GU J Sci 30(3): 139-159 (2017)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Odd Burr Power Lindley Distribution with Properties and Applications

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Article Info

Received: 10/02/2017
Accepted: 03/05/2017

Keywords

Maximum likelihood
Estimation
Simulation
Quantile function
Moment

Abstract

We introduce a four-parameter distribution, called odd Burr power Lindley distribution, which extends the Lindley distribution and has increasing, upside-down and bathtub shapes for the hazard rate function. Our purpose is to provide a generalization that may be useful to still more complex situations. It includes as special sub-models some well-known distributions such as Lindley, power Lindley, odd log-logistic Lindley, among others. Several statistical properties of the distribution are explored. A simulation study is performed to assess the maximum likelihood estimations of introduced distribution parameters in terms of bias and mean square error, estimated average length and coverage probability.

1. INTRODUCTION

The Lindley distribution is a very well-known distribution that has been extensively used over the past decades for modeling data in reliability, biology, insurance, finance, and lifetime analysis. The Lindley distribution was introduced by Lindley (1958) to analyze failure time data. The motivation for introducing the Lindley distribution arises from its ability to model failure time data with increasing, decreasing, unimodal and bathtub shaped hazard rates. This distribution represents a good alternative to the exponential failure time distributions that suffer from not exhibiting unimodal and bathtub shaped failure rates.

The need for extended forms of the Lindley distribution arises in many applied areas. The emergence of such distributions in the statistics literature is only very recent. For some extended forms of the Lindley distribution and applications, the reader is referred to Kumaraswamy Lindley (Cakmakyapan and Ozel, 2014), beta odd log-logistic Lindley (Cordeiro et al., 2015), generalized Lindley (Nadarajah et al., 2011), quasi Lindley distribution (Shanker and Mishra, 2013), inverse Lindley (Sharma et al., 2015), power Lindley (Ghitany et al. 2013). The pdf and cdf of the Lindley distribution are, respectively, given by

$$g(y;\lambda)=\frac{\lambda^2}{1+\lambda}(1+y)e^{-\lambda y}, \qquad y>0,\ \lambda>0 \tag{1}$$

$$G(y;\lambda)=1-\left(1+\frac{\lambda y}{1+\lambda}\right)e^{-\lambda y}, \quad y>0,\ \lambda>0 \tag{2}$$

It can be seen that this distribution is a mixture of Exponential (λ) and gamma $(2,\lambda)$ distributions. Using the transformation $X=Y^{\frac{1}{\beta}}$, Ghitany et al. (2013) derived the power Lindley (PL) distribution given by

$$g(x;\lambda,\beta)=\frac{\lambda^2\beta}{1+\lambda}(1+x^{\beta})x^{\beta-1}e^{-\lambda x^{\beta}}, \qquad x>0,\ \lambda,\beta>0 \tag{3}$$

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$$G(x;\lambda,\beta)=1-\left(1+\frac{\lambda}{1+\lambda}x^{\beta}\right)e^{-\lambda x^{\beta}}, \quad x>0, \; \lambda,\beta>0 \tag{4}$$

The PL distribution does not provide enough flexibility for analyzing different types of lifetime data. To increase the flexibility for modelling purposes, it will be useful to consider further alternatives to this distribution. Therefore, the aim of this study is to introduce a new distribution using the Lindley distribution. Recently, an important class of univariate distributions is the odd Burr generalized family (OBu-G for short) proposed by Alizadeh et al. (2016) with two extra shape parameters. Motivated by OBu-G family, the main aim of this paper is to provide an extension of the PL distribution using the Burr distribution. So, we propose the new Odd Burr Power Lindley ("OBu-PL" for short) distribution by adding three extra parameters to the Lindley distribution. The objectives of the research are to study some structural properties of the proposed distribution.

The paper is organized as follows. In Section 2, we introduce the OBu-PL distribution and provide plots of the density and hazard rate functions. Shapes, quantile function, moments are also obtained. Estimation by the method of maximum likelihood and an explicit expression for the observed information matrix are presented in Section 3. A simulation study is conducted in Section 4. Applications to real data sets are considered in Section 5. Finally, Section 6 presents concluding remarks.

2. MAIN PROPERTIES

2.1. Probability Density and Cumulative Density Functions

The cdf of OBu-G family introduced by Alizadeh et al. (2016) is given by

$$F(x;a,b)=1-\left\{1-\frac{G(x)^{a}}{G(x)^{a}+\bar{G}(x)^{a}}\right\}^{b}, \quad a,b>0, \; x>0 \tag{5}$$

and the pdf

$$f(x;a,b)=\frac{abg(x)G(x)^{a-1}\bar{G}(x)^{ab-1}}{\left\{G(x)^{a}+\bar{G}(x)^{a}\right\}^{b+1}}, \quad a,b>0, \quad x>0 \tag{6}$$

where $\bar{G}(x)=1-G(x)$. OBu-G contains two important family, odd log-logistic-G by Gleaton and Lynch (2006) and proportional hazard rate family by Gupta and Gupa (1998). OBu-G provides more flexibility for density and hazard functions for each special case. Further, for integer b, we consider a system formed by b independent components following the Odd-log-logistic family (Gleaton and Lynch, 2006) given by

$$\Pi(x;a)=\frac{G(x)^{a}}{G(x)^{a}+\bar{G}(x)^{a}}, \quad a>0 \tag{7}$$

Suppose the system fails if at least one of the b components fails and let X denote the lifetime of the entire system. Then, the cdf of X is

$$F(x;a,b)=1-\left(1-\frac{G(x)^{a}}{G(x)^{a}+\bar{G}(x)^{a}}\right)^{b}, \quad a,b>0 \tag{8}$$

Altun et al. (2016) studied Odd Burr Lindley (OBu-L), if X follows OBu-L, then $Y=X^{\beta}$ has OBu-PL proposed model. Inserting (4) in (5), the cdf of the OBu-PL with four parameters ($a,b,\lambda,\beta>0$) is defined as

$$F(x;a,b,\lambda,\beta)=1-\left\{1-\frac{\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a}}{\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a}+\left[\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a}}\right\}^{b} \quad (9)$$

The corresponding pdf of the OBu-PL is given by

$$f(x;a,b,\lambda,\beta)=\frac{ab\beta x^{\beta-1}\left(1+x^{\beta}\right)e^{-\lambda x^{\beta}}\,\frac{\lambda^{2}}{1+\lambda}\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a-1}\left[\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{ab-1}}{\left\{\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a}+\left[\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a}\right\}^{b+1}} \quad (10)$$

where λ and β are scale parameters and the shape parameters a and b govern the skewness of (10). A random variable X with the pdf (10) is denoted by $X \sim \text{OBu-PL}(a,b,\lambda,\beta)$. Some special cases of the OBu-PL distribution is presented in Table 1.

Table 1. *Some Special Cases of the OBu-PL distribution*

a	b	β	Reduced Distribution
1	1	1	Lindley
1	1	-	Power Lindley
-	1	-	Odd Log-Logistic Power Lindley
-	1	1	Odd Log-Logistic Lindley
-	-	1	Odd Burr Lindley

Some of the possible shapes of density functions in (10) for selected parameter values are illustrated in Figure 1. As seen from Figure 1, the density function can take various forms depending on the parameter values. The pdf of the OBu-PL distribution is unimodal. It increases and decreases for various values of the parameters giving the shapes obtained in Figure 1. It is evident that the OBu-PL distribution is much more flexible than the PL distribution, i.e. the additional shape parameters a and b allow for a high degree of flexibility of the OBu-PL distribution.

