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Alpha Power Inverted Exponential Distribution: Properties and Application

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Abstract

In this study, we introduce a new distribution based on the inverted exponential distribution called as "Alpha Power Inverted Exponential" distribution. Some of the statistical properties are provided such as hazard rate function, quantile function, skewness, kurtosis, and order statistics. Model parameters are obtained by the maximum likelihood. We prove empirically importance and flexibility of the new distribution in modeling with real data applications.

1. INTRODUCTION

In spite of the fact that there are many statistical distributions in literature, it is always possible to develop both more flexible and more suitable specific real world scenarios. In statistics, the exponential distribution widely used to describe the time between events in a Poisson process. It plays an important role only with continuous memoryless random distribution. On the other hand, it has a constant failure rate. It is rare to see an event that has a constant failure rate. The exponential distribution becomes unsuitable for modeling real life situations in engineering, mechanical and electronic systems, business, insurance etc. with bathtub and inverted bathtub failure rates. The inverted bathtub hazard rate describes the relative failure rate which initially increases, come to a head after some time, and so decreases over time. To make up this disadvantage, Keller and Kamath [1] introduced inverted exponential (IE) distribution which have the inverted bathtub hazard rate. Then, it was studied in detail as a lifetime model [2]. In the life distribution, the random variable $X = 1/Y$ possesses IE distribution when the random variable Y has an exponential distribution.

The IE distribution is commonly used for analyzing biology, engineering and medicine [3]. Afterwards, many authors have proposed distributions using the IE distribution in statistical literature. Bayes estimators of the parameter and reliability function of the IE distribution were obtained by Singh et al. [4]. Dey [5] derived Bayes estimators of the parameter for the IE distribution. Abouammoh and Alshingiti [6] proposed a generalized IE (GIE) distribution. Oguntunde et al. [7] proposed exponentiated GIE (EGIE) distribution and Singh et al. [8] obtained the maximum likelihood and Bayes estimators of the parameters of for the GIE distribution. Other studies are Kumaraswamy inverse exponential (KIE) distribution by Oguntunde et al. [9], transmuted inverse exponential (TIE) distribution by Oguntunde et al. [10] and Weibull inverse exponential (WIE) distribution by Oguntunde [11].

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The probability density function (pdf) of the IE distribution is given as follows:

$$f_{IE}(x)=\frac{\lambda}{x^2}\exp\left(-\frac{\lambda}{x}\right)\quad x>0,\lambda>0 \tag{1}$$

and corresponding cumulative distribution function (cdf) as follows:

$$F_{IE}(x)=\exp\left(-\frac{\lambda}{x}\right)\quad x>0,\lambda>0 \tag{2}$$

where $\lambda>0$ is the scale parameter.

There have been many methods to obtain more flexible distributions (see Lee et al. [13]). In recent years, Mahdavi and Kundu [12] have proposed a new method called as Alpha Power Transformation (APT) for introducing an extra parameter to a family of distributions. This parameter provides more flexibility to the proposed family. Mahdavi and Kundu [12] used APT method to the exponential distribution and obtain the alpha power exponential (APE) distribution. Dey et al. [14] proposed a three-parameter distribution, known as alpha power transformed generalized exponential (αPTGE) distribution. Dey et al. [15] introduced alpha power transformed Weibull (APTW) distribution which contains APE distribution for $l=1$ and alpha power transformed Rayleigh (APTR) distribution $l=2$.

Let $f(x)$ and $F(x)$ be the pdf and the cdf of a continuous random variable X, respectively. The APT of $F(x)$ for $x\in R$ is defined as follows:

$$F_{APT}(x)=\begin{cases}\dfrac{\alpha^{F(x)}-1}{\alpha-1}, & if\ \alpha>0,\ \alpha\neq 1\\ F(x), & if\ \alpha=1,\end{cases} \tag{3}$$

and the corresponding pdf as follows:

$$f_{APT}(x)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}f(x)\alpha^{F(x)}, & if\ \alpha>0,\ \alpha\neq 1\\ f(x), & if\ \alpha=1.\end{cases} \tag{4}$$

