GU J Sci 31(2): 660-675 (2018)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Exponentiated Transmuted Power Function Distribution: Theory & Applications

Rana Muhammad USMAN[1], Muhammad Ahsan ul HAQ[1,2*], Nurbanu BURSA[3], Gamze ÖZEL[3]

[1]*College of Statistical and Actuarial Sciences, University of the Punjab, Lahore, Pakistan.*

[2]*Quality Enhancement Cell (QEC), National College of Arts, Lahore, Pakistan.*

[3]*Department of Statistics, Faculty of Science, Hacettepe University, Turkey.*

Article Info

Received: 07/05/2017
Accepted: 29/03/2018

Keywords

Power function dist
Rényi entropy
MLE

Abstract

This paper introduces a new generalization of Transmuted Power Function distribution named as Exponentiated Transmuted Power Function distribution with its fundamental properties. The expressions of failure and survival rate functions on the basis of their graphs are provided. We compute moments, moment generating function, quantile function. Then, Rényi entropy is discussed and the expressions of the order statistics are derived. Parameters of the proposed distribution are estimated using the maximum likelihood method. Real lifetime data application shows the flexibility of the proposed distribution and its better fit as compared to some existing models with the confidence that the model provide better performance to deal with the problems related to electronics and engineering reliability.

1. INTRODUCTION

Statistical models are commonly used to predict the real-life data. Although literature presents many univariate models with appropriate real life examples, some models are more flexible to explain the real life phenomenon with respect to failure rate and reliability analysis. From last few decades, Power function (PF) distribution is commonly used to explain the limited and scarce data sets [20]. The PF distribution is a special case of beta distribution and also considered as an inverse Pareto distribution [10]. This distribution is preferred for the better fit as compared to lognormal, Weibull, exponential and Gaussian distribution due to its simplicity, applicability and therefore attractiveness to reliability engineering. It is frequently used in social sciences, physics, economics, engineering and many other fields [16].

The fundamental statistical properties of the PF distribution are studied [1-2, 4, 9, 13-14]. Characterization and estimation of parameters of the PF distribution by [1-2]. Reliability analysis for the PF distribution [3]. Further, authors estimate the parameters using different methods, Bayesian analysis of the PF distribution under double prior [20]. Robust estimators for the PF distribution by [19]. Then, some generalized forms of the PF distribution have been formed to obtain more flexible and appropriate model for the real life data. For example, beta PF distribution [7], Weibull PF distribution [21], Kumaraswamy PF distribution [18], exponentiated Kumaraswamy PF distribution [6], McDonald Power function [11] and transmuted Weibull power function distribution [12].

The probability density function (pdf) and cumulative distribution function (cdf) of the transmuted PF distribution [10] are, respectively, given as follows:

$$f\left(y;\alpha,\beta,\gamma\right)=\frac{\alpha y^{\alpha-1}}{\beta^{\alpha}}\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right],\quad 0<y<\beta,\alpha>0,\gamma>0, \tag{1}$$

*Corresponding author, e-mail: ahsanshani36@gmail.com

$$F(y)=\left(\frac{y}{\beta}\right)^{\alpha}\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right], \tag{2}$$

where α and γ are shape parameters and β is scale parameter.

The exponentiated family of distribution is derived by using the cdf of an arbitrary parent distribution by a shape parameter say θ>0. The pdf of the generator is given by

$$g(y;\xi,\theta)=\theta f(y;\xi)\{F(y;\xi)\}^{\theta-1} \tag{3}$$

and its correspondingly cdf is

$$G(y;\xi,\theta)=\{F(y;\xi)\}^{\theta}, \tag{4}$$

where ξ represents the parameters of baseline distribution and $\theta>0$ represents another shape parameter.

The motivation of this study is to obtain more flexible model (A model which can deal with various kind of data sets) and to get goodness of fit to deal with the real-life data related to the electronics and engineering reliability. Furthermore, the basic motivations for using the new distribution in practice are the following: 1. to make the kurtosis more flexible compared to the baseline model; 2. to produce a skewness for symmetrical distributions; 3. to construct heavy-tailed distributions that are not longer-tailed for modeling real data; 4. to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped; 5. to define special models with all types of the hazard functions.

The purposes of our study are to extend the transmuted PF distribution to its standard exponentiated transmuted PF distribution, explain its properties and show flexibility on the basis of real life examples. The rest of the study is organized as follows: Section 2 introduces exponentiated transmuted PF distribution with its shapes and reliability analysis. Section 3 elaborates some fundamental properties including moments, incomplete moments, generating function, quantile function, random number generation, mean deviation, mode, entropies, order statistics, and maximum likelihood estimation. Section 4 gives application for justifying the flexibility of model as compared to other existing models and Section 5 concludes the study.

2. EXPONENTIATED TRANSMUTED POWER FUNCTION DISTRIBUTION

If $F(y;\alpha,\beta,\gamma)$ is the cdf of transmuted PF distribution with parameter α,γ are shape parameters and β is scale parameter, then Eq. (2) yields a new model ET-PF cumulative distribution $(for \beta\geq y\geq 0)$, say $G(y)=G(y;\alpha,\beta,\gamma,\theta)$, reduces to

$$G(y)=\left(\frac{y}{\beta}\right)^{\alpha\theta}\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta} \tag{5}$$

where β is a scale parameter, $\alpha>0,\gamma>0$ and $\theta>0$ are three positive shape parameters. The corresponding pdf of the ET-PF distribution is obtained by inserting (1) and (2) in Eq. (3) as

$$f(y)=\frac{\alpha\theta}{\beta}\left(\frac{y}{\beta}\right)^{\alpha\theta-1}\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta-1}, \quad 0<y<\beta \tag{6}$$

Sub-models of the ET-PF distribution are given below:

1. If θ=1, the ET-PF distribution converts into the transmuted PF distribution TPF(α,β,γ).
2. If γ=1, we obtain the exponentiated PF distribution EPF(α,β,θ).
3. If θ=γ=1, we get the PF distribution PF(α,β).

