GU J Sci 30(4): 432-440 (2017)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

A Kantorovich Type Generalization of the Szàsz Operators via Two Variable Hermite Polynomials

Serdal Yazıcı[1], Bayram Çekim[2,*]

[1]*Gazi University , Faculty of Science, Department of Mathematics, 06100, Beşevler, Ankara, Turkey.*

[2]*Gazi University, Faculty of Science, Department of Mathematics, 06100, Beşevler, Ankara, Turkey.*

Article Info

Received:15/05/2017
Accepted:18/09/2017

Keywords

Hermite polynomial, Kantorovich type generalization, Modulus of continuity, Voronovskaya type asymptotic formula.

Abstract

The purpose of this paper is to give the Kantorovich generalization of the operators via two variable Hermite polynomials which are introduced by Krech [1] and to research approximating features with help of the classical modulus of continuity, the class of Lipschitz functions, Voronovskaya type asymptotic formula, second modulus of continuity and Peetre's K-functional for these operators.

1. INTRODUCTION

Heretofore, many authors has studied on linear positive operators and properties of their approximation, see for example [2, 6-11, 17, 18]. In addition to fact that authors working on the approximation theory with help of linear positive operators have been given linear positive operators via some orthogonal polynomials, see for example [1, 3, 4, 5, 12]. Therefore, we are going to define the Kantorovich type of the operators being made up of one of orthogonal polynomials.

Firstly, we recall H_k which is two variable Hermite polynomial (see [13]) defined by

$$H_k(n,\alpha) = k!\sum_{s=0}^{\left[\frac{k}{2}\right]} \frac{n^{k-2s}\alpha^s}{(k-2s)!s!}. \tag{1.1}$$

Furthermore, the generating function of two variable Hermite polynomials is the as follows (see [13])

$$\sum_{k=0}^{\infty} H_k(n,\alpha)\frac{t^k}{k!} = e^{nt+\alpha t^2}. \tag{1.2}$$

Secondly, Krech has presented the Szász operators including two variable Hermite polynomials

(see [1]) as

*Corresponding author, e-mail:bayramcekim@gazi.edu.tr

$$G_n^{\alpha}(f;x) := e^{-(nx+\alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n,\alpha) f\left(\frac{k}{n}\right), \tag{1.3}$$

where $n=1,2,3,...$, $\alpha \ge 0$ and $x \in [0,\infty)$.

Now, we introduce a Kantorovich type generalization of G_n^{α}.

2. KANTOROVICH TYPE GENERALIZATION OF OPERATORS G_n^{α}

In this section, the Kantorovich type generalization of G_n^{α} has been defined by

$$S_n^{\alpha}(f;x) := n e^{-(nx+\alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n,\alpha) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \tag{2.1}$$

where $n=1,2,3,...$, $\alpha \ge 0$, $x \in [0,\infty)$ and $f \in C[0,\infty)$ for which the corresponding series is convergent, in here $C[0,\infty)$ is the space of continuous functions on $[0,\infty)$.

Lemma 1. The operators given by (2.1) yield the following equalities.

i. $S_n^{\alpha}(1;x) = 1$,

ii. $S_n^{\alpha}(t;x) = x + \frac{4\alpha x^2 + 1}{2n}$,

iii. $S_n^{\alpha}(t^2;x) = x^2 + \frac{4\alpha x^3 + 2x}{n} + \frac{12\alpha^2 x^4 + 18\alpha x^2 + 1}{3n^2}$,

iv. $S_n^{\alpha}(t^3;x) = x^3 + \frac{12\alpha x^4 + 9x^2}{2n} + \frac{24\alpha^2 x^5 + 48\alpha x^3 + 7x}{2n^2} + \frac{120\alpha^2 x^4 + 32\alpha^3 x^6 + 64\alpha x^2 + 1}{4n^3}$,

v. $S_n^{\alpha}(t^4;x) = x^4 + \frac{8\alpha x^5 + 8x^3}{n} + \frac{24\alpha^2 x^6 + 60\alpha x^4 + 15x^2}{n^2} + \frac{32\alpha^3 x^7 + 144\alpha^2 x^5 + 108\alpha x^3 + 6x}{n^3}$

$$+ \frac{840\alpha^2 x^4 + 560\alpha^3 x^6 + 80\alpha^4 x^8 + 210\alpha x^2 + 1}{5n^4}.$$