The APT of survival function $S_{APT}(x)$ and the hazard rate function $h_{APT}(x)$ are, respectively, given by

$$S_{APT}(x)=\begin{cases}\dfrac{\alpha}{\alpha-1}\left(1-\alpha^{F(x)-1}\right), & if\ \alpha\neq 1\\ 1-F(x), & if\ \alpha=1,\end{cases} \tag{5}$$

and

$$h_{APT}(x)=\begin{cases}f(x)\dfrac{\alpha^{F(x)-1}}{1-\alpha^{F(x)-1}}\log\alpha, & if\ \alpha\neq 1\\ \dfrac{f(x)}{S(x)}, & if\ \alpha=1.\end{cases} \tag{6}$$

In this study, we propose Alpha Power Inverted Exponential (APIE) distribution motivated by IE distribution and the APT method that mentioned above in Section 2. In section 2, statistical properties of the APIE distribution are obtained including skewness, kurtosis, order statistics, survival, hazard rate and quantile functions. Section 4 provides the maximum likelihood estimation of model parameters and real data is used to evaluate the performance of the proposed distribution. Finally, the study is completed in Section 5.

2. APIE DISTRIBUTION

Motivated by APT method, we obtain APIE distribution. The random variable X has a two-parameter APIE distribution if the cdf of X for x>0 as follows:

$$F_{APIE}(x)=\left\{\begin{array}{ll}\dfrac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-1}, & if\ \alpha>0,\ \alpha\neq 1\\ \exp\left(-\dfrac{\lambda}{x}\right), & if\ \ \alpha=1,\end{array}\right\} \tag{7}$$

and the corresponding pdf is obtained as

$$f_{APIE}(x)=\left\{\begin{array}{ll}\dfrac{\log\alpha}{\alpha-1}\dfrac{\lambda}{x^2}\exp\left(-\dfrac{\lambda}{x}\right)\alpha^{\exp\left(-\frac{\lambda}{x}\right)}, & if\ \alpha>0,\ \alpha\neq 1\\ \dfrac{\lambda}{x^2}\exp\left(-\dfrac{\lambda}{x}\right), & if\ \ \alpha=1\end{array}\right\} \tag{8}$$

where $\lambda>0$, $\alpha>0$ are scale and shape parameters, respectively.

Figure 1 shows the density function of the APIE distribution for several values of parameters.

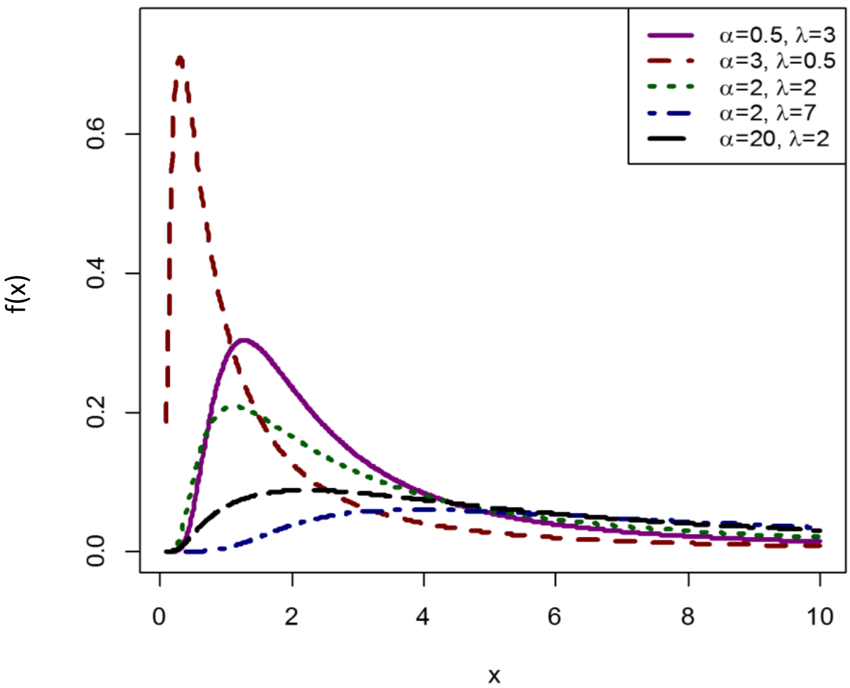

Figure 1. *The pdf of the APIE distribution for different parameter values*

As seen in Figure 1, the pdf of APIE is flexible and has various shapes for the several values of parameters.