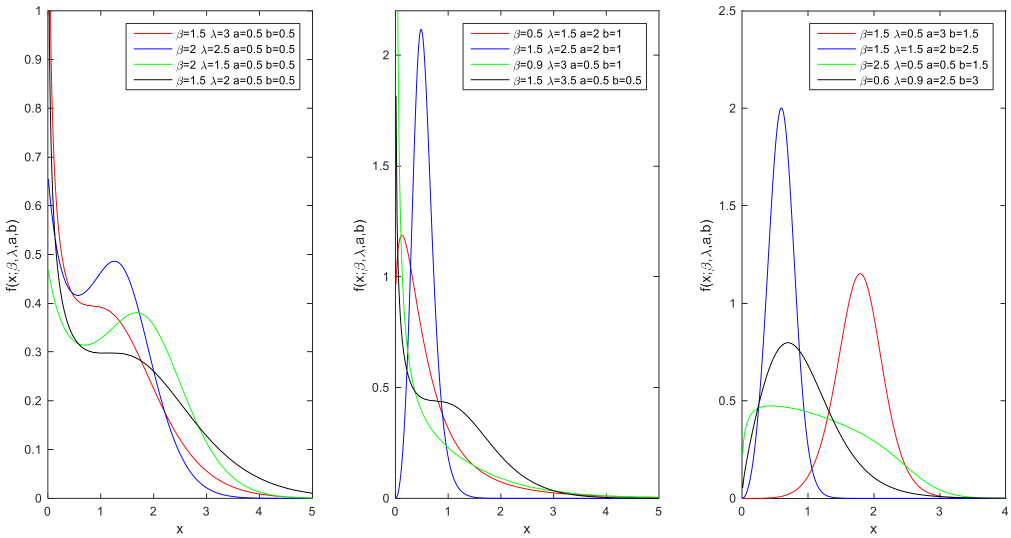

Figure 1. *Plot of the pdf for several values of parameters.*

2.2. Survival and Hazard Rate Functions

We obtain the survival function corresponding to (9) as

$$S(x;a,b,\lambda,\beta)=\left\{1-\frac{\left[1-\left(1+\dfrac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a}}{\left[1-\left(1+\dfrac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a}+\left[\left(1+\dfrac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a}}\right\}^{b} \tag{11}$$

In reliability studies, the hazard rate function (hrf) is an important characteristic and fundamental to the design of safe systems in a wide variety of applications. Therefore, we discuss these properties of the OBu-PL distribution. The hrf of X takes the form

$$h(x;a,b,\lambda,\beta)=\frac{ab\beta\dfrac{\lambda^{2}}{1+\lambda}x^{\beta-1}(1+x^{\beta})e^{-\lambda x^{\beta}}\left(1-\left(1+\dfrac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right)^{a-1}}{\left\{\left[1-\left(1+\dfrac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a}+\left[\left(1+\dfrac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{a}\right\}\left[\left(1+\dfrac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]} \tag{12}$$

Plots for the hrf of the OBu-PL distribution for several parameter values are displayed in Figure 2. Figure 2 shows that the hrf of the OBu-PL distribution can have very flexible shapes, such as increasing, upside-down, and bathtub. This attractive flexibility makes the hrf of the OBu-L distribution useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

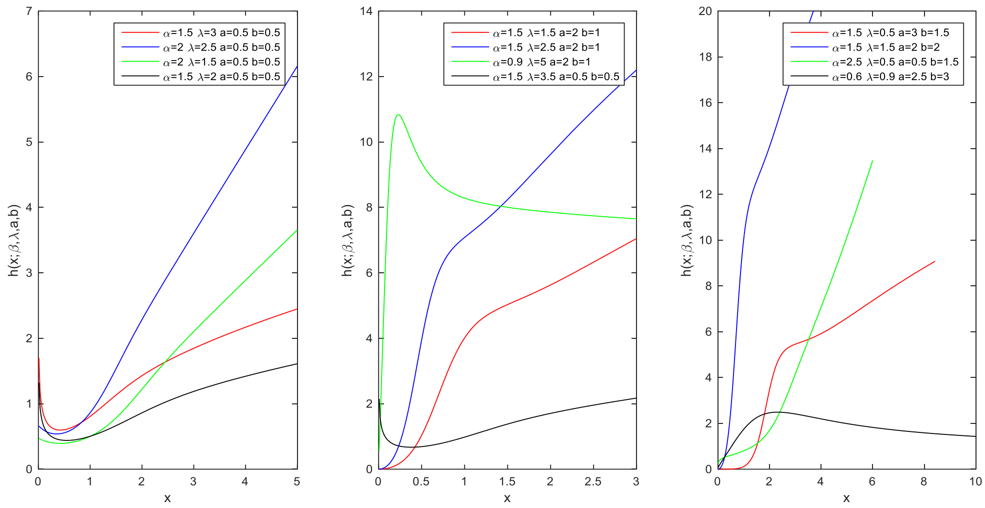

Figure 2. *Plots for the hrf for several values of parameters.*

2.3. Asymptotic

Proposition 1. The asymptotic of cdf, pdf, and hrf of the OBu-PL distribution as $x \to 0$ are given by

$$F(x) \sim b(\lambda x^{\beta})^{a} \quad \text{as } x \to 0,$$
$$f(x) \sim ab\beta\lambda^{a} x^{a\beta-1} \quad \text{as } x \to 0,$$
$$h(x) \sim \frac{a\beta}{x} \quad \text{as } x \to 0.$$

Proposition 2. The asymptotic of cdf, pdf, and hrf of the OBu-PL as $x \to \infty$ are given by

$$1-F(x) \sim \left(\frac{a\lambda}{1+\lambda}\right)^{b} x^{b\beta} e^{-b\lambda x^{\beta}} \quad \text{as } x \to \infty,$$
$$f(x) \sim b\left(\frac{a\lambda}{1+\lambda}\right)^{b} x^{b\beta-1} e^{-b\lambda x^{\beta}} \quad \text{as } x \to \infty,$$
$$h(x) \sim \frac{b}{x} \quad \text{as } x \to \infty.$$

These equations show the effect of parametrs on tail of OBu-PL distribution.

2.4. Quantile Function

Let $X \sim OBu-PL(a,\ b,\ \lambda, \beta)$, the quantile function, say $Q(p)$, is defined by $F(Q(p)) = p$ .Then, we can obtain $Q(p)$ as the root of the following equation

$$\left[1+\lambda+\lambda Q(p)^{\beta}\right] e^{-\lambda Q(p)^{\beta}} = \frac{(1+\lambda)(1-p)^{\frac{1}{ab}}}{(1-p)^{\frac{1}{ab}} + \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}}}$$

for $0 < p < 1$. Substituing $Z(p) = -1-\lambda-\lambda Q(p)^{\beta}$, we can rewrite it as

$$Z(p)e^{Z(p)} = \frac{-(1+\lambda)e^{-1-\lambda}(1-p)^{\frac{1}{ab}}}{(1-p)^{\frac{1}{ab}} + \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}}}$$

Hence, the equation of $Z(p)$ is

$$Z(p) = W\left[\frac{-(1+\lambda)e^{-1-\lambda}(1-p)^{\frac{1}{ab}}}{(1-p)^{\frac{1}{ab}} + \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}}}\right]$$

where $W[.]$ is the Lambert function (Corless et al., 1996). Then, we obtain

$$Q(p) = \left\{-1-\frac{1}{\lambda}-\frac{1}{\lambda}W\left[\frac{-(1+\lambda)e^{-1-\lambda}(1-p)^{\frac{1}{ab}}}{(1-p)^{\frac{1}{ab}} + \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}}}\right]\right\}^{\frac{1}{\beta}} \qquad (13)$$

The particular case of (13) for $a = b = 1$ has been derived recently by Jórda (2010). Here, we also propose different algorithms for generating random data from the OBu-PL distribution as follows:

Algorithm 1. (Mixture form of the Lindley distribution)

(a) Generate $U_i \sim \text{Uniform}(0,1)$, $i = 1,2,...,n$
(b) Generate $V_i \sim \text{Exponential}(\lambda)$, $i = 1,2,...,n$
(c) Generate $W_i \sim \text{Gamma}(2,\lambda)$, $i = 1,2,...,n$
(d) If $\frac{\left[1-(1-U_i)^{\frac{1}{b}}\right]^{\frac{1}{a}}}{(1-U_i)^{\frac{1}{ab}} + \left[1-(1-U_i)^{\frac{1}{b}}\right]^{\frac{1}{a}}} \le \frac{\lambda}{1+\lambda}$, set $X_i = V_i^{\frac{1}{\beta}}$, otherwise set $X_i = W_i^{\frac{1}{\beta}}$, $i = 1,2,...,n$