Figures 1 (a) and (b) represent the pdf of the ET-PF distribution graphically on different combination of parameters α,γ and θ for fixed β=2 and β=1, respectively. As seen from Figures 1 (a) and (b), observed distribution have different subfamilies depends on value of θ. The density function can take various forms depending on the parameter values. Both unimodal and monotonically decreasing and increasing shapes appear to be possible. It is evident that the ET-PF distribution is much more flexible than the T-PF and PF distributions.

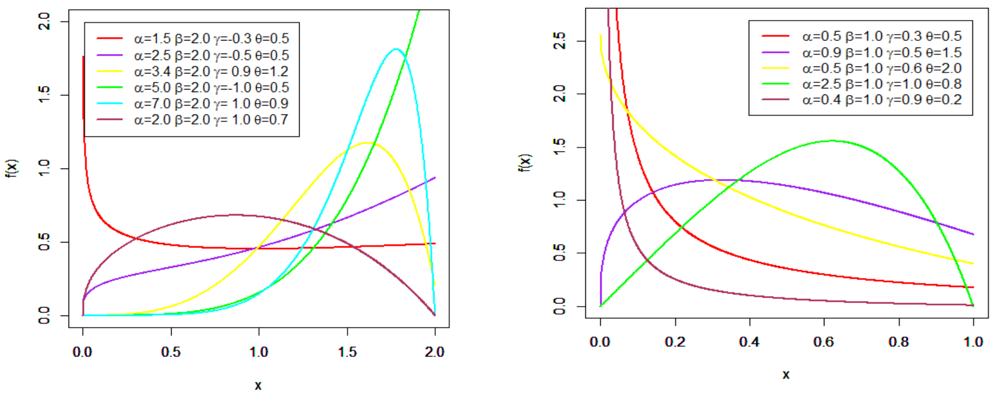

Figure 1. *Plots of the pdf for ET-PF distribution with some selected parameter values*

Due to the complex behavior of density function following expansion is used. For $0<\alpha<1$, we have

$$(1+a)^t = \sum_{i=0}^{\infty}\binom{t}{i}a^i \text{ for } t>0 \text{ and} |a|<1$$

where, $\binom{t}{i}=\dfrac{n(n-1)\ldots\ldots(t-i-1)}{i!}$

By using above expansions, the pdf from Eq. (6) can also be written in its simplified form as

$$f(y)=\alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}y^{\alpha(\theta+j)-1}\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right] \quad (7)$$

where $\pi_{i,j}=(-1)^j\dfrac{\gamma^i}{\beta^{\alpha(\theta+j)}}\binom{\theta-1}{i}\binom{i}{j}, \quad i<\theta-1 \text{ and } j<i.$

2.1. Survival and hazard functions

An important measure to explain and characterize the phenomenon of life is hazard function. Therefore, the survival function S(y) and hazard function h(y) of the ET-PF distribution are, respectively, given as

$$S(y)=1-\left(\frac{y}{\beta}\right)^{\alpha\theta}\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta} \tag{8}$$

$$h(y)=\frac{\dfrac{\alpha\theta}{\beta}\left(\dfrac{y}{\beta}\right)^{\alpha\theta-1}\left[1+\gamma-2\gamma\left(\dfrac{y}{\beta}\right)^{\alpha}\right]\left[1+\gamma-\gamma\left(\dfrac{y}{\beta}\right)^{\alpha}\right]^{\theta-1}}{1-\left(\dfrac{y}{\beta}\right)^{\alpha\theta}\left[1+\gamma-\gamma\left(\dfrac{y}{\beta}\right)^{\alpha}\right]^{\theta}} \tag{9}$$

Figure 2 explains the behavior of hazard function of the ET-PF distribution for several parameter values.

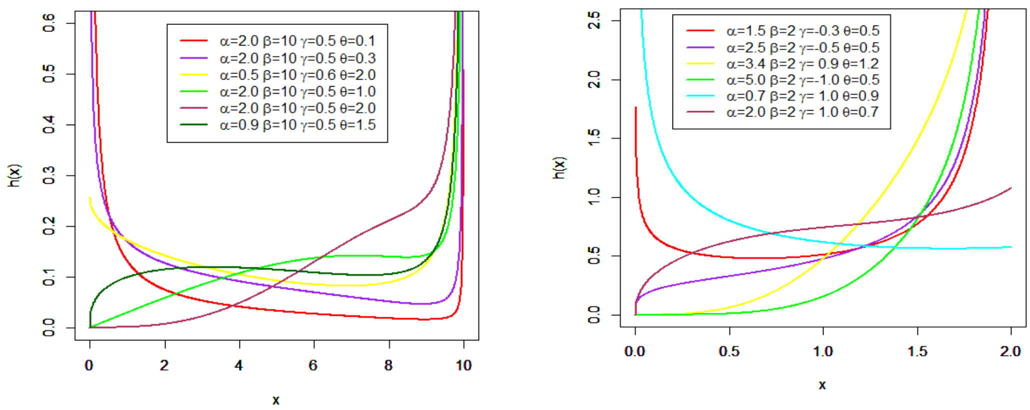

Figure 2. *Plots of hazard function of the ET-PF distribution for some selected parameter values*

Figure 2 shows that the distribution has increasing, bathtub behavior, upside-down bathtub behavior and exponentially decreasing behavior starts from y-axis. It seems that the ET-PF distribution is much appropriate in explaining the death rate and existence rate for the lifetime of the certain product. These different shapes show that the hazard rate function of the ET-PF is useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

3. MATHEMATICAL PROPERTIES

Algebraic expressions explain capable the structural quantities for a distribution as compared to the numerical illustration of density functions. Under this concept, we derive some expressions for important properties of the ET-PF distribution.