3. MAIN PROPERTIES

3.1. Survival and Hazard Rate Functions

Now, we will provide the survival and hazard rate functions of the APIE distribution. The survival function of the APIE distribution for x>0 is given as

$$S_{APIE}(x)=\left\{\begin{array}{ll}\dfrac{\alpha}{\alpha-1}\left(1-\alpha^{\exp\left(-\frac{\lambda}{x}\right)-1}\right), & if\ \alpha\neq 1\\ 1-\exp\left(-\dfrac{\lambda}{x}\right), & if\ \alpha=1.\end{array}\right\} \tag{9}$$

Other important characteristic of the APIE distribution is the hazard rate function which is given by

$$h_{APIE}(x)=\left\{\begin{array}{ll}\dfrac{\lambda}{x^2}\exp\left(-\dfrac{\lambda}{x}\right)\dfrac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)-1}}{1-\alpha^{\exp\left(-\frac{\lambda}{x}\right)-1}}\log\alpha, & if\ \alpha\neq 1\\ \dfrac{\frac{\lambda}{x^2}\exp\left(-\frac{\lambda}{x}\right)}{1-\exp\left(-\frac{\lambda}{x}\right)}, & if\ \alpha=1\end{array}\right\} \tag{10}$$

Here, λ and α indicate the scale parameter and shape parameter, respectively.

Plots of hazard rate function for the APIE distribution are shown in Figure 2 for some values of parameters.

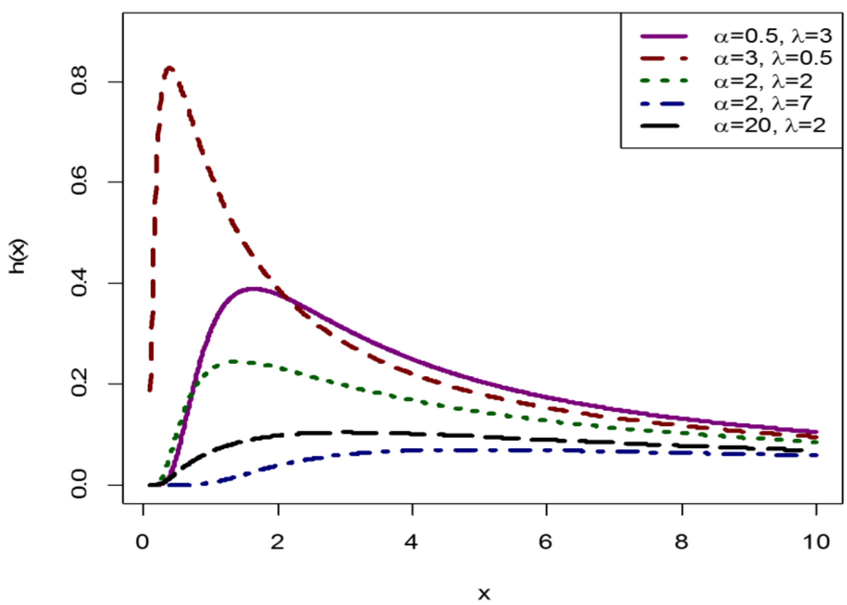

Figure 2. *The hazard rate function of the APIE distribution for different parameter values*

As seen in Figure 2 the hazard rate function of the APIE distribution is flexible for different values of parameters.

3.2. Quantile Function

Quantile function is important in statistics and this function is described by the inverse of the cdf given by

$$Q(u) = \inf\{x \in R : u \le F(x)\}$$
$$= F^{-1}(x)$$

Let $F(x) = u$. Then, from Equation (7), we have

$$\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)} - 1}{\alpha - 1} = u\,, \tag{11}$$

$$x = \frac{-\lambda}{\log\left(\frac{\log(u(\alpha-1)+1)}{\log\alpha}\right)}. \tag{12}$$

Therefore, the quantile function of the APIE distribution is defined as

$$Q(u) = \frac{-\lambda}{\log\left(\frac{\log(u(\alpha-1)+1)}{\log\alpha}\right)} \tag{13}$$

where u ~ Uniform(0,1). The p^{th} quantile function of X~APIE(λ,α) distribution is shown below

$$X = \frac{-\lambda}{\log\left(\frac{\log(u(\alpha-1)+1)}{\log\alpha}\right)}, \quad 0 < u < 1\,.$$