Observe that the operators are well-defined for all test function $e_i(\mathrm{t}) = \mathrm{t}^i$ for $i = 0,1,2,3,4.$

Lemma 2. The operators given by (2.1) yield the following equalities.

i. $\psi_1 = S_n^{\alpha}((t-x)^1;x) = \frac{4\alpha x^2 + 1}{2n}$,

ii. $\psi_2 = S_n^{\alpha}((t-x)^2;x) = \frac{x}{n} + \frac{12\alpha^2 x^4 + 18\alpha x^2 + 1}{3n^2}$,

iii. $\psi_3 = S_n^{\alpha}((t-x)^3;x) = \dfrac{12\alpha x^3 + 5x}{2n^2} + \dfrac{120\alpha^2 x^4 + 32\alpha^3 x^6 + 64\alpha x^2 + 1}{4n^3}$,

iv. $\psi_4 = S_n^{\alpha}((t-x)^4;x) = \dfrac{3x^2}{n^2} + \dfrac{24\alpha^2 x^5 + 44\alpha x^3 + 5x}{n^3} + \dfrac{840\alpha^2 x^4 + 560\alpha^3 x^6 + 80\alpha^4 x^8 + 210\alpha x^2 + 1}{5n^4}$.

Now, we can give Theorem 1 for approximation properties of the operators S_n^{α} using the well known Korovkin theorem with the help of Lemma 1.

Theorem 1. Let the operator defined by S_n^{α} in (2.1) and $f \in C_B[0,\infty)$. So, $S_n^{\alpha}(f;x)$ is uniformly convergent to $f(x)$ on $[0,b]$ where $C_B[0,\infty)$ is the space of uniformly continuous and bounded functions on $[0,\infty)$.

Proof. Follow the standart procedure in [16], see also [14,15].

3. APPROXIMATION PROPERTIES OF OPERATORS S_n^{α}

In this section, we present the rate of convergence of the operators with the help of the usual and second modulus of continuity, Lipschitz class functions, Peetre's K - functional and Voronovskaya type formula. Firstly, we remind some definitions as follows.

Let $Lip_M(\beta)$ be Lipschitz class of order β. If $f \in Lip_M(\beta)$, the inequality

$$|f(t) - f(u)| \leq M|t-u|^{\beta} \tag{3.1}$$

holds, where $t,u \in [0,\infty)$, $0 < \beta \leq 1$ and $M > 0$. The classical modulus of continuity of $f \in C_B[0,\infty)$ is denoted by

$$\omega(f;\delta) = \sup_{|h|\leq\delta}\{|f(x+h) - f(x)| : x \in [0,\infty)\}, \tag{3.2}$$

where $\delta > 0$.

Furthermore, a vector space, $C_B{}^2[0,\infty) = \{f \in C_B[0,\infty) : f', f'' \in f \in C_B[0,\infty)\}$, is normed space with following norm that

$$\|f\|_{C_B{}^2[0,\infty)} = \|f\|_{C_B[0,\infty)} + \|f'\|_{C_B[0,\infty)} + \|f''\|_{C_B[0,\infty)} \tag{3.3}$$

for every $f \in C_B{}^2[0,\infty)$. We can remind Peetre's K-functional of the function $f \in C_B[0,\infty)$ that is as follows

$$K(f;\delta) = \inf_{g \in C_B{}^2[0,\infty)}\left\{\|f-g\|_{C_B[0,\infty)} + \delta\|g\|_{C_B{}^2[0,\infty)}\right\} \tag{3.4}$$

for $\delta>0$. We define the second-order modulus of smoothness of function $f\in C_B[0,\infty)$ by

$$\omega_2(f;\delta)=\sup_{0<h\le\delta}\left\{\left|f(x+2h)-2f(x+h)+f(x)\right|:x\in[0,\infty)\right\} \tag{3.5}$$

for $\delta>0$. Moreover, we have the inequality that is relation between Peetre's K-functional and ω_2 as following that

$$K(f;\delta)\le M\left\{\omega_2(f;\sqrt{\delta})+\min(1,\delta)\|f\|_{C_B[0,\infty)}\right\} \tag{3.6}$$

for all $\delta>0$ and M is positive constant.

Theorem 2. The operators S_n^{α} defined in (2.1) verify the following inequality

$$\left|S_n^{\alpha}(f;x)-f(x)\right|\le M\omega\left(f;\delta_n\right), \tag{3.7}$$

where $f\in C_B[0,\infty)$, $x\in[0,b]$, M is a constant and $\delta_n=\frac{1}{\sqrt{n}}$.