Algorithm 2. (Mixture form of the PL distribution)

(a) Generate $U_i \sim \text{Uniform}(0,1)$, $i = 1,2,...,n$
(b) Generate $Y_i \sim \text{Weibull}(\beta,\lambda)$, $i = 1,2,...,n$
(c) Generate $Z_i \sim \text{Generalized Gamma}(2,\beta,\lambda)$, $i = 1,2,...,n$
(d) If $\frac{\left[1-(1-U_i)^{\frac{1}{b}}\right]^{\frac{1}{a}}}{(1-U_i)^{\frac{1}{ab}} + \left[1-(1-U_i)^{\frac{1}{b}}\right]^{\frac{1}{a}}} \le \frac{\lambda}{1+\lambda}$, set $X_i = Y_i$, otherwise set $X_i = Z_i$, $i = 1,2,...,n$

Algorithm 3. (Inverse Method)

(a) Generate $U_i \sim \text{Uniform}(0,1),\ i = 1,2,...,n$
(b) Set

$$X_i = \left\{-1-\frac{1}{\lambda}-\frac{1}{\lambda}W\left[\frac{-(1+\lambda)e^{-1-\lambda}(1-U_i)^{\frac{1}{ab}}}{(1-p)^{\frac{1}{ab}}+\left[1-(1-U_i)^{\frac{1}{b}}\right]^{\frac{1}{a}}}\right]\right\}^{\frac{1}{\beta}}$$

2.5. Extreme Value

Let $X_1,...,X_n$ be a random variable from (10) and $\bar{X} = (X_1 + ... + X_n)/n$ denote the sample mean, then by central limit theorem, the distribution $\sqrt{n}\left[\bar{X} - E(X)\right]/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$. Sometimes, one would be interested in the asymptotic of the extreme value, $M_n = \max\{X_1,...,X_n\}$ and $m_n = \min\{X_1,...,X_n\}$. For (1), it can be seen that

$$\lim_{t\to\infty}\frac{F(tx)}{F(t)} = x^{a\beta}$$

and

$$\lim_{t\to\infty}\frac{1-F(tx)}{1-F(t)} = e^{-ab\lambda x^{\beta}}.$$

Thus, it follows from Theorem 1.6.2 in Leadbetter et al. (1983) that they must be norming constant $a_n, b_n, c_n > 0$ and $d_n > 0$ such that

$$\Pr\left[a_n(M_n - b_n) \le x\right] \to e^{-ab\lambda x^{\beta}}$$

and

$$\Pr\left[c_n(m_n - d_n) \le x\right] \to 1 - e^{-x^{a\beta}}$$

as $n \to \infty$. Using Corollary 1.6.3 of Leadbetter et al. (1983), we can obtain the form of normalizing constants a_n, b_n, c_n and d_n.

2.6. Expansions

In this subsection, we provide alternative mixture representations for the pdf and cdf of X. Despite the fact that the pdf and cdf of OBu-PL require mathematical functions that are widely available in modern statistical packages, frequently analytical and numerical derivations take advantage of power series for the pdf. Some useful expansions for (6) can be derived by using the concept of power series. Let $H_k(x)$ denote the cdf of the exponential PL with parameters λ, β and k. We obtain the cdf of OBu-PL as

$$F(x)=1-\sum_{i=0}^{\infty}(-1)^i \frac{G(x)^{ai}}{\left[G(x)^a+\bar{G}(x)^a\right]^i} \tag{14}$$

where $\alpha_k=\sum_{j=k}^{\infty}(-1)^{j+k}\binom{ai}{j}\binom{j}{k}$, $\left[G(x)^a+\bar{G}(x)^a\right]^i=\sum_{k=0}^{\infty}\beta_k G(x)^k$, and $\beta_k=h_k(\alpha,i)$ which is defined in Appendix A. Then, we can write

$$\frac{G(x)^{ai}}{\left[G(x)^a+\bar{G}(x)^a\right]^i}=\frac{\sum_{k=0}^{\infty}\alpha_k G(x)^k}{\sum_{k=0}^{\infty}\beta_k G(x)^k}=\sum_{k=0}^{\infty}\gamma_k G(x)^k$$

where $\gamma_0=\frac{\alpha_0}{\beta_0}$ and for $k\geq 1$ we have $\gamma_k=\beta_0^{-1}\left[\alpha_k-\beta_0^{-1}\sum_{r=1}^{k}\beta_r\gamma_{k-r}\right]$. Then, we obtain

$$F(x)=1-\sum_{i,k=0}^{\infty}(-1)^i\gamma_k(\alpha,i)G(x)^k=1-\sum_{k=0}^{\infty}a_k^* G(x)^k=\sum_{k=0}^{\infty}b_k^* G(x)^k \tag{15}$$

where $a_k^*=\sum_{i=0}^{\infty}(-1)^i\gamma_k(\alpha,i)$, $b_0^*=1-a_0^*$ and $b_k^*=-a_k^*$ for $k\geq 1$.

Then, we have

$$F(x)=\sum_{k=0}^{\infty}b_k^*\Pi_k(x) \tag{16}$$

and

$$f(x)=\sum_{k=0}^{\infty}b_{k+1}^*\pi_{k+1}(x) \tag{17}$$

where $\Pi_k(x)$ and $\pi_k(x)$ denote the cdf and pdf of exp-PL, respectively.

2.7. Moments

Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis). Now, we obtain ordinary and incomplete moments of the OBu-PL distribution. Nadarajah et al. (2011) defined and computed

$$A\left(a_1,a_2,a_3,a_4;\lambda,\beta\right)=\int_0^{\infty}x^{a_1}\left(1+x^{\beta}\right)^{a_2}\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{a_4}e^{-a_3x^{\beta}}dx \tag{18}$$

which can be used to produce ordinary moments $\left(\mu_n'\right)$. Then, we have

$$A(a_1,a_2,a_3,a_4;\lambda,\beta)=\sum_{\ell,r=0}^{\infty}\sum_{k=0}^{\ell}(-1)^{\ell}\binom{a_4}{\ell}\binom{\ell}{k}\binom{a_2}{r}\left(\frac{\lambda}{1+\lambda}\right)^{\ell}\frac{\Gamma\left(\frac{a_1+1}{\beta}+k+r\right)}{\beta(\lambda\ell+a_3)^{\frac{a_1+1}{\beta}+k+r}} \quad (19)$$

From using (18) and (19), we obtain

$$\mu_n'=E\left[X^n\right]=\frac{\lambda^2\beta}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\,b_{k+1}^{*}A(n+\beta-1,1,\lambda,k;\lambda,\beta) \quad (20)$$

The ordinary moments of the OBu-PL distribution can be calculated directly from (20). We now provide a formula for the conditional moments of the OBu-PL distribution. Nadarajah et al. (2011) defined and computed the following equation for the conditional moments. From (18), we have

$$B(a_1,a_2,a_3,a_4;y,\gamma,\lambda,\beta)=\int_0^y x^{a_1}\left(1+x^{\beta}\right)^{a_2}\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{a_4}e^{-a_3x^{\beta}}dx$$
$$=\sum_{\ell,r=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{\ell}\binom{a_4}{\ell}\binom{\gamma}{k}\binom{a_2}{\gamma}\left(\frac{\lambda}{1+\lambda}\right)^{\ell}\frac{\gamma\left(\frac{a_1+1}{\beta}+k+r,\frac{y^{\frac{1}{\beta}}}{\lambda\ell+a_3}\right)}{\beta(\lambda\ell+a_3)^{\frac{a_1+1}{\beta}+k+r}} \quad (21)$$

where $\gamma(x,z)=\int_0^z t^{z-1}e^{-t}dt$ denotes the incomplete gamma function.