3.1. Moments

The rth moments has much importance in statistical analysis and in real life applications. Moreover, it helps to explain central tendencies, dispersions, skewness and kurtosis and some other characteristics of the observed model. By the matter of fact, it is essential to develop rth moment for a new proposed distribution.

Theorem 1: Let Y be a random variable (r.v.) from the ET-PF distribution. Then, its rth moment is given by

$$\mu'_r = \alpha\theta\beta^r \sum_{i=0}^{\infty}\sum_{j=0}^{i}\varphi_{i,j}\left[\frac{(1+\gamma)}{r+\alpha(\theta+j)} - \frac{2\gamma}{r+\alpha(\theta+j+1)}\right] \tag{10}$$

Proof: The r^{th} moment of Y can be obtained from

$$\mu'_r = \int_0^{\beta} y^r f(x;a,b,\beta)\,dx$$

$$= \alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^{\infty} y^{r+\alpha(\theta+j)-1}\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]dx$$

$$= (1+\gamma)\alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^{\beta} y^{r+\alpha(\theta+j)-1}dx - \frac{2\gamma\alpha\theta}{\beta^{\alpha}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^{\beta} y^{r+\alpha(\theta+j)+\alpha-1}dx$$

On inserting the value of $\pi_{i,j} = (-1)^j \frac{\gamma^i}{\beta^{\alpha(\theta+j)}}\binom{\theta-1}{i}\binom{i}{j}$ and after simplification, we get

$$= \alpha\theta\beta^r \sum_{i=0}^{\infty}\sum_{j=0}^{i}\varphi_{i,j}\left[\frac{(1+\gamma)}{r+\alpha(\theta+j)} - \frac{2\gamma}{r+\alpha(\theta+j+1)}\right]$$

where $\varphi_{i,j} = (-1)^j \gamma^i \binom{\theta-1}{i}\binom{i}{j}$

By inserting r =1, we get the mean of the ET-PF distribution as

$$\mu'_1 = \alpha\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{i}\varphi_{i,j}\left[\frac{(1+\gamma)}{1+\alpha(\theta+j)} - \frac{2\gamma}{1+\alpha(\theta+j+1)}\right]$$

3.2. Incomplete Moments

Incomplete moments are mostly used to obtain Lorenz and Bonferroni curves, to evaluate mean residual life and mean waiting time, to find mean deviation about mean and median. Moreover, it has much application in other areas such as reliability, insurance, demography, and economics.

Theorem 2: Let Y has the ET-PF distribution. Then, its r^{th} incomplete moment is given by

$$\varphi_r(y) = \alpha\theta\beta^r \sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\left[\frac{(1+\gamma)t^{r+\alpha(\theta+j)}}{r+\alpha(\theta+j)} - \frac{2\gamma t^{r+\alpha(\theta+j+1)}}{\beta^{\alpha}\left(r+\alpha(\theta+j+1)\right)}\right] \tag{11}$$

Proof: The rth incomplete moment of Y follows ET-PF distribution with pdf given in Eq. (7) can be obtained from

$$\varphi_r(y) = \int_0^{y} t^r f(t)\,\mathrm{d}t = \alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^{y} t^{r+\alpha(\theta+j)-1}\left[1+\gamma-2\gamma\left(\frac{t}{\beta}\right)^{\alpha}\right]dt$$

$$=(1+\gamma)\alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^y t^{r+\alpha(\theta+j)-1}dt-\frac{2\gamma\alpha\theta}{\beta^\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^y t^{r+\alpha(\theta+j)+\alpha-1}dt$$

By inserting the value of $\pi_{i,j}$ and after integration, we get

$$=\alpha\theta\beta^r\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\pi_{i,j}\left[\frac{(1+\gamma)t^{r+\alpha(\theta+j)}}{r+\alpha(\theta+j)}-\frac{2\gamma t^{r+\alpha(\theta+j+1)}}{\beta^\alpha\left(r+\alpha(\theta+j+1)\right)}\right]$$

3.3. Moment generating function

Here, we present a closed form of moment generating function (mgf) M(t) for the ET-PF distribution as follows:

$$M_Y(t)=\alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}(-1)^{\alpha(\theta+j)-1}\left[\frac{(1+\gamma)\Gamma\left(\alpha(\theta+j),-\beta t\right)}{t^{\alpha(\theta+j)}}-\frac{2\gamma(-1)^\alpha\Gamma\left(\alpha(\theta+j+1),-\beta t\right)}{\beta^\alpha t^{\alpha(\theta+j+1)}}\right] \tag{12}$$

Proof: The mgf of the ET-PF distribution is obtained as

$$M_y(t)=E\left(e^{ty}\right)=\int_0^\beta e^{ty}f(y;a,b,\beta)dy$$

$$=\alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^\beta y^{\alpha(\theta+j)-1}\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^\alpha\right]e^{ty}dy$$

$$=(1+\gamma)\alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^\beta y^{\alpha(\theta+j)-1}e^{-(-ty)}dy-\frac{2\gamma\alpha\theta}{\beta^\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^\beta y^{\alpha(\theta+j)+\alpha-1}e^{-(-ty)}dy.$$

Put $-ty=z$ and $y=-\frac{z}{t}$ where $dy=-\frac{1}{t}dz.$ Then, we have

$$M_y(t)=(1+\gamma)\alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^{-\beta t}\left(-\frac{z}{t}\right)^{\alpha(\theta+j)-1}e^{-z}\left(-\frac{1}{t}\right)dz-\frac{2\gamma\alpha\theta}{\beta^\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_{i,j}\int_0^{-\beta t}\left(-\frac{z}{t}\right)^{\alpha(\theta+j)+\alpha-1}e^{-z}\left(-\frac{1}{t}\right)dz$$

After integration and simplification, we get the mgf in the form of the incomplete gamma function.