Proof. We know that modulus of continuity of function $f\in C_B[0,\infty)$ verifies the following inequality

$$|f(t)-f(x)|\le\omega(f;\delta)\left(\frac{|t-x|}{\delta}+1\right). \tag{3.8}$$

Using (3.8), Cauchy-Schwarz inequality and Lemma 2, we have

$$\begin{aligned}\left|S_n^{\alpha}(f;x)-f(x)\right|&\le S_n^{\alpha}(|f(t)-f(x)|;x)\\&\le\omega(f;\delta)\left(1+\frac{1}{\delta}S_n^{\alpha}(|t-x|;x)\right)\\&\le\omega(f;\delta)\left(1+\frac{1}{\delta}\sqrt{S_n^{\alpha}((t-x)^2;x)}\right)\\&\le\omega(f;\delta)\left(1+\frac{1}{\delta}\sqrt{\psi_2}\right)\\&\le M\omega\left(f;\delta_n\right),\end{aligned}$$

where $M=1+\sqrt{b+12\alpha^2b^4+18\alpha b^2+1}$ and $\delta_n=\frac{1}{\sqrt{n}}$.

Theorem 3. If $f\in Lip_M(\beta)$, then we have

$$\left|S_n^{\alpha}(f;x)-f(x)\right|\le M^{*}(\delta_n)^{\frac{\beta}{2}}, \tag{3.9}$$

where $x\in[0,b]$, M^{*} is constant and $\delta_n=\frac{1}{n}$.

Proof. From $f\in Lip_M(\beta)$ and linearity property of S_n^{α}, we obtain

$$\left|S_n^{\alpha}(f;x)-f(x)\right|\le S_n^{\alpha}(|f(t)-f(x)|;x)$$

$$\le MS_n^{\alpha}(|t-x|^{\beta};x).$$

On the basis of Lemma 2 and Hölder's inequality firstly for integral and then for sum via

p=$\frac{\beta}{2}$, $q=\frac{2-\beta}{2}$, we obtain

$$\left|S_n^{\alpha}(f;x)-f(x)\right|\le MS_n^{\alpha}(|t-x|^{\beta};x)$$

$$\le M(S_n^{\alpha}((t-x)^2;x))^{\frac{\beta}{2}}$$

$$\le M(\psi_2)^{\frac{\beta}{2}}$$

$$\le M^{*}(\delta_n)^{\frac{\beta}{2}},$$

where $M^{*}=M\left(b+12\alpha^2b^4+18\alpha b^2+1\right)^{\frac{\beta}{2}}$ and $\delta_n=\frac{1}{n}$.

Theorem 4. Let K be Peetre's K-functional. The operators S_n^{α} defined in (2.1) verify the following inequality

$$\left|S_n^{\alpha}(f;x)-f(x)\right|\le 2K(f,\delta_n), \tag{3.10}$$

where $f\in C_B[0,\infty), x\in[0,b]$ and $\delta_n=\frac{b}{n}+\frac{4\alpha b^2+1}{2n}+\frac{12\alpha^2b^4+18\alpha b^2+1}{3n^2}$.

Proof . From the Taylor's series expansion of the function $g\in C_B{}^2[0,\infty)$, we have

$$g(t)=g(x)+g'(x)(t-x)+g''(c)\frac{(t-x)^2}{2},\ c\in(x,t).$$

When we apply the operators S_n^{α} to both sides of the aforementioned equality and recall the linearity property of the operators S_n^{α}, we obtain

$$S_n^{\alpha}(g;x)-g(x)=g'(x)S_n^{\alpha}((t-x);x)+\frac{g''(c)}{2}S_n^{\alpha}\left((t-x)^2;x\right).$$

By Lemma 2, we have

$$\left|S_n^{\alpha}(g;x)-g(x)\right|\leq g'(x)\psi_1+\frac{g''(c)}{2}\psi_2$$

$$\leq g'(x)\frac{4\alpha x^2+1}{2n}+\frac{g''(c)}{2}\left(\frac{x}{n}+\frac{12\alpha^2x^4+18\alpha x^2+1}{3n^2}\right)$$

$$\leq \|g'\|_{C_B[0,\infty)}\frac{4\alpha x^2+1}{2n}+\frac{\|g''\|_{C_B[0,\infty)}}{2}\left(\frac{x}{n}+\frac{12\alpha^2x^4+18\alpha x^2+1}{3n^2}\right)$$