In particular, the first three quantiles, Q_1, Q_2, Q_3 for the APIE distribution, are obtained by setting u=0.25 (25^{th} Percentile), u=0.50 (50^{th} Percentile) and u=0.75 (75^{th} Percentile), in Equation (13), respectively. The median (Q_2) is obtained from Equation (13) by substituting u=0.5. Therefore, the median is obtained for the APIE distribution as follows:

$$M = \frac{-\lambda}{\log\left(\frac{\log\left(\alpha/2 + 1/2\right)}{\log\alpha}\right)}.$$

Here, 25^{th} percentile and 75^{th} percentile are given by, respectively,

$$Q_1 = \frac{-\lambda}{\log\left(\frac{\log\left(\alpha/4 + 1/4\right)}{\log\alpha}\right)} \quad 0 < u < 1, \qquad Q_3 = \frac{-\lambda}{\log\left(\frac{\log\left(3\alpha/4 + 3/4\right)}{\log\alpha}\right)}, \quad 0 < u < 1.$$

3.3. Skewness and Kurtosis

The coefficient of skewness is a measure of symmetry and the coefficient of kurtosis is also a measure of whether the data are heavy tailed or thin tailed. The Bowley's skewness [17] is based on quartiles as follows:

$$S=\frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)},$$

and the Moors' kurtosis [18] is given below

$$K=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)-Q\left(\frac{3}{8}\right)+Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)},$$

where $Q(.)$ represents the quantile function.

Note that the Bowley's skewness and the Moors' kurtosis can be obtained by $Q(u)$ which is given in Equation (13).

It is important to state that the distribution is symmetric for S = 0. When S > 0, the distribution is positively (right-skewed). For S < 0, the distribution is left-skewed (negatively-skewed). Similarly, as long as K increases the tail of the distribution brings about heavier. A normal distribution has kurtosis exactly 3. If compared to a normal distribution, when $K>3(K<3)$ its tails are longer (shorter) and central peak is higher (lower).

Table 1. *Skewness and kurtosis of the APIE distribution for different parameter values*

Parameters		Skewness	Kurtosis	Median	25th Per.	75th Per.
λ	α					
0.5	2	0.4815	1.7679	2.1471	1.0157	5.3801
	3	0.4817	1.7683	2.4997	1.1550	6.3442
	7	0.4795	1.7613	3.3951	1.5286	8.7013
	20	0.4769	1.7530	4.7537	2.1398	12.1348
2	2	0.4815	1.7679	8.5884	4.0630	21.5203
	3	0.4817	1.7683	9.9990	4.6201	25.3771
	7	0.4795	1.7613	13.5807	6.1145	34.8052
	20	0.4769	1.7530	19.0148	8.5593	48.5395
3	2	0.4815	1.7679	12.8826	6.0945	32.2804
	3	0.4817	1.7683	14.9985	6.9301	38.0656
	7	0.4795	1.7613	20.3711	9.1718	52.2078
	20	0.4769	1.7530	28.5222	12.8389	72.8093
7	2	0.4815	1.7679	30.0594	14.2206	75.3210
	3	0.4817	1.7683	34.9966	16.1703	88.8199
	7	0.4795	1.7613	47.5326	21.4008	121.8184
	20	0.4769	1.7530	66.5519	29.9576	169.8884

The skewness, kurtosis, median and Q_1, Q_3 of the APIE distribution for the several values of the parameters are listed in Table 1. Table 1 indicates that skewness and kurtosis are positive for all values of parameters. For α=2, α=3, the kurtosis and skewness increase whereas these values decrease for α=7 and α=20. The kurtosis and skewness do not vary for all attempted values of parameters. Note that the APIE distribution is right skewed and leptokurtic for all values of the parameters.