3.4. Quantile function and random number generation

The quantile function of variable Y belong to the ET-PF distribution can be obtained from Eq. (5) as $F(y)=q$ and $y=F^{-1}(q)$, where q is a uniform random variate with unit interval (0, 1). After simplification, the quantile function is given by

$$y_q=\beta\left[\frac{(1+\gamma)\pm\sqrt{(1+\gamma)^2-4\gamma q^{\frac{1}{\theta}}}}{2\gamma}\right]^{\frac{1}{\alpha}} \tag{13}$$

3.5. Mode

The mode of the ET-PF distribution for variable Y is obtained as

$$\text{Mode}=\beta\left[(1+\gamma)\left\{\frac{3(\alpha\theta+\alpha-1)\pm\sqrt{\left[9(\alpha\theta+\alpha-1)^2-8(\alpha\theta-1)(\alpha\theta+2\alpha-1)\right]}}{4\gamma(\alpha\theta+2\alpha-1)}\right\}\right]^{\frac{1}{\alpha}} \tag{14}$$

Proof: We can find mode by using its definition as $f^{'}(x)=df(x)/dx=0.$ As

$$f'(x)=\frac{\alpha\theta}{\beta^{\alpha\theta}}(y)^{\alpha\theta-1}\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta-2}\left[\begin{array}{l}(\alpha\theta-1)\left\{1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right\}\left\{1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right\}\\ -2\alpha\gamma\left(\frac{y}{\beta}\right)^{\alpha}\left\{1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right\}\\ -(\theta-1)\alpha\gamma\left(\frac{y}{\beta}\right)^{\alpha}\left\{1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right\}\end{array}\right]=0$$

As $\frac{\alpha\theta}{\beta^{\alpha\theta}}(y)^{\alpha\theta-1}\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta-2}>0$ for all $\alpha,\theta,\beta,\gamma>0$ and $0<y<\beta$. Therefore, after simplification we get

$$\left[(\alpha\theta-1)(1+\gamma)^2-3\gamma(1+\gamma)(\alpha\theta+\alpha-1)\left(\frac{y}{\beta}\right)^{\alpha}+2\gamma^2(\alpha\theta+2\alpha-1)\left(\frac{y}{\beta}\right)^{2\alpha}\right]=0$$

By using quadratic formula, the result follows that

$$Mode=\beta\left[(1+\gamma)\left\{\frac{3(\alpha\theta+\alpha-1)\pm\sqrt{\left[9(\alpha\theta+\alpha-1)^2-8(\alpha\theta-1)(\alpha\theta+2\alpha-1)\right]}}{4\gamma(\alpha\theta+2\alpha-1)}\right\}\right]^{\frac{1}{\alpha}}$$

The second derivative may be used if required.

3.6. Mean deviation

Mean deviation is a tool to evaluate the dispersion in a population by measuring the totality of absolute deviation from mean and median. Mean deviation about mean and median is defined as respectively

$$\delta_{\mu}(y)=E\left(\left|Y-\mu_1^{'}\right|\right)=\int_0^{\beta}\left|y-\mu_1^{'}\right|f(y)\,dy=2\mu_1^{'}F\left(\mu_1^{'}\right)-2\varphi_1\left(\mu_1^{'}\right) \tag{15}$$

$$\delta_M(y) = E(|y-M|) = \int_0^{\beta} |y-M| \, f(y)\, dy = \mu_1^{'} - 2\varphi_1(M) \tag{16}$$

where $\mu_1^{'} = E(Y)$ can be obtained from Eq. (10), $F(\mu_1^{'})$ can be calculated from Eq. (5) and $\phi_1(\mu_1^{'})$ is the first incomplete moment computed from Eq. (11).

3.7. Skewness and kurtosis

Skewness is the measure of the asymmetry of the probability distribution and kurtosis is the measure of peakedness of the density function. Both measures are the descriptive measures of the shape of the probability distribution. Skewness and kurtosis can be easily determined by the following expressions based on first four mean moments calculated by Eq. (11) or Eq. (12)

$$\gamma_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} \qquad \text{and} \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} \tag{17}$$

3.8. Rényi Entropy

The entropy of a random variable Y is used to measure the variation of the uncertainty. Mostly, the Rényi entropy is used as a common measure of entropy.

Theorem: If the random variable Y is defined as Eq. (5), then the Rényi entropy is given by

$$I_R(\delta) = \frac{\delta \log \alpha}{1-\delta} + \frac{\delta \log \theta}{1-\delta} + \log \beta + \frac{1}{1-\delta} \log\left[\sum_{i,k,s=0}^{\infty} \sum_{j=0}^{i} \omega_{i,j,k,s} \gamma^{i+j} 2^s\right] \tag{18}$$

where $\delta > 0$ and for $\delta \neq 1$

Proof: If Y has the ET-PF distribution, then the Rényi entropy is defined as

$$I_R(\delta) = \frac{1}{1-\delta} \log\left[I(\delta)\right] \tag{18a}$$

$$I(\delta) = \int_0^{\beta} f^{\delta}(y)\, dy = \int_0^{\beta} \frac{(\alpha\theta)^{\delta}}{\beta^{\alpha\theta\delta}} (y)^{\delta(\alpha\theta-1)} \left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\delta} \left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\delta(\theta-1)} dy$$