$$\leq\left(\frac{x}{n}+\frac{4\alpha x^2+1}{2n}+\frac{12\alpha^2x^4+18\alpha x^2+1}{3n^2}\right)\left(\|g'\|_{C_B[0,\infty)}+\frac{\|g''\|_{C_B[0,\infty)}}{2}\right)$$

$$\leq\left(\frac{x}{n}+\frac{4\alpha x^2+1}{2n}+\frac{12\alpha^2x^4+18\alpha x^2+1}{3n^2}\right)\|g\|_{C_B^2[0,\infty)}$$

$$\leq\left(\frac{b}{n}+\frac{4\alpha b^2+1}{2n}+\frac{12\alpha^2b^4+18\alpha b^2+1}{3n^2}\right)\|g\|_{C_B^2[0,\infty)}.$$

Now, let $f\in C_B[0,\infty)$. We use the above inequality as follow

$$\left|S_n^{\alpha}(f;x)-f(x)\right|=\left|S_n^{\alpha}(f;x)-S_n^{\alpha}(g;x)+S_n^{\alpha}(g;x)-g(x)+g(x)-f(x)\right|$$

$$\leq S_n^{\alpha}(|f-g|;x)+|f(x)-g(x)|+\left|S_n^{\alpha}(g;x)-g(x)\right|$$

$$\leq 2\|f-g\|_{C_B[0,\infty)}+2\|g\|_{C_B^2[0,\infty)}\left(\frac{b}{n}+\frac{4\alpha b^2+1}{2n}+\frac{12\alpha^2b^4+18\alpha b^2+1}{3n^2}\right).$$

By applying infimum both sides of this inequality for $g\in C_B{}^2[0,\infty)$, we have

$$\left|S_n^{\alpha}(f;x)-f(x)\right|\leq 2K(f,\delta_n),$$

where $\delta_n=\frac{b}{n}+\frac{4\alpha b^2+1}{2n}+\frac{12\alpha^2b^4+18\alpha b^2+1}{3n^2}$.

Theorem 5. For the operators (2.1), the following inequality holds

$$\left|S_n^{\alpha}(f;x)-f(x)\right|\le 2M\left\{\omega_2\left(f,\sqrt{\lambda_n}\right)+\min\left(1,\lambda_n\right)\|f\|\right\}, \tag{3.11}$$

where $f\in C_B[0,\infty), x\in[0,b]$, M is a positive constant that is independent of n and $\lambda_n=\frac{b}{n}+\frac{4\alpha b^2+1}{2n}+\frac{12\alpha^2b^4+18\alpha b^2+1}{3n^2}$.

Proof. By using Theorem 4, we obtain $\left|S_n^{\alpha}(f;x)-f(x)\right|\le 2K(f,\lambda_n)$. The proof is completed by choosing $\delta=\lambda_n$ in (3.6).

Theorem 6. Let $f\in C_B{}^2[0,\infty)$ and $x\in[0,\infty)$ is a fixed point. Then, we have

$$\lim_{n\to\infty} n\left[S_n^{\alpha}(f;x)-f(x)\right]=\frac{1}{2}\left[(4\alpha x^2+1)f'(x)+xf''(x)\right]. \tag{3.12}$$

Proof. By Taylor formula for the function f, we get

$$f(t)=f(x)+f'(x)(t-x)+f''(x)\frac{(t-x)^2}{2}+(t-x)^2\mu(t,x),$$

where $\mu(.,x)\in C_B[0,\infty)$ and $\lim\limits_{t\to x}\mu(t,x)=0$.

When we apply the operators S_n^{α} to both sides of the aforementioned equality and recall the linearity property of the operators S_n^{α}, we obtain

$$S_n^{\alpha}(f;x)-f(x)=f'(x)S_n^{\alpha}((t-x);x)+\frac{f''(x)}{2}S_n^{\alpha}((t-x)^2;x)+S_n^{\alpha}\left((t-x)^2\mu(t,x);x\right).$$

By Lemma 2, we get

$$S_n^{\alpha}(f;x)-f(x)=f'(x)\psi_1+\frac{f''(x)}{2}\psi_2+S_n^{\alpha}\left((t-x)^2\mu(t,x);x\right). \tag{3.13}$$