3.4. Order Statistics

Order statistics are encountered though many areas of statistical theory and practice. Let $X_1, X_2, ..., X_n$ be a random sample from any APIE distribution. Let $X_{i:n}$ indicate the i^{th} order statistics. Now, we derive the pdf of the i^{th} order statistics $X_{i:n}\left(1 \le i \le n\right)$ for APIE distribution given by

$$f_{i:n}\left(x\right)=\frac{n!}{\left(i-1\right)!\left(n-i\right)!} f\left(x_i\right)\left[F\left(x_i\right)\right]^{i-1}\left[1-F\left(x_i\right)\right]^{n-i},$$

where $f\left(x\right)$ and $F\left(x\right)$ are given in Equation (8) and (7), respectively. Therefore, the pdf for the i^{th} order statistics becomes

$$f_{i:n}\left(x\right)=\frac{n!}{\left(i-1\right)!\left(n-i\right)!}\frac{\log\alpha}{\alpha-1}\frac{\lambda}{x^2}\exp\left(-\frac{\lambda}{x}\right)\alpha^{\exp\left(-\frac{\lambda}{x}\right)}\left[\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-1}\right]^{i-1}\left[1-\left[\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-1}\right]\right]^{n-i}. \tag{14}$$

From Equation (14), for i=1, the pdf of the minimum order statistics of the APIE distribution is shown as

$$f_{1:n}\left(x\right)=\frac{n!}{\left(n-1\right)!} f\left(x_1\right)\left[1-F\left(x_1\right)\right]^{n-1}=nf\left(x_1\right)\left[1-F\left(x_1\right)\right]^{n-1},$$

$$f_{1:n}\left(x\right)=\frac{n!}{\left(n-1\right)!}\frac{\log\alpha}{\alpha-1}\frac{\lambda}{x^2}\exp\left(-\frac{\lambda}{x}\right)\alpha^{\exp\left(-\frac{\lambda}{x}\right)}\left[1-\left[\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-1}\right]\right]^{n-1}, \tag{15}$$

and likewise, the pdf of the maximum order statistics $\left(i=n\right)$ of the APIE distribution as follows:

$$f_{n:n}\left(x\right)=\frac{n!}{\left(n-1\right)!} f\left(x_n\right)\left[F\left(x_n\right)\right]^{n-1}=nf\left(x_n\right)\left[F\left(x_n\right)\right]^{n-1},$$

$$f_{n:n}\left(x\right)=\frac{n!}{\left(n-1\right)!}\frac{\log\alpha}{\alpha-1}\frac{\lambda}{x^2}\exp\left(-\frac{\lambda}{x}\right)\alpha^{\exp\left(-\frac{\lambda}{x}\right)}\left[\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-1}\right]^{n-1}. \tag{16}$$

4. APPLICATION

In this section, we perform two applications of the APIE model to prove empirically its potentiality. We used two data sets that were pre-modeled by different distribution. Then, we provide a comparison of fits of other competitive models. So as to compare the fits of the APIE model with other competing distributions, we consider Akaike Information Criteria (AIC), Corrected Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hannan-Quinn (HQIC), and log-likelihood (LL).

The data sets are modelled with different distributions in some previous studies. These distributions are the weighted Lindley (WL), Lindley (L) distributions from Shanker et al. [19]; three-parameter weighted Lindley (TPWL) distribution from Shanker et al. [20]; IE, generalized inverted exponential (GIE), inverse Rayleigh (IR) distributions from Sharma et al. [22] and Singh et al. [21] and inverse Lindley (IL) distribution from Sharma et al. [22].

The first data set describes the survival times of 55 patients. These patients who treated radiotherapy suffer from Head and Neck cancer disease. This data set which is reported by Efron [23] and descriptive statistics are given in Table 2 and 3, respectively.

Table 2. *The first data set*

6.537 10.42 14.48 16.10 22.70 3441.55 4245.28 49.40 53.62 63 64 83 84 91 108 112 129 133 133 139 140 140 146 149 154 157 160 160 165 146 149 154 157 160 160 165 173 176 218 225 241 248 273 277 297 405 417 420 440 523 583 594 1101 1146 1417

Table 3. *Descriptive statistics for the first data set*

	Mean	Median	Mode	St. D.	Variance	Skewness	Kurtosis	25^{th} P.	75^{th} P.
Data 1	375.2	157	160	737.4	543750.2	4.22	18.859	112	277

It can be noticed from Table 3 that the first data set is right-skewed and leptokurtic with regard to the coefficients of skewness and kurtosis.