After simplification final expression is

$$I(\delta) = (\alpha\theta)^{\delta} \sum_{i,k,s=0}^{\infty} \sum_{j=0}^{i} \omega_{i,j,k,s} \gamma^{i+j} 2^s \beta^{1-\delta} \tag{18b}$$

$$where\, \omega_{i,j,k,s} = (-1)^{k+s} \binom{\delta}{i} \binom{\delta(\theta-1)}{j} \binom{i}{k} \binom{j}{s} \; for\, i < \delta, k < i\, and\; s < j$$

Substituting Eq. (18b) in Eq. (18a), the result follows. Note that the q entropy (Hq) is defined by

$$\mathrm{H_q} = \frac{1}{\mathrm{q}-1}\log\left(1-\left(1-\mathrm{q}\right)\mathrm{I_R}\left(\delta\right)\right) \tag{19}$$

Substitution of Eq. (18) completes the proof.

3.9. Order Statistics

Let Y be r.v. and its ordered values is denoted as $Y_{(1)}, Y_{(2)}, Y_{(3)}, \ldots\ldots, Y_{(n)}$. The pdf of order statistics is obtained using the function

$$f_{s:n}(y) = \frac{n!}{(s-1)!(n-s)!}\left[F(y)\right]^{s-1}\left[1-F(y)\right]^{n-s} f(y)$$

The density of the s^{th} ordered statistics follows the ET-PF distribution is derived as follows

$$f_{s:n}(x) = \frac{n!}{(s-1)!(n-s)!}\left[\left(\frac{y}{\beta}\right)^{\alpha\theta}\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta}\right]^{s-1}\left[1-\left(\frac{y}{\beta}\right)^{\alpha\theta}\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta}\right]^{n-s}$$
$$\times\frac{\alpha\theta}{\beta}\left(\frac{y}{\beta}\right)^{\alpha\theta-1}\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta-1}$$

The pdf of the first-order statistics is obtained as

$$f_{1:n}(x) = n\left[1-\left(\frac{y}{\beta}\right)^{\alpha\theta}\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta}\right]^{n-1}\frac{\alpha\theta}{\beta}\left(\frac{y}{\beta}\right)^{\alpha\theta-1}\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta-1}$$

The density of the largest order statistic is given by

$$f_{n:n}(x) = n\left[\left(\frac{y}{\beta}\right)^{\alpha\theta}\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta}\right]^{n-1}\frac{\alpha\theta}{\beta}\left(\frac{y}{\beta}\right)^{\alpha\theta-1}\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta-1}$$

3.10. Estimation

Many estimation methods have argued in literature but maximum likelihood estimation (MLE) method provides maximum information about the properties of estimated parameters and mostly used. Moreover, normal approximation of these estimators can frankly be managed systematically and mathematically for large sample theory. Consequently, the MLE has adopted to estimate the unknown parameters $(\alpha, \beta, \gamma\,\text{and}\,\theta)$ of the ET-PF distribution.

Let Y have vector of parameters $(\alpha, \beta, \gamma\,\text{and}\,\theta)^T$ with size *n*. The sample likelihood function is achieved as

$$\prod_{i=1}^{n} f(y) = \frac{\alpha^n \theta^n}{\beta^{n\alpha\theta}} \prod_{i=1}^{n} y^{\alpha\theta-1} \left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]^{\theta-1}$$

The log-likelihood function is

$$L = n\log\alpha + n\log\theta - n\alpha\theta\log\beta + (\alpha\theta-1)\sum\log y + \sum\log\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right] + (\theta-1)\sum\log\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right] \quad (20)$$

Now we have to maximize the above log-likelihood function given in Eq. (20) to get MLEs of unknown parameters of exponentiated transmuted PF distribution. For this purpose, we take the first derivative of the above log-likelihood equation with respect to parameters and equate to zero respectively.

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - n\theta\log\beta + \theta\sum\log y - \sum\frac{2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\log\left(\frac{y}{\beta}\right)}{\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]} - (\theta-1)\sum\frac{\gamma\left(\frac{y}{\beta}\right)^{\alpha}\log\left(\frac{y}{\beta}\right)}{\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]} = 0 \quad (21)$$

$$\frac{\partial L}{\partial \beta} = -\frac{n\alpha\theta}{\beta} + \sum\frac{2\alpha\gamma y\left(\frac{y}{\beta}\right)^{\alpha-1}\frac{1}{\beta^2}}{\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]} + (\theta-1)\sum\frac{\alpha\gamma y\left(\frac{y}{\beta}\right)^{\alpha-1}\frac{1}{\beta^2}}{\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]} = 0 \quad (22)$$

$$\frac{\partial L}{\partial \gamma} = \sum\frac{1-2\left(\frac{y}{\beta}\right)^{\alpha}}{\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]} + (\theta-1)\sum\frac{1-\left(\frac{y}{\beta}\right)^{\alpha}}{\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]} = 0 \quad (23)$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} - n\alpha\log\beta + \alpha\sum\log y + \sum\log\left[1+\gamma-\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right] = 0 \quad (24)$$