From using equations (18) and (21), we obtain nth incomplete moment of the OBu-PL is found to be

$$m_n(y)=E\left[X^n\middle|X<y\right]=\frac{\lambda^2\beta}{1+\lambda}\sum_{k=0}^{\infty}(k+1)b_{k+1}^{*}B(n+\beta-1,\lambda,k;y,\lambda,\beta) \quad (22)$$

Skewness measures the degree of the long tail and kurtosis is a measure of the degree of tail heaviness. For the OBu-PL, Galton's skewness can be computed by using quantile function in (13) as

$$S=\frac{Q(3/4)-2Q(2/4)+Q(1/4)}{Q(3/4)-Q(1/4)}$$

and the Moors' kurtosis is based on octiles as

$$K=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)},$$

where $Q(\cdot)$ represents the quantile function. When the distribution is symmetric, $S=0$ and when the distribution is right (or left) skewed $S>0(\text{or } S<0)$. As K increases, the tail of the distribution becomes heavier. These measures are less sensitive to outliers and they exist even for distributions without moments.

We present skewness and kurtosis of the OBu-PL distribution for various values of parameters in Table 2.

Table 2. *First four ordinary moments, kurtosis and skewness of the OBu-PL distribution for various values of parameters.*

$\lambda=0.5, \beta=1$	μ'_1	μ'_2	μ'_3	μ'_4	Skewness	Kurtosis
$a=0.5 \quad b=1$	4.61	49.33	735.58	13593.73	2.12	5.59
$a=0.5 \quad b=1.5$	2.80	20.88	223.75	2984.73	2.35	6.85
$a=0.5 \quad b=2$	1.92	10.94	93.28	1021.46	2.58	8.53
$a=0.5 \quad b=5$	0.45	0.85	2.68	11.19	3.42	15.49
$a=1 \quad b=1$	3.37	19.23	151.36	1499.81	1.79	4.06
$a=1 \quad b=1.5$	2.40	9.59	52.13	354.46	1.76	3.85
$a=1 \quad b=2$	1.88	6.01	25.82	137.10	1.75	3.80
$a=1 \quad b=5$	0.88	1.34	2.70	6.76	1.74	3.76
$a=2 \quad b=1$	2.85	10.10	43.41	224.25	1.35	2.20
$a=2 \quad b=1.5$	2.37	6.77	22.50	85.52	1.28	1.87
$a=2 \quad b=2$	2.09	5.20	14.80	47.16	1.25	1.74
$a=2 \quad b=5$	1.46	2.50	4.79	10.04	1.21	1.61

Table 2 reveals that for $a<1$, kurtosis and skewness increase when b increases. For $a \geq 1$, the kurtosis and skewness decrease when b increases. Plots for skewness and kurtosis based on Moors's and Galton's measures are presented in Figure 3.

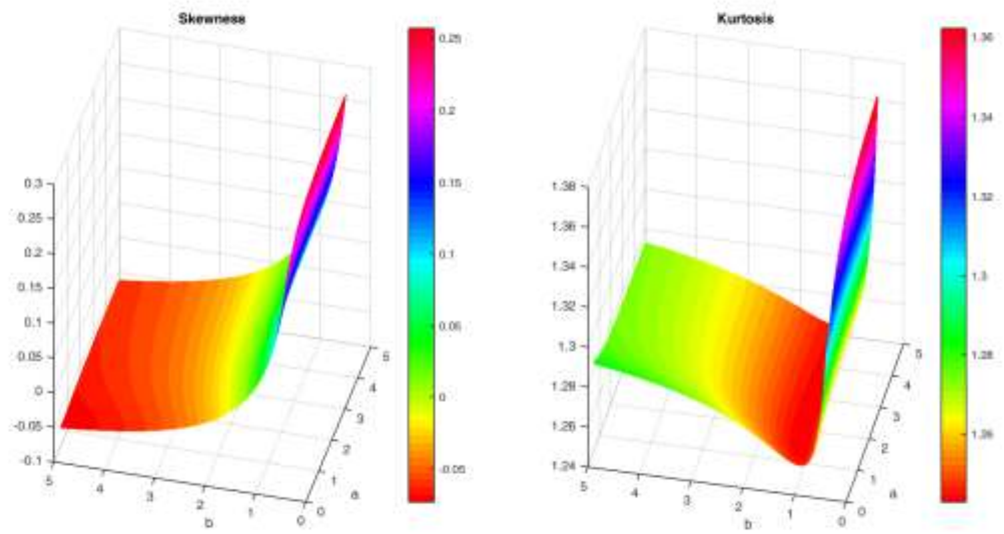

Figure 3. *Plots of Galton skewness and Moor kurtosis of OBu-PL distribution for several values of parameters.*

3. ESTIMATION

Several approaches for parameter estimation have been proposed in the literature but the maximum likelihood method is the most commonly employed. Here, we consider estimation of the unknown parameters of the OBu-PL distribution by the method of maximum likelihood. Let $x_1, x_2, ..., x_n$ be observed values from the OBu-PL distribution with parameters a, b, λ and β. The log-likelihood function for (a, b, λ, β) is given by

$$\log L = n\log\left(\frac{ab\lambda^2\beta}{1+\lambda}\right) + (\beta-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log(1+x_i^\beta) + (ab-1)\sum_{i=1}^{n}\log(t_i) + (a-1)\sum_{i=1}^{n}\log(1-t_i)$$
$$-(b+1)\sum_{i=1}^{n}\log\left\{t_i^a + (1-t_i)^a\right\} \quad (23)$$

where $t_i = 1-\left(1+\frac{\lambda}{1+\lambda}x_i^\beta\right)e^{-\lambda x_i^\beta}$.

The derivatives of the log-likelihood function with respect to the parameters a, b, λ and β are given by, respectively,

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\log(1-t_i) + b\sum_{i=1}^{n}\log(t_i) - (b+1)\left[\sum_{i=1}^{n}\frac{t_i^a\log(t_i) + (1-t_i)^a\log(1-t_i)}{t_i^a + (1-t_i)^a}\right]$$
$$\frac{\partial \log L}{\partial b} = \frac{n}{b} + a\sum_{i=1}^{n}\log(t_i) - \sum_{i=1}^{n}\log\left[t_i^a + (1-t_i)^a\right] \quad (24)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{2n}{\lambda} - \frac{n}{1+\lambda} + (ab-1)\sum_{i=1}^{n}\frac{t_i^{(\lambda)}}{t_i} + (1-a)\sum_{i=1}^{n}\frac{t_i^{(\lambda)}}{1-t_i} - a(b+1)\sum_{i=1}^{n}t_i^{(\lambda)}\frac{t_i^{a-1} - (1-t_i)^{a-1}}{t_i^a - (1-t_i)^a}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\frac{x_i^\beta\log(x_i)}{1+x_i^\beta} + (ab-1)\sum_{i=1}^{n}\frac{t_i^{(\beta)}}{t_i} + (1-a)\sum_{i=1}^{n}\frac{t_i^{(\beta)}}{1-t_i} - a(b+1)\sum_{i=1}^{n}t_i^{(\beta)}\frac{t_i^{a-1} - (1-t_i)^{a-1}}{t_i^a - (1-t_i)^a}$$

where

$$t_i^{(\lambda)} = \frac{\partial t_i}{\partial \lambda} = \frac{-1}{(1+\lambda)^2}x_i^\beta e^{-\lambda x_i^\beta} + x_i^\beta\left(1+\frac{\lambda}{1+\lambda}x_i^\beta\right)e^{-\lambda x_i^\beta}$$

and

$$t_i^{(\beta)} = \frac{\partial t_i}{\partial \beta} = -\frac{\lambda}{(1+\lambda)}x_i^\beta e^{-\lambda x_i^\beta}\log(x_i) + \lambda x_i^\beta\left(1+\frac{\lambda}{1+\lambda}x_i^\beta\right)e^{-\lambda x_i^\beta}\log(x_i)$$

The MLEs of (a,b,λ,β), say $(\hat{a},\hat{b},\hat{\lambda})$, are the simultaneous solutions of the equations $\frac{\partial \log L}{\partial a} = 0$, $\frac{\partial \log L}{\partial b} = 0$, and $\frac{\partial \log L}{\partial \lambda} = 0$.