By Cauchy-Schwarz inequality, we have

$$nS_n^{\alpha}\left((t-x)^2\mu(t,x);x\right)\le\left(n^2S_n^{\alpha}\left((t-x)^4;x\right)\right)^{\frac{1}{2}}\left(S_n^{\alpha}\left(\mu^2(t,x);x\right)\right)^{\frac{1}{2}}.$$

It is clear that $\mu^2(x,x)=0$ and $\mu^2(t,x)$ is bounded. Then, we get

$$\lim_{n\to\infty}S_n^{\alpha}\left(\mu^2(t,x);x\right)=\mu^2(x,x)=0.$$

So, we obtain

$$\lim_{n\to\infty} nS_n^{\alpha}\left((t-x)^2\mu(t,x);x\right)=0. \tag{3.14}$$

Now, we can write the following equality from (3.13) and (3.14)

$$\lim_{n\to\infty} n\left[S_n^{\alpha}(f;x)-f(x)\right]=\frac{1}{2}\left[(4\alpha x^2+1)f'(x)+xf''(x)\right].$$

The proof is done.

ACKNOWLEDGMENT

The authors are grateful to the referees for their valuable comments and suggestions which improved the quality and the clarity of the paper.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Krech, G., “A note on some positive linear operators associated with the Hermite polynomials’’, Carpathian J. Math., 32 (1): 71-77, (2016).

[2] Szász, O., “Generalization of S. Bernstein’s polynomials to the infinite interval’’, J. Res. Nat. Bur. Stand., 45: 239-245, (1950).

[3] Sucu, S., İçöz, G.,Varma, S., “On some extensions of Szász opertors including Boas-Buck type polynomials’’, Abstr. Appl. Anal.,Vol.2012, Article ID 680340: 15 pages, (2012).

[4] Varma, S., Sucu, S., İçöz, G., “Generalization of Szász operators involving Brenke type polynomials’’, Comput. Math. Appl., 64(2): 121-127, (2012).

[5] Varma, S., Taşdelen, F., “Szász type operators involving Charlier polynomials’’, Mathematical and Computer Modeling, 56: 118-122, (2012).

[6] Atakut, Ç., Büyükyazici, İ., “Stancu type generalization of the Favard Szász operators’’, Appl. Math. Lett., 23(12): 1479-1482, (2010).

[7] Ciupa, A., “A class of integral Favard-Szász type operators’’, Stud. Univ. Babes-Bolyai Math., 40(1): 39-47, (1995).

[8] Gadzhiev, A. D.,“ The convergence problem for sequence of positive linear operators on unbounded sets and theorem analogues to that of P.P.Korovkin’’, Sov. Math. Dokl., 15(5): 1436-1453, (1974).

[9] Stancu, D. D. “Approximation of function by a new class of polynomial operators’’, Rev. Rourn. Math. Pures et Appl., 13(8): 1173-1194, (1968).

[10] Gupta, V., Vasishtha, V., Gupta, M. K., “Rate of convergence of the Szász-Kantorovich-Bezier operators for bounded variation function’’, Publ. Ins. Math.(Beograd)(N.S.), 72: 137-143, (2006).

[11] Atakut, Ç., İspir, N., “Approximation by modified Szász-Mirakjan operators on weighted spaces’’, Proc. Indian Acad. Sci. Math., 112: 571-578, (2012).

[12] Taşdelen, F., Aktaş, R., Altın, A., “A Kantorovich type of Szász operators including Brenke type polynomials’’, Abstract and Applied Analysis, Vol.2012: 13 pages, (2012).

[13] Appell, P., Kampe de Feriet, J., Hypergeometriques et Hyperspheriques: Polynomes d’Hermite, Gauthier-Villars, Paris, 1926.

[14] DeVore, R. A. and Lorentz, G.G., Constructive Approximation, Springer-Verlag, Berlin, 1993.

[15] Ditzian, Z. and Totik, V., Moduli of smoothness, Springer-Verlag, New York, 1987.

[16] Korovkin, P. P., “On convergence of linear positive operators in the space of continuous functions” (Russian), Doklady Akad. Nauk. SSSR (NS) 90: 961–964, (1953).

[17] Özarslan, M. A. and Duman, O., “Approximation properties of Poisson integrals for orthogonal expansions”, Taiwanese J. Math. 12: 1147-1163, (2008).

[18] Toczek, G. and Wachnicki, E., “On the rate of convergence and the Voronovskaya theorem for the Poisson integrals for Hermite and Laguerre expansions”, J. Approx. Theory, 116: 113-125, (2002).