The exact solution of above-derived ML estimator for unknown parameters is not possible. So it is well-situated to use nonlinear optimization algorithms such as a Newton-Raphson algorithm to maximize the above likelihood function numerically. We can use R (optimal function or maxBFGS function), or MATHEMATICA (Maximize function). After application of large sample property of MLEs, $\hat{\theta}$ can be treated as being approximately normal with mean θ and variance-covariance matrix equal to the inverse of the expected information matrix, i.e. $\sqrt{n\left(\hat{\theta}-\theta\right) \rightarrow N\left(0,\, nI^{-1}(\theta)\right)}$, $I(\theta)$ is the information matrix then its inverse of matrix is $I^{-1}(\theta)$ provide the variances and covariance's. The $I\left(\hat{\theta}\right)$ variance-covariance matrix is actually equal to the inverse of the expected information matrix $I^{-1}\left(\hat{\theta}\right)$ is given as

$$I(\hat{\varphi}) = \begin{bmatrix} J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\theta} & J_{\alpha\gamma} \\ J_{\beta\alpha} & J_{\beta\beta} & J_{\beta\theta} & J_{\beta\gamma} \\ J_{\theta\alpha} & J_{\theta\beta} & J_{\theta\theta} & J_{\theta\gamma} \\ J_{\gamma\alpha} & J_{\gamma\beta} & J_{\gamma\theta} & J_{\gamma\gamma} \end{bmatrix}$$

4. SIMULATION STUDY

In this section, we examine the performance of the ET-PF distribution by carrying out various simulations for different example sizes and different parameter values by using Monte Carlo Simulation using R language. Quantile function is used to generate random data from ET-PF distribution. The simulation study is repeated N=5000 times each with sample size *n=50*, *100*, *200* and the parameter values I: $\alpha = 0.5,\ \beta = 10,\ \theta = 6,\ \gamma = 0.7$ and II: $\alpha = 2,\ \beta = 7,\ \theta = 5,\ \gamma = 1$ Two quantities are computed in this simulation: Average bias of the MLE $\hat{\theta}$ of the parameter $\alpha, \beta, \theta, \gamma$ and Root mean square error (RMSE) of the MLE $\hat{\theta}$defined as

$$Average\,Bias = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}-\theta\right)$$

and

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}-\theta\right)^2}$$

The parameter combinations for the simulation study are shown in Table 1. The values in Table 1 indicate that as the sample size increases the biases and RMSEs of the estimates decrease.

Table 1. *Monte Carlo simulation results*

		Set-I		Set-II	
Parameter	Sample size	Average Bias	RMSE	Average Bias	RMSE
$\boldsymbol{\alpha}$	50	-0.174	0.287	-0.231	0.276
	100	-0.122	0.285	-0.142	0.271
	200	-0.067	0.298	-0.081	0.277
$\boldsymbol{\beta}$	50	3.682	5.498	4.232	5.521
	100	2.638	4.445	2.543	3.985
	200	1.794	3.428	1.512	2.813
$\boldsymbol{\theta}$	50	0.255	0.871	0.245	0.594
	100	0.196	0.752	0.251	0.572
	200	0.101	0.602	0.163	0.448
$\boldsymbol{\gamma}$	50	0.219	0.694	0.476	1.154
	100	0.081	0.393	0.061	0.466
	200	0.035	0.243	0.025	0.297

5. REAL DATA APPLICATIONS

In this section, we provide three applications based on real data sets from different areas to illustrate the flexibility of the ET-PF distribution in contrast to other models including power, transmuted-power and transmuted-Rayleigh distributions. The MLEs of the parameters are determined fort the ET-PF distribution and three other models with goodness-of-fit statistics are computed for checking the adequacy of the all four models.

Data set 1: The first real data set consists of the survival times of guinea pigs injected with different doses of tubercle bacilli [5]. It is well known that guinea pigs have high susceptibility to human tuberculosis and that is why they were used in that study. In this study, we used the data of animals in the same cage that under the same regimen; the data includes 72 observations. The data are: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 376.

Data set 2: The second real data set refers to air conditioning failures given by [8]. These data were analyzed a lot of researchers and for an alternative approach, failure data also used in this study. The data are: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

Data set 3: The third real data set is about traffic and consisting of the length of intervals between times at which vehicles pass a point on a road [15]. The data are: 2.5, 2.6, 2.6, 2.7, 2.8, 2.8, 2.9, 3,3, 3.1, 3.2, 3.4, 3.7, 3.9, 3.9, 3.9, 4.6, 4.7, 5, 5.6, 5.7, 6, 6, 6.1, 6.6, 6.9, 6.9, 7.3, 7.6, 7.9, 8, 8.3, 8.8, 9.3, 9.4, 9.5, 10.1, 11, 11.3, 11.9, 11.9, 12.3, 12.9, 12.9, 13, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6, 21.4, 22.8, 23.7, 23.7, 24.7, 29.7, 30.6, 31, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42, 44.8, 49.8, 51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, 119.8. Table 2 gives summary about descriptive statistics of each data set.

Table 2. *Descriptive statistics for the data sets*

n	Minimum	Median	Mean	Maximum	Variance	Skewness	Kurtosis
72	12	70	99.829	376	6580.122	1.796	5.614
30	1	22	59.600	261	5167.421	1.694	4.967
83	2.500	12.300	21.757	119.800	579.530	1.897	6.498

These data sets are modeled by ET-PF distribution and compared with the power, transmuted-power and transmuted-Rayleigh distributions. Their associated densities are given by:

The pdf of the PF distribution is

$$f(y) = \frac{\alpha y^{\alpha-1}}{\beta^{\alpha}}; 0 < y < \beta, \alpha > 0$$

The pdf of the TPF [10] distribution is

$$f(y) = \frac{\alpha y^{\alpha-1}}{\beta^{\alpha}}\left[1+\gamma-2\gamma\left(\frac{y}{\beta}\right)^{\alpha}\right]; 0 < y < \beta, \alpha > 0, |\gamma| \le 1$$

The pdf of the TR distribution [17] is

$$f(y) = \frac{y}{\alpha^2} exp\left(-\frac{y^2}{2\alpha^2}\right)\left[1-\gamma-2exp\left(-\frac{y^2}{2\alpha^2}\right)\right]; y > 0, \alpha > 0, |\gamma| \le 1$$

The MLEs of the parameters are computed using a script of the R-language, the Adequacy Model. In Adequacy Model package, there exists many maximization algorithms like NR (Newton-Raphson), BFGS (Broyden-Fletcher-Goldfrab-Shanno), BHHH (Berndt-Hall-HAll-Hausmann), SANN (Simulated-Annealing), NM (Nelder-Mead) and Limited-Memory quasi-Newton code for Bound-constrained optimization (L-BFGS-B) [22]. Here, the MLEs are computed using SANN method.