4. SIMULATION STUDY

In this section, we evaluate the performance of the MLEs of the parameters of OBu-PL model by means of a simulation study. Inverse transform algorithm is used to generate random data from the OBu-PL distribution. The used algorithm can be found in Section 2.4. The precision of the MLEs is discussed by means of bias, mean square error (MSE), estimated average length (AL) and coverage probability (CP). We generated N=1000 samples of sizes $n=50,55,...,1000$ from OBu-PL distribution with $a=0.5, b=0.5, \lambda=2, \beta=2$. We obtained MLEs of the parameters for each generated sample and standard errors of MLEs are obtained by inverting observed information matrix. The estimated bias, MSEs, CPs and ALs can be obtained using following equations:

$$Bias_{\alpha}(n)=\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i-\alpha)$$

$$MSE_{\alpha}(n)=\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i-\alpha)^2$$

$$CP_{\alpha}(n)=\sum_{i=1}^{N}I\left(\hat{\alpha}_i-1.95996s_{\hat{\alpha}_i},\hat{\alpha}_i+1.95996s_{\hat{\alpha}_i}\right)$$

$$AL_{\alpha}(n)=\frac{3.919928}{N}\sum_{i=1}^{N}s_{\hat{\alpha}_i}$$

where $\alpha=(a,b,\lambda,\beta)$. The numerical results of simulation are shown in the plots of Figures 4-7. It is clear from these plots that the estimated biases and MSEs decrease when the sample n increases. The coverage probabilities of all parameters are near to 0.95 and approaches to nominal value when the sample size increases. Further, the average length of all parameters decreases when the sample size increases. The results are obtained for selected parameters but similar results can be obtained for other parameter combinations.