The goodness-of-fit statistics and the MLEs of parameters are presented in Tables 4 and 5, respectively.

Table 4. *The goodness-of-fit statistics for the first data set*

Distribution	AIC	CAIC	BIC	HQIC	LL
APIE	757.2254	757.4567	761.3463	758.8306	376.613
IE	773.3742	773.4494	775.4346	774.1767	385.687
GIE	773.1815	773.4127	777.3024	770.1445	384.591
IR	840.1341	840.2094	842.0660	838.6152	419.067
L	765.75	765.82	767.81	764.2312	381.875

Table 5. *The MLEs for the first data set*

Distribution	Estimated Parameters
APIE	51.58173, 23.77790
IE	59.12589
GIE	0.7770681, 49.2410155
IR	741.3652
L	0.008804

On the basis of Table 4, it is quite obvious that APIE distribution provides the overall best fit. For this reason, the proposed distribution can be selected as an adequate distribution when comparing to other distributions to explain the first data set.

The empirical density function and density function of the APIE distribution are showed in Figure 3. It is quite clear from Figure 3 that the APIE model is suitable for the first data set.

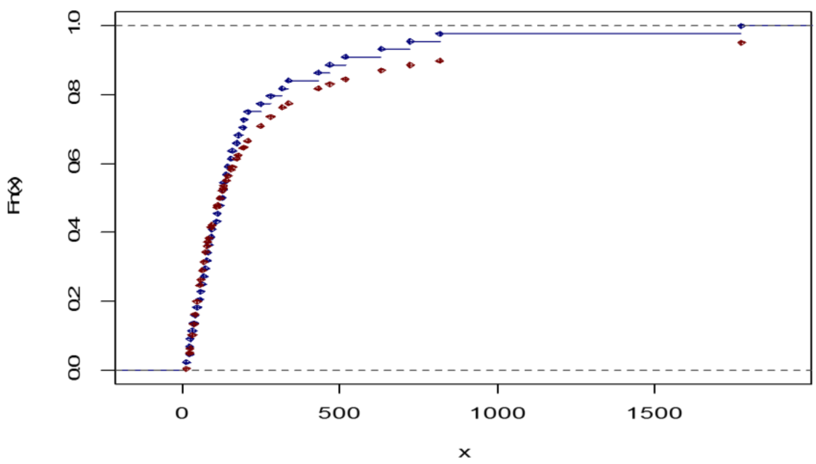

Figure 3. *Plots of empirical cdf and APIE distribution cdf for the first data set.*

The second data set shown in Table 6 includes 44 survival times of patients get Head and Neck cancer disease. Patients are treated using a combination of radiotherapy and chemotherapy (RT+CT). The second data used in this paper was given by Efron [23].

The data set and its descriptive statistics are presented in Table 6 and Table 7, respectively.

Table 6. *The second data set*

12.20 23.56 23.74 25.87 31.98 37 41.35 47.38 55.46 58.36 63.47 68.46 78.26 74.47 81.43 84 92 94 110 112 119 127 130 133 140 146 155 159 173 179 194 195 209 249 281 319 339 432 469 519 633 725 817 1776

Table 7. *Descriptive statistics for the second data set*

	Mean	Median	Mode	St. D.	Variance	Skewness	Kurtosis	25^{th} P.	75^{th} P.
Data 2	223.5	128.5	12.2	305.4	93286.4	3.504	15.387	64.71	239

It can be noticed from Table 7 that the second data set is also right-skewed and leptokurtic with the coefficients of skewness and kurtosis.

The performance of the compared distributions are demonstrated in Table 8. The result shows that when the compare with the other distribution, the APIE distribution has the lowest values of AIC, BIC, CAIC, HQIC and the highest value of LL. From the result, we conclude that the APIE distribution is a very flexible distribution to model right-skewed data sets.

Table 8. *The goodness-of-fit statistics for the second data set*

Distribution	AIC	CAIC	BIC	HQIC	LL
APIE	562.8453	563.138	566.4137	559.7090	279.423
IE	571.0622	572.8689	572.8689	569.4935	284.531
GIE	572.4309	572.7226	576.0443	569.2930	284.215
IL	690.2096	690.3052	691.9938	688.6415	344.105
IR	962.7151	962.8112	964.4993	961.1475	480.358
TPWL	569.4500	570.05	568.3803	564.7445	281.725

The MLEs of parameters for the second data set is presented in Table 9.