Table 3, 4 and 5 describe the estimated parameters of ET-PF distribution with goodness of fit measures such as log-likelihood (Log-L), Akaike Information Criterion ($AIC = 2p - 2ln(L)$), Bayesian Information Criterion ($BIC = pln(n) - 2ln(L)$) and Kolmogrov-Smirnov (K-S) test for all data sets. In information criterions, L is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters.

Table 3. *Estimates and goodness of fit measures under considered distributions based on data set 1*

Model	ML Estimates	LogL	AIC	BIC	K-S Statistics	p-value
Power	α=0.747 β= 280.633	397.342	798.684	803.210	0.273	0.000
Transmuted-Power	α =0.890 β = 357.233 γ=0.831	398.406	802.812	809.600	0.232	0.001
Transmuted-Rayleigh	α=86.227 γ =0.517	393.396	790.792	795.317	0.157	0.061
Exponentiated Transmuted-Power	α =-0.156 β =342.899 θ=5.508 γ =1.171	381.921	771.841	780.892	0.137	0.141

Table 4. *Estimates and goodness of fit measures under considered distributions based on data set 2*

Model	ML Estimates	LogL	AIC	BIC	K-S Statistics	p-value
Power	α=0.414 β=318.298	157.068	318.137	320.939	0.232	0.079
Transmuted-Power	α=0.607 β=268.507 γ=0.725	152.531	311.061	315.265	0.182	0.276
Transmuted-Rayleigh	α=65.078 γ=0.618	175.859	355.719	358.521	0.438	0.000
Exponentiated Transmuted-Power	α=-0.074 β=258.734 θ=8.318 γ=1.268	149.291	306.187	312.187	0.118	0.798

Table 5. *Estimates and goodness of fit measures under considered distributions based on data set 3*

Model	ML Estimates	LogL	AIC	BIC	K-S Statistics	p-value
Power	α=0.454 β=113.087	359.338	722.677	727.514	0.233	0.000
Transmuted-Power	α=0.647 β=114.848 γ=0.879	346.975	699.950	707.206	0.167	0.019
Transmuted-Rayleigh	α=25.470 γ=0.685	379.003	762.005	766.843	0.346	0.000

Exponentiated Transmuted-Power	α=-0.118 β=123.359 θ=1.315 γ=1.678	322.995	653.991	663.666	0.061	0.921

The values of the statistics Log-L, AIC and BIC of ET-PF distribution are comparatively smaller than the other distributions on three data sets. Therefore, the results show that ET-PF distribution provides a significantly better fit than the other models. So, it could be chosen as the best model.

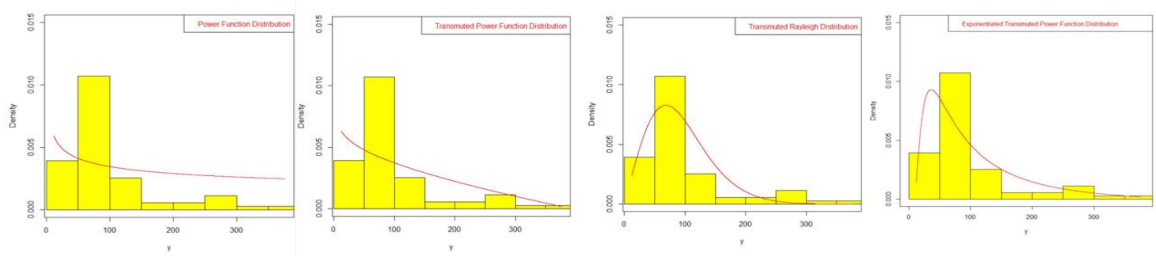

Figure 3. *Plots of the estimated pdf and cdf of the distributions for the first data set*

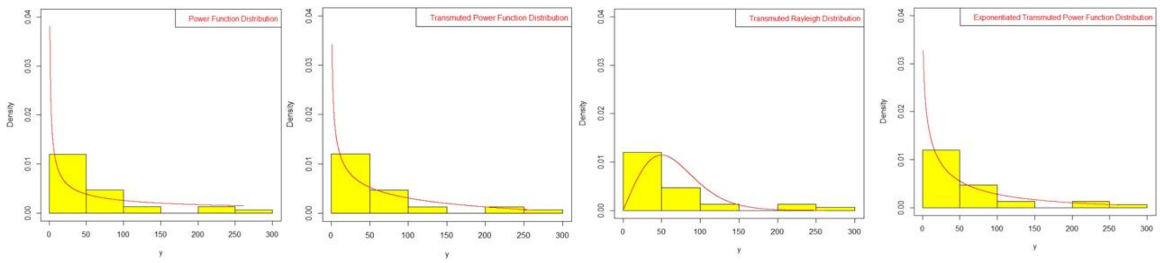

Figure 4. *Plots of the estimated pdf and cdf of the distributions for the second data set*