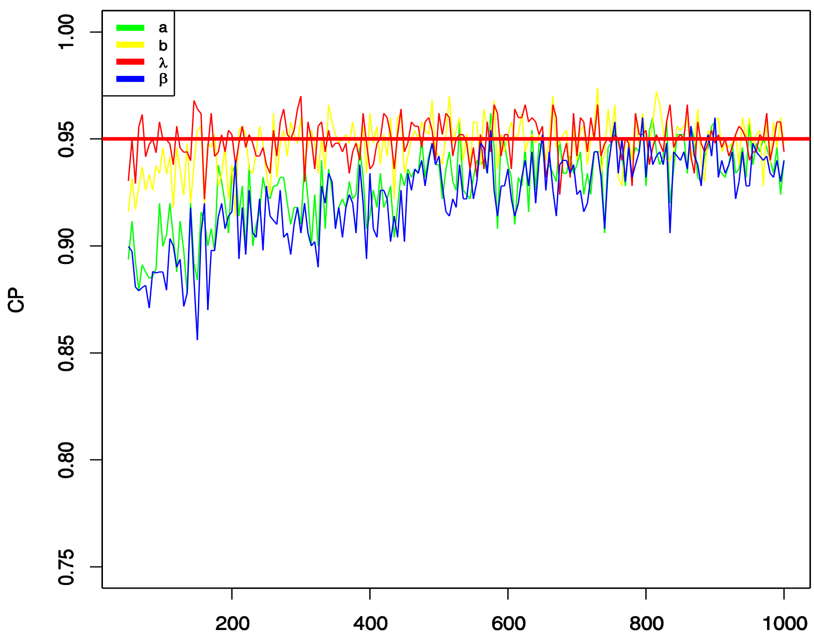

Figure 4. *Estimated CPs of selected parameter vector*

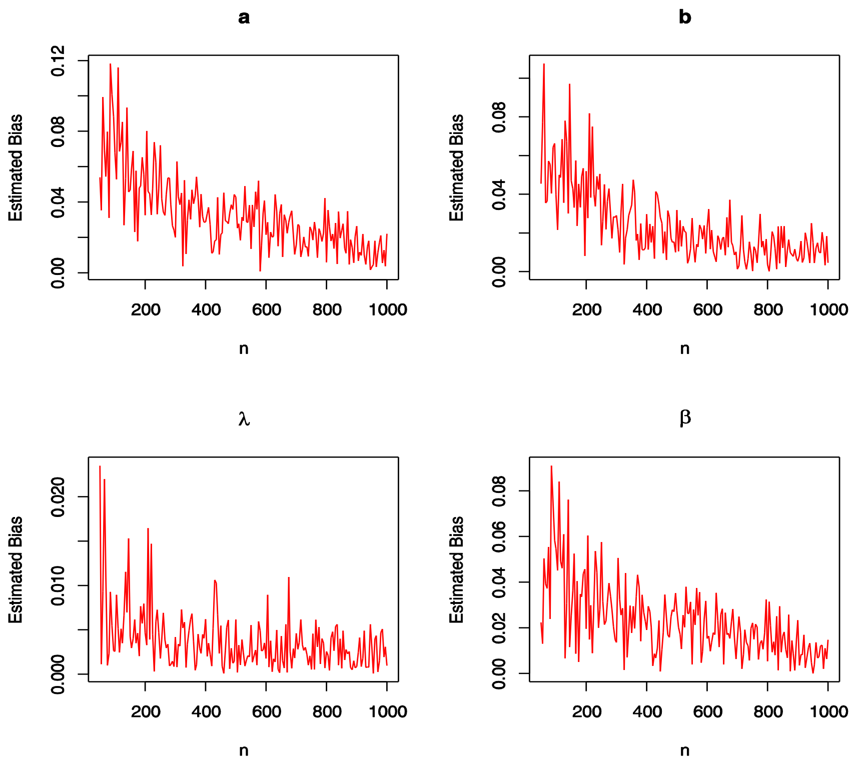

Figure 5. *Estimated bias of selected parameter vector*

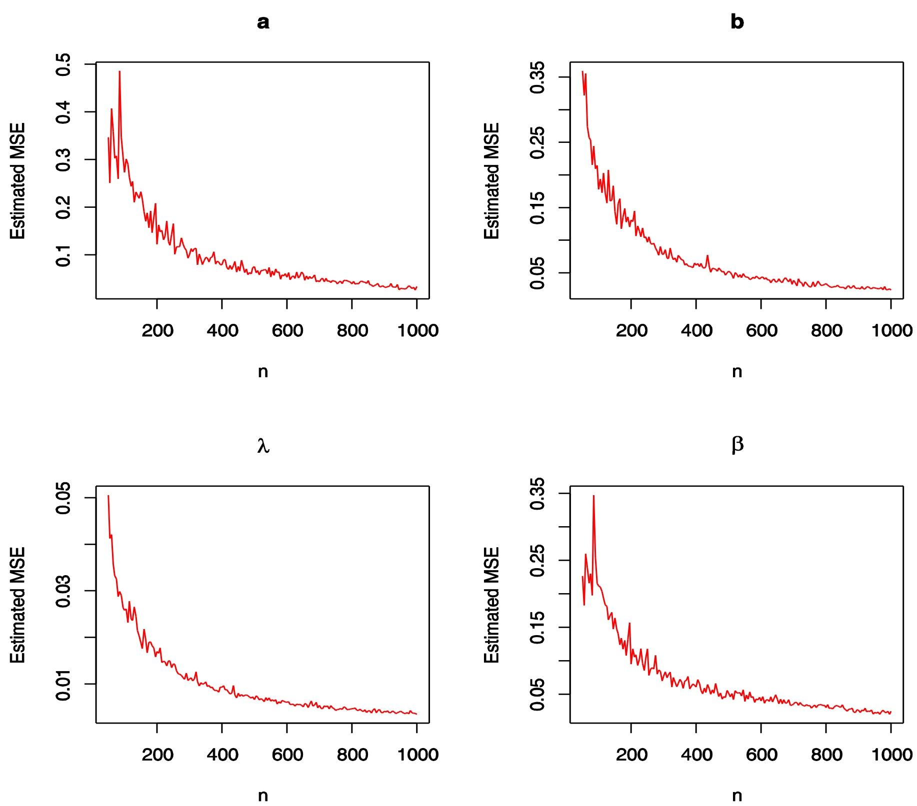

Figure 6. *Estimated MSEs of selected parameter vector*

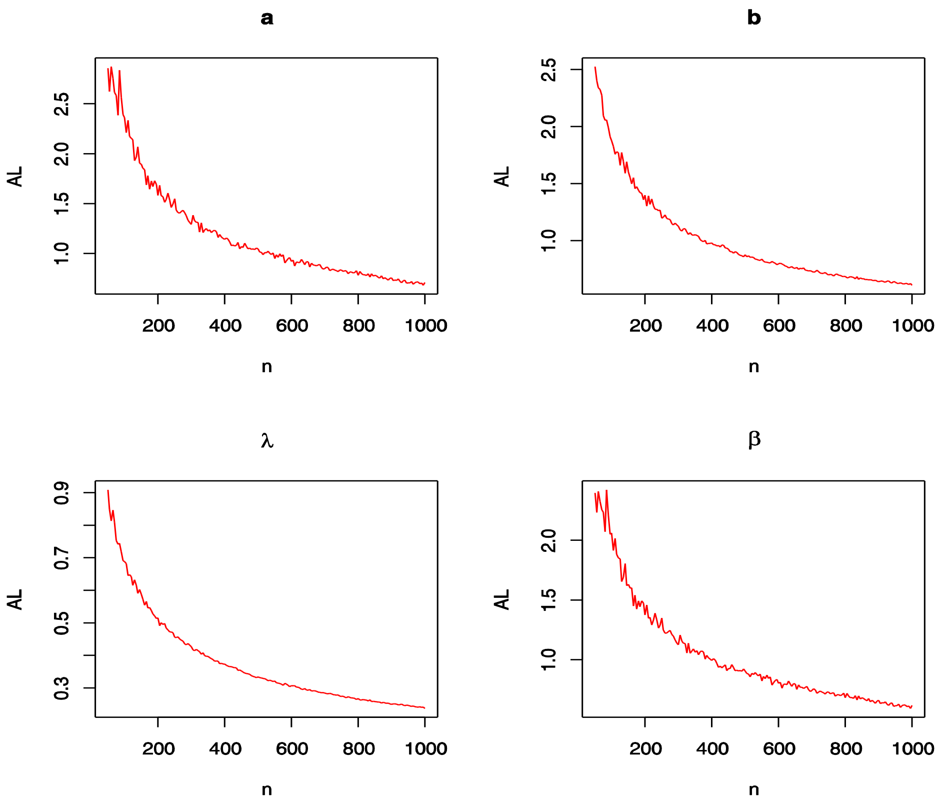

Figure 7. *Estimated ALs of selected parameter vector*

5. APPLICATION

In this section, real data modeling performance of OBu-PL distribution is compared with several well-known distributions given in Table 4. The two real data sets are used to prove modeling ability of OBu-PL distribution. R statistical software is used for computations. Cramer-von Mises (W*) and Anderson-Darling (A*) statistics, log-likelihood values, AIC, CAIC, BIC and HQIC values are obtained for all models and used to decide best model. The smaller value of these statistics indicates that the better fit to data.

Table 4. *Fitted distributions and their abbreviations*

Distribution	Abbreviation	References
Exponentiated Generalized Power Lindley	EG-PL	Cordeiro et al. (2013)
Kumaraswamy Power Lindley	Kum-PL	Oluyede et al. (2016)
Power Lindley	PL	Ghitany et al. (2013)
Odd Log-logistic Power Lindley	OLL-PL	Alizadeh et al. (2017)
Beta Power Lindley	B-PL	Eugene et al. (2002)
Odd Burr Power Lindley	OBu-PL	Proposed

The first data set is from Weisberg (2005) and it represents the sum of skin folds in 202 athletes collected at the Australian Institute of Sports. Table 5 gives Cramer-von Mises (W*) and Anderson-Darling (A*) statistics and log-likelihood values for the all fitted distributions. Based on Table 5, it is clear that OBu-PL distribution provides the overall best fit and therefore could be chosen as the more adequate model from other models for explaining the data set.

Table 5. *Fitting summary of distributions for first data set*

Models	*a*	*b*	λ	β	A	W	-LogL
PL			0.002	1.584	3.067	0.523	968.001
			0.0001	0.015			
OLL-PL	458.219		1.097	0.005	2.234	0.363	962.950
	2.746		0.0006	0.0002			
K-PL	2.571	0.229	0.012	1.486	1.773	0.300	957.906
	0.666	0.085	0.002	0.071			
EG-PL	0.2007	4.646	0.177	1.021	1.903	0.321	958.573
	0.0145	0.6016	0.002	0.002			
B-PL	39.690	0.163	0.133	1.071	0.709	0.103	948.855
	9.283	0.014	0.003	0.006			
OBu-PL	3.226	0.086	0.008	1.498	0.503	0.075	947.962
	0.797	0.032	0.001	0.059			

More information can be provided by a histogram of the data with fitted lines of probability density functions for all distributions. Figure 8 also suggests that the OBu-PL fits skewed data very well. Figure 9 displays plots of the fitted density, cumulative and survival functions with P-P plot for the OBu-PL model. They reveal a good adjustment for the data of the estimated density, cumulative and survival functions of the OBu-PL distribution.

