_n, SSx_n), d(Bz^*, Tz^*), \right. \right. \\ \left. \left. d(SSx_n, Bz^*), d(ASx_n, Tz^*) \right\} \right\}^\lambda.$$

Letting $n \rightarrow \infty$, we can obtain

$$d(Sz, Tz^*) \leq \left\{ \max \left\{ d(Sz, z), d(Sz, Sz), d(z, Tz^*), \right. \right. \\ \left. \left. d(Sz, z), d(Sz, Tz^*) \right\} \right\}^\lambda \\ = \left\{ \max \{ d(z, Tz^*), 1 \} \right\}^\lambda \\ = d^\lambda(z, Tz^*)$$

which implies that $d(z, Tz^*) = 1$ i.e. $Tz^* = z$.

Since $\{B, T\}$ is R-weakly commuting maps of type (P), we have

$$d(Tz, Bz) = d(TTz^*, BBz^*) \leq Rd(Bz^*, Tz^*) = Rd(z, z) = R.$$

But $R > 0$ and $d(x, y) \geq 1$. Therefore, $d(Bz, Tz) = 1$, so $Bz = Tz$.

Lastly, we have

$$d(Sx_{2n}, Tz) \leq \left\{ \max \left\{ d(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), \right. \right. \\ \left. \left. d(Sx_{2n}, Bz), d(Ax_{2n}, Tz) \right\} \right\}^\lambda.$$

Letting $n \rightarrow \infty$, we can obtain

$$d(z, Tz) \leq \left\{ \max \{ d(z, Tz), d(z, z), d(Tz, Tz), d(z, Tz), d(z, Tz) \} \right\}^\lambda \\ \Rightarrow d(z, Tz) \leq \left\{ \max \{ d(z, Tz), 1 \} \right\}^\lambda \\ = d^\lambda(z, Tz)$$

which implies that $d(z, Tz) = 1$ i.e. $Tz = z$.

Now, $z = Tz \in T(X) \subseteq A(X)$, so $\exists z^{**} \in X$ such that $z = Az^{**}$. Then

$$d(Sz^{**}, z) = d(Sz^{**}, z) \\ \leq \left\{ \max \left\{ d(Az^{**}, Bz), d(Az^{**}, Sz^{**}), d(Bz, Tz), \right. \right. \\ \left. \left. d(Sz^{**}, Bz), d(Az^{**}, Tz) \right\} \right\}^\lambda$$

$$\begin{aligned}
&= \left\{ \max \left\{ d(z, z), d(z, Sz^{**}), d(Bz, Bz), \right. \right. \\
&\quad \left. \left. d(Sz^{**}, z), d(z, z) \right\} \right\}^\lambda \\
&= \left\{ \max \{ d(Sz^{**}, z), 1 \} \right\}^\lambda \\
&= d^\lambda(Sz^{**}, z)
\end{aligned}$$

this implies that $d(Sz^{**}, z) = 1$ i.e. $Sz^{**} = z$.

Since $\{S, A\}$ is R-weakly commuting maps of type (P), we have

$$d(Az, Sz) = d(AAz^{**}, SSz^{**}) \leq Rd(Sz^{**}, Az^{**}) = Rd(z, z) = R$$

so $Az = Sz$.

We obtain $Sz = Tz = Az = Bz = z$, so z is a common fixed point of S, T, A and B .

Case 4: Suppose that T is continuous, we can obtain the same result by the way of case 3.

In addition, we prove that S, T, A and B have a unique common fixed point. Suppose that $w \in X$ is also a common fixed point of S, T, A and B , then

$$\begin{aligned}
d(z, w) &= d(Sz, Tw) \\
&\leq \left\{ \max \{ d(Az, Bw), d(Az, Sz), d(Bw, Tw), d(Sz, Bw), d(Az, Tw) \} \right\}^\lambda \\
&= \left\{ \max \{ d(z, w), 1 \} \right\}^\lambda \\
&= d^\lambda(z, w)
\end{aligned}$$

this implies that $d(z, w) = 1$ i.e. $z = w$, which is a contradiction.

Hence, S, T, A and B have a unique common fixed point.

Example 2. Let $X = R$ be a usual metric space. Define the mappings $d : X \times X \rightarrow R^+$ by $d(x, y) = e^{|x-y|}$ for all $x, y \in X$. Clearly (X, d) is a complete multiplicative metric space. Consider the following mappings:

$$Sx = x, Tx = \frac{1}{2}x, Bx = 3x, Ax = 2x \text{ for all } x \in X$$

- $SX = TX = BX = AX = X$, so $SX \subset BX, TX \subset AX$;
- $\{A, S\}, \{B, T\}$ are R-weakly commuting maps of type (P);
- One of S, T, A and B is continuous;

d) Let $\lambda = \frac{1}{3}$, then

$$\begin{aligned} d(Sx, Ty) &\leq \left\{ \max \left\{ d(Ax, By), d(Bx, Tx), d(By, Ty), d(Sx, By), d(Ax, Ty) \right\} \right\}^\lambda \\ &\Rightarrow e^{\left| x - \frac{1}{2}y \right|} \leq \left\{ \max \left\{ e^{|3x-2y|}, e^{|2x|}, e^{\left| \frac{3}{2}y \right|}, e^{|2y-x|}, e^{\left| 3x - \frac{1}{2}y \right|} \right\} \right\}^\lambda \\ &= \max \left\{ e^{|3x-2y|\lambda}, e^{|2x|\lambda}, e^{\left| \frac{3}{2}y \right|\lambda}, e^{|2y-x|\lambda}, e^{\left| 3x - \frac{1}{2}y \right|\lambda} \right\}. \end{aligned}$$

Because $y = \ln x$ is an increasing mapping, so

$$\Leftrightarrow \left| x - \frac{1}{2}y \right| = \max \left\{ |3x-2y|\lambda, |2x|\lambda, \left| \frac{3}{2}y \right|\lambda, |2y-x|\lambda, \left| 3x - \frac{1}{2}y \right|\lambda \right\}.$$

There are three situations: (i) $x \geq \frac{1}{2}y \geq 0$ or

$\frac{1}{2}y \geq x \geq 0$; (ii) $\frac{1}{2}y < x < 0$ or $x < \frac{1}{2}y < 0$;

(iii) $x > 0, y < 0$ or $x < 0, y > 0$. No matter what kind of situation, inequality (iii) is true. So all the condition of main theorem are true, then we obtain $SO = TO = AO = BO = 0$, so 0 is the unique common fixed point of S, T, A and B .

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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