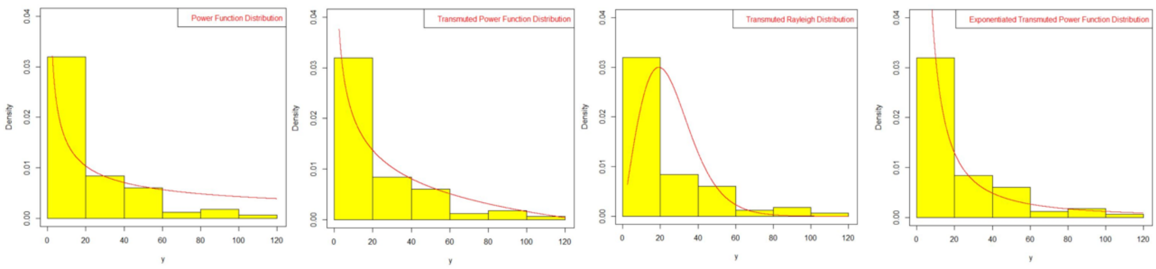

Figure 5. *Plots of the estimated pdf and cdf of the distributions for the third data set*

More information is provided by a usual comparison of the histograms of the data sets with the fitted pdf. The plots of fitted distributions and the histograms of the data sets are given Figure 3, 4 and 5. They indicate that the ET-PF distribution provides more adequate fit than the other distributions.

6. CONCLUSION

Various univariate lifetime distributions have been constructed for better fit real life data belong to numerous fields. This study explain a new life time distribution, the exponentiated transmuted power function distribution, by using exponentiated-G generator. Some fundamental properties have studied including mean deviation, moments, entropies and order statistics. It provides a flexible distribution then some existing models. We hope this model will have greater interest in several fields of research and broader application in electronic industry.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Ahsanullah, M., & Kabir, A. L. "A characterization of the power function distribution". The Canadian Journal of Statistics/La Revue Canadienne de Statistique, 2(1): 95-98, (1974).

[2] Ahsanullah, M. "Estimation of the parameters of a power function distribution by record values". Pakistan Journal of Statistics, 5(2): 189-194, (1989).

[3] Ali, M. M., & Woo, J. "Inference on reliability P (Y< X) in a power function distribution". Journal of Statistics and Management Systems, 8(3): 681-686, (2005).

[4] Balakrishnan, N., & Nevzorov, V. "A primer on statistical distributions". Hoboken, New Jersey: A John Wiley & Sons: Inc, (2003).

[5] Bjerkedal, T. "Acquisition of Resistance in Guinea Pies infected with Different Doses of Virulent Tubercle Bacilli". American Journal of Hygiene, 72(1): 130-48, (1960).

[6] Bursa, N., & Ozel, G. "The exponentiated Kumaraswamy-power function distribution". Hacettepe Journal of Mathematics and Statistics, 46(2): 277-292, (2017).

[7] Cordeiro, G. M., & dos Santos Brito, R. "The beta power distribution". Brazilian Journal of Probability and Statistics, 26(1): 88-112, (2012).

[8] Cox, D. R., & Lewis, P. A. W. "The statistical analysis of series of events London", Waiting Time Paradoxes and Order Statistics, 745, (1966).

[9] Forbes, C., Evans, M., Hastings, N., & Peacock, B. "Statistical distributions", John Wiley & Sons, (2011).

[10] Haq, M. A., Butt, N. S., Usman, R. M., & Fattah, A. A. "Transmuted power function distribution". Gazi University Journal of Science, 29(1): 177-185, (2016a).

[11] Haq, M. A., Elgarhy, M., Hashmi, S., Ozel, G., & ul Ain, Q. "Transmuted Weibull Power Function Distribution: its Properties and Applications". Journal of Data Science, 16(2): 397-418, (2018a).

[12] Haq, M. A., Usman, R. M., Bursa, N., & Öze, G. "McDonald Power Function Distribution with Theory and Applications". International Journal of Statistics & Economics, 19(2): 89-107, (2018b).

[13] Johnson, R. A. Miller and Freund's. "Probability and Statistics for Engineers", Prentice Hall, (1994).

[14] Johnson, N. L., Kotz, S., & Balakrishnan, N. "Continuous univariate distributions", Wiley, New York, (1995).

[15] Jorgensen, B. "Statistical properties of the generalized inverse Gaussian distribution". Science & Business Media, 9: (2012).

[16] Meniconi, M., & Barry, D. “The power function distribution: A useful and simple distribution to assess electrical component reliability”. Microelectronics Reliability, 36(9): 1207-1212, (1996).

[17] Merovci, F.. “Transmuted rayleigh distribution”. Austrian Journal of Statistics, 42(1): 21-31, (2016).

[18] Oguntunde, P. E., Odetunmibi, O., & Okagbue, H. I.. “The Kumaraswamy-power distribution: A generalization of the power distribution”. International Journal of Mathematical Analysis, 9(13): 637-645, (2015).

[19] Shakeel, M., ul Haq, M. A., Hussain, I., Abdulhamid, A. M., & Faisal, M. “Comparison of two new robust parameter estimation methods for the power function distribution”. PloS One, 11(8): 1-11, (2016).

[20] Sultan, R., Sultan, H., & Ahmad, S. “Bayesian analysis of power function distribution under double priors”. Journal of Statistics Applications & Probability, 3(2): 239-249, (2014).

[21] Tahir, M., Alizadeh, M., Mansoor, M., Cordeiro, G. M., & Zubair, M. “The Weibull-power function distribution with applications”. Hacettepe Journal of Mathematics and Statistics, 45(42): 1208-1230, (2014).

[22] Team, R. C. R: “A language and environment for statistical computing”. R Foundation for Statistical Computing, Vienna, Austria. 2013, (2014).