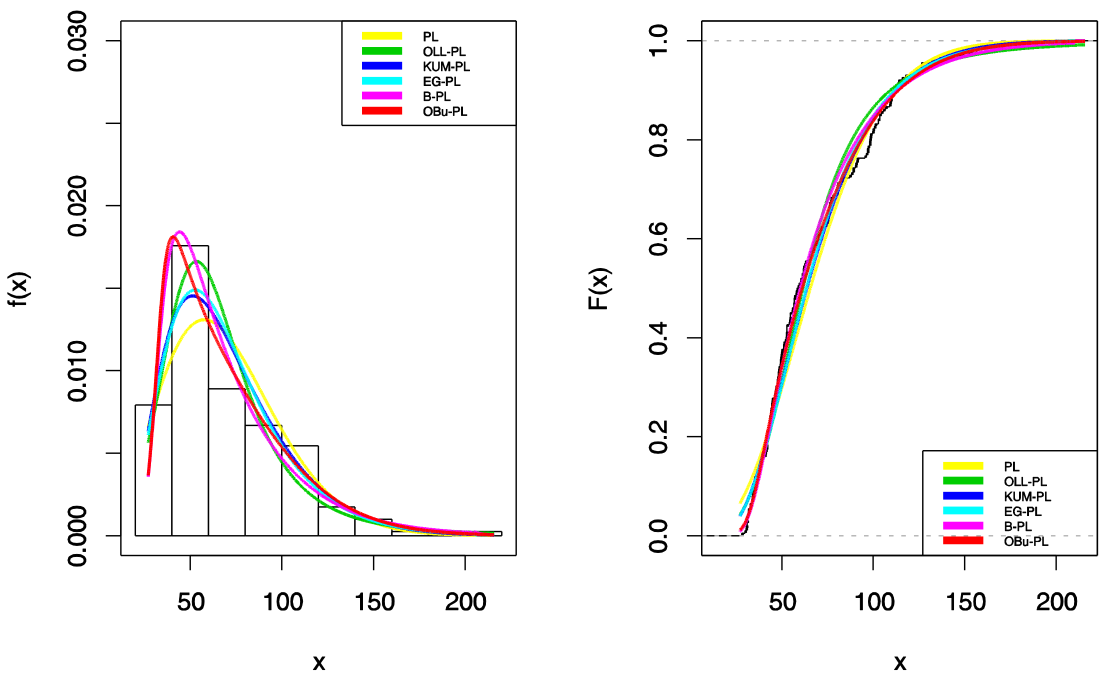

Figure 8. *Fitted densities of distributions for first data set*

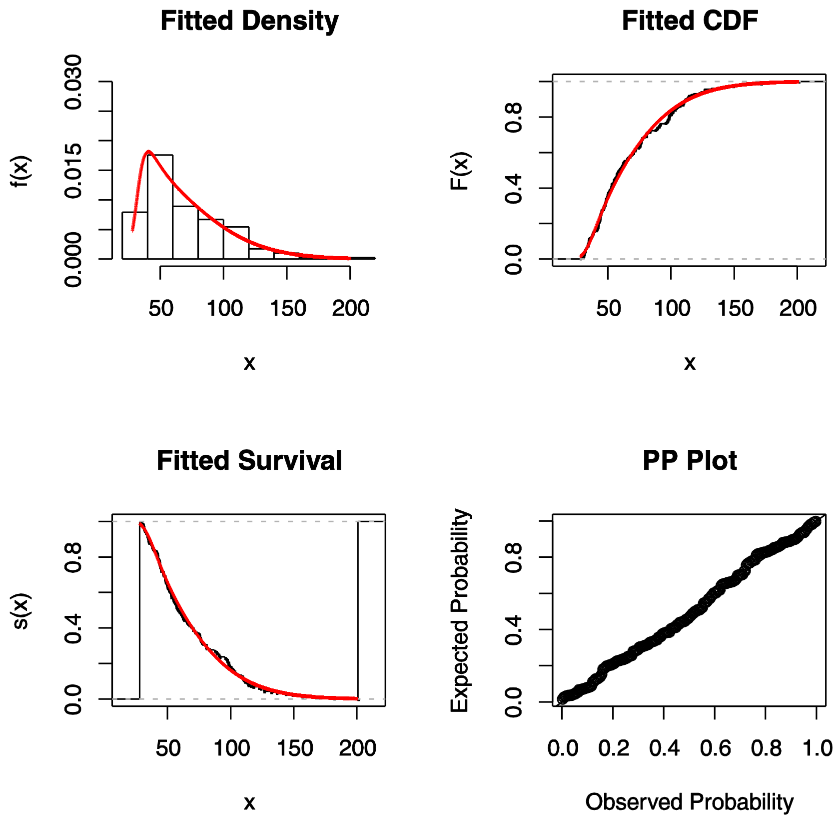

Figure 9. *Plots for fitted functions of the OBu-PL model for first data set.*

The second data set (n = 40) is from Jorgensen (1982) and it represents the active repair times (h) for an airborne communication transceiver. The estimates of the parameters and the numerical values of the statistics are listed in Table 7.

Table 7. *Fitting summary of distributions for second data set*

Models	*a*	*b*	λ	β	A	W	-LogL
PL			0.586	0.798	1.090	0.153	95.942
			0.099	0.082			
OLL-PL	2.420		0.838	0.408	0.701	0.101	92.974
	1.195		0.128	0.181			
KUM-PL	5.224	0.158	3.382	0.761	0.555	0.072	90.924
	0.026	0.025	0.002	0.002			
EG-PL	0.206	12.551	10.950	0.369	0.488	0.068	90.774
	0.029	4.623	0.0175	0.01			
B-PL	19.760	0.149	5.385	0.603	0.346	0.045	89.207
	8.549	0.025	0.004	0.003			
OBu-PL	16.362	0.025	1.555	0.647	0.154	0.019	86.805
	14.930	0.026	0.179	0.148			

As seen from Table 7, OBu-PL model gives the lowest values for the statistics as compared to other models. Figure 10 indicates that the OBu-PL distribution provides a better fit to the data than all other models. Moreover, Table 8 reveals that OBu-PL model has the lowest values for AIC, CAIC, BIC and HQIC among all fitted models. Then OBu-PL model can be chosen as best model for both data sets.

Table 8. *AIC, CAIC, BIC and HQIC values of fitted models for both data sets*

Data set 1					Data set 2				
Models	AIC	CAIC	BIC	HQIC	Models	AIC	AICC	BIC	HQIC
PL	1940.01	1940.06	1946.61	1942.67	PL	195.885	196.209	199.263	197.106
OLL-PL	1931.91	1932.02	1941.82	1935.91	OLL-PL	191.950	192.617	197.017	193.782
KUM-PL	1923.81	1924.01	1937.04	1929.16	KUM-PL	189.848	190.991	196.603	192.291
EG-PL	1925.14	1925.14	1938.38	1930.51	EG-PL	189.549	190.692	196.304	191.991
B-PL	1905.71	1905.91	1918.94	1911.06	B-PL	186.414	187.557	193.169	188.856
OBu-PL	1903.92	1904.12	1917.15	1909.27	OBu-PL	181.611	182.753	188.366	184.053

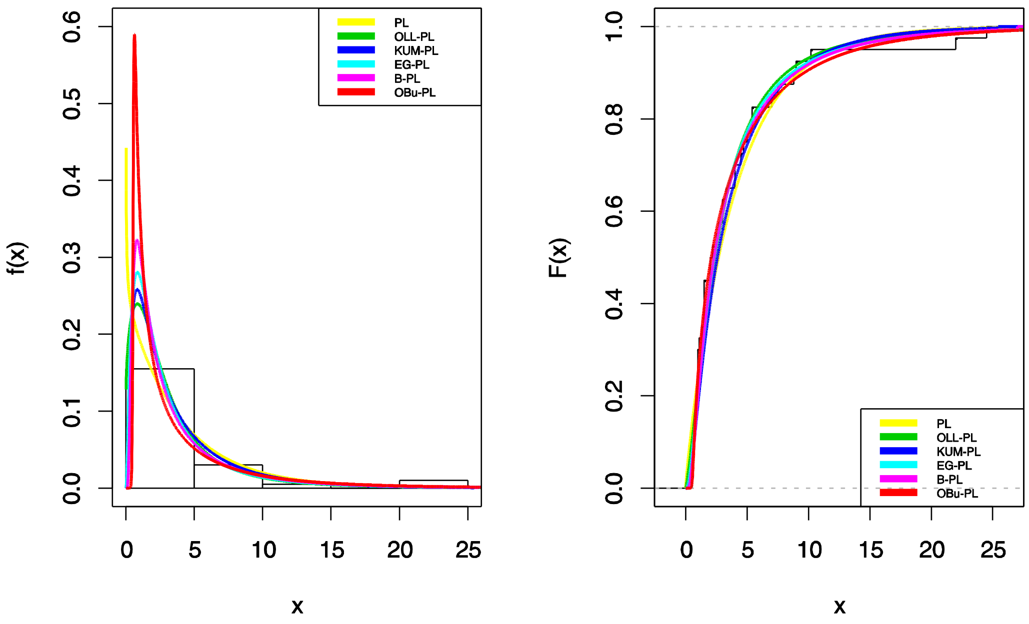

Figure 10. *Fitted densities of distributions for strengths of life of fatigue fracture data set.*

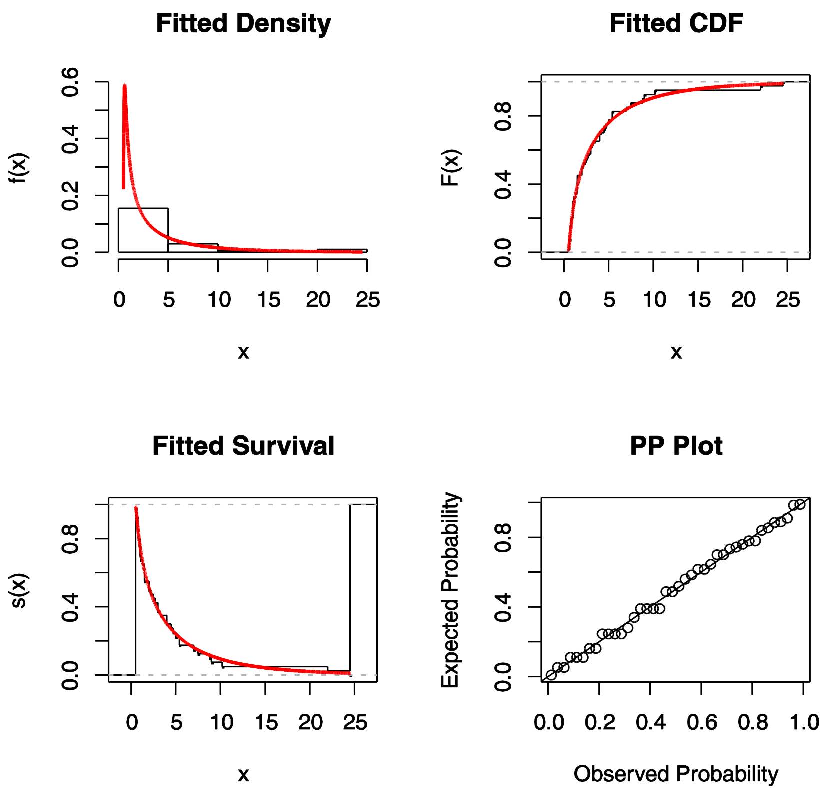

Figure 11. *Plots for fitted functions of the OBu-PL model for strengths of life of fatigue fracture data set.*

The histogram of the third data and the fitted pdf, cdf and survival function of the OBu-PL model are displayed in Figure 11. It is clear from Figure 11 that the OBu-PL model provides better fits than other models.