Table 9. *The MLEs for the second data set*

Distribution	Estimated Parameters
APIE	2.191138, 61.800462
IE	75.3793
GIE	1.1799, 83.8998
IL	77.6755
IR	2547.4170
TPWL	0.0047801, 0.0484017, -0.077115

The cumulative density functions of the empirical distributions and APIE distribution are demonstrated in Figure 4. As is seen from Figure 4, the APIE model can be chosen for the second data set.

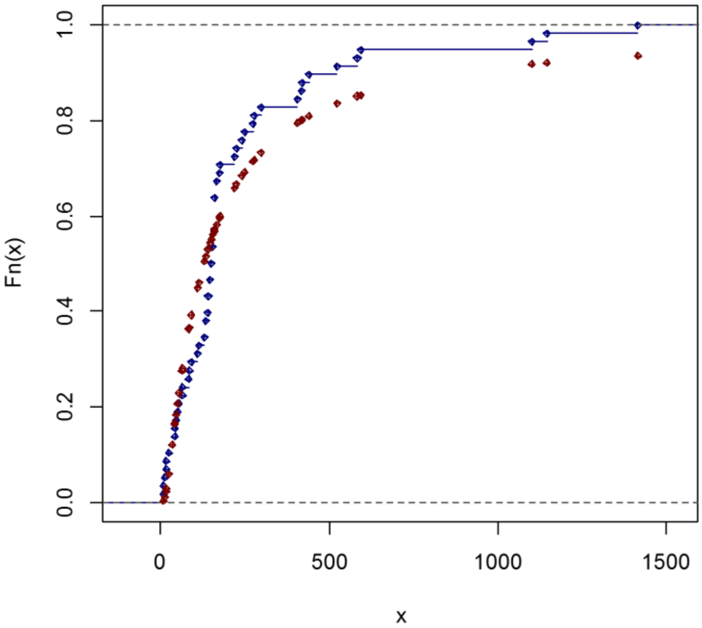

Figure 4. *Plot of empirical cdf and APIE distribution cdf for the second data set*

5. CONCLUSION

In this study, the Alpha Power Inverted Exponential (APIE) distribution is obtained. Some important statistical properties of the APIE distribution are obtained including survival, hazard rate and quantile functions, skewness, kurtosis. As seen from the plots of the hazard rate function the proposed distribution could be useful to model data sets with increasing and decreasing failure rates. Then, we provide two real data application and show that the APIE distribution is the better than the other compared distributions for the right-skewed data sets.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Keller, A.Z., Kamath, A.R.R., Perera, U.D., “Reliability analysis of CNC machine tools”, Reliability Engineering, 3(6): 449-473, (1982).

[2] Lin, C.T., Duran, B.S., Lewis, T.O., “Inverted gamma as a life distribution”, Microelectronics Reliability, 29(4): 619-626, (1989).

[3] Oguntunde, P.E., Adejumo, A., Owoloko, E.A., “On the exponentiated generalized inverse exponential distribution”, World Congress on Engineering, London, 80-83, (2017).

[4] Singh, S.K., Singh, U., Kumar, D., “Bayes estimators of the reliability function and parameter of inverted exponential distribution using informative and non-informative priors”, Journal of Statistical Computation and Simulation, 83(12): 2258-2269, (2013).

[5] Dey, S., “Inverted exponential distribution as a life distribution model from a bayesian viewpoint”, Data Science Journal, 6: 107-113, (2007).

[6] Abouammoh, A.M., Alshingiti, A.M., "Reliability estimation of generalized inverted exponential distribution", Journal of Statistical Computation and Simulation, 79(11): 1301-1315, (2009).

[7] Oguntunde, P.E., Adejumo, A., Balogun, O.S., "Statistical properties of the exponentiated generalized inverted exponential distribution", Applied Mathematics, 4(2): 47-55, (2014).

[8] Singh, S.K., Singh, U., Kumar, M., "Estimation of parameters of generalized inverted exponential distribution for progressive Type-II censored sample with