6. CONCLUSION

In this study, a new four-parameter distribution is introduced. A characteristic of the OBu-PL distribution is that its hrf can be increasing, bathtub-shaped, and unimodal depending on its parameter values. Several properties of the new distribution such as pdf, hrf, and moments are obtained. The MLE procedure is presented. Real data applications and a simulation study indicate the flexibility and capacity of the proposed distribution in data modeling. The new model provides consistently a better fit than the other models, namely: exponentiated generalized power Lindley, Kumaraswamy power Lindley, power Lindley, odd log-logistic power Lindley and beta power Lindley distributions. In view of the density function and hrf shapes, it seems that the proposed model can be considered as a suitable candidate model in reliability analysis, biological systems, data modeling, and related fields.

CONFLICT OF INTEREST

No conflict of interest was declared by the author.

REFERENCES

[1] Altun, G., Alizadeh, M., Altun, E., & Ozel, G. (2017). Odd Burr Lindley distribution with properties and applications. Hacettepe Journal of Mathematics and Statistics, 46(2), 255-276.

[2] Alizadeh, M., Cordeiro, G. M., Nascimento, A.D.C., M.C.S., Lima, Ortega, E.M.M, Odd-Burr generalized family of distributions with some applications, 2016, Journal of Statistical Computation and Simulation, DOI:10.1080/00949655.2016.1209200.

[3] Alizadeh, M., MirMostafaee, S. M. T. K., & Ghosh, I. (2017). A new extension of power Lindley distribution for analyzing bimodal data. CHILEAN JOURNAL OF STATISTICS, 8(1), 67-86.

[4] Andrews, D. F., Herzberg, A. M., 1985, Data: A Collection of Problems from Many Fields for the Student and Research Worker, Springer Series in Statistics, New York.

[5] Cakmakyapan, S., Ozel, G., 2014, A new customer lifetime duration distribution: The Kumaraswamy Lindley distribution, International Journal of Trade, Economics and Finance, 5 (5), 441-444.

[6] Cordeiro, G. M., Alizadeh, M., Tahir, M. H., Mansoor, M., Bourguignon, M., Hamedani, G. G., 2015, The Beta Odd Log-Logistic Generalized Family of Distributions, Hacettepe Journal of Mathematics and Statistics, 45, 73, DOI: 10.15672/HJMS.20157311545.

[7] Cordeiro, G. M., Ortega, E. M., & da Cunha, D. C. (2013). The exponentiated generalized class of distributions. Journal of Data Science, 11(1), 1-27.

[8] Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J., Knuth, D. E., 1996, On the Lambert W Function, Adv. Comput. Math. 5, 329-359.

[9] Eugene, N., Lee, C., & Famoye, F. (2002). Beta-normal distribution and its applications. Communications in Statistics-Theory and methods, 31(4), 497-512.

[10] Gupta, R. C., Gupta, P. L., & Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. Communications in Statistics-Theory and methods, 27(4), 887-904.

[11] Gleaton, J. U., & Lynch, J. D. (2006). Properties of generalized log-logistic families of lifetime distributions. Journal of Probability and Statistical Science, 4(1), 51-64.

[12] Ghitany, M.E., Al-Mutairi, D.K., Balakrishnan, N., Al-Enezi, L.J., 2013, Power Lindley distribution and associated inference, Computational Statistics and Data Analysis, 64, 20-33.

[13] Jorgensen, B., 1982, Statistical properties of the generalized inverse Gaussian distribution. New York: Springer-Verlag.

[14] Jórda, P., 2010, Computer generation of random variables with Lindley or Poisson-Lindley distribution via the Lambert W function, Mathematics and Computers in Simulation, 81, 851-859.

[15] Leadbetter M. R., Lindgren G., Rootzn H., 1983, Extremes and Related Properties of Random

[16] Lindley, D. V., 1958, Fiducial distributions and Bayes theorem, Journal of the Royal Statistical Society, 20 (1), 102–107.

[17] Nadarajah, S., Bakouch, H. S., & Tahmasbi, R., 2011. A generalized Lindley distribution. Sankhya B, 73, 2, 331-359.

[18] Oluyede, B. O., Yang, T., Makubate, B., 2016, A New Class of Generalized Power Lindley Distribution with Application to Lifetime Data, Asian Journal of Mathematics and Applications, 1-34.

[19] Sequences and Processes, Springer Statist. Ser., Springer, Berlin.

[20] Shanker, R., Mishra, A. 2013, A quasi Lindley distribution. African Journal of Mathematics and Computer Science Research, 6, 4, 64-71.

[21] Sharma, V.K., Singh, S.K., Singh, U., Agiwal, V., 2015, The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data, 32, 3, 162-173.

[22] Smith, R. L. and Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Applied Statistics, 36, 3, 358–369.

[23] Weisberg, S. (2005). Applied linear regression (Vol. 528). John Wiley & Sons.

Appendix A

First, by expanding z^{λ} in Taylor series, we have

$$z^{\lambda}=\sum_{k=0}^{\infty}(\lambda)_k\,(z-1)^k\big/k!=\sum_{i=0}^{\infty}f_i z^i \tag{1.A}$$

$$f_i = f_i(\lambda)=\sum_{k=i}^{\infty}\frac{(-1)^{k-i}}{k!}\binom{k}{i}(\lambda)_k$$

and $(\lambda)_k=\lambda(\lambda-1)...(\lambda-k+1)$ is the descending factorial.

Second, we use throughout an equation of Gradshteyn and Ryzhik (2007) for a power series raised to a positive integer i given by

$$\left(\sum_{j=0}^{\infty}a_j\upsilon^j\right)^i=\sum_{j=0}^{\infty}c_{i,j}\upsilon^j \tag{2.A}$$

where the coefficients $c_{i,j}$ (for j=1,2,...) are obtained from the recurrence equation (for $j\geq 1$)

$$c_{i,j}=(ja_0)^{-1}\sum_{m=1}^{j}\left[m(j+1)-j\right]a_m c_{i,j-m} \tag{3.A}$$

and $c_{i,0}=\alpha_0^i$. Hence, the coefficients $c_{i,j}$ can be calculated directly from $c_{i,0},...,c_{i,j-1}$ and, therefore, from $a_0,...,a_j$. They can be given explicitly in terms of the a_j , although it is not necessary for programming numerically our expansions in any algebraic or numerical software.

We now obtain an expansion for $\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^c$. We can write,

$$\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]=\sum_{j=0}^{\infty}t_j G(x)^j\,,$$

where $t_j=t_j(\alpha)\sum_{j=r}^{\infty}(-1)^{r+j}\binom{\alpha}{j}\binom{j}{r}+(-1)^j\binom{\alpha}{j}$. Then using (1.A) we can write

$$\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^c=\sum_{i=0}^{\infty}f_i\left(\sum_{j=0}^{\infty}t_j G(x)^j\right)^i \tag{4.A}$$

where $f(i) = f_i(c)$. Finally, using equations (23) and (24), we obtain

$$\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{c} = \sum_{j=0}^{\infty} h_j G(x)^{j} \tag{5.A}$$

where $h_j = h_j(\alpha, c) = \sum_{j=0}^{\infty} h_j G(x)^{j}$ and $m_j = (jt_0)^{-1} \sum_{m=1}^{j} \left[m(j+1) - j\right] t_m m_{i,j-m}$ (for $j \geq 1$) and $m_{i,0} = t_0^{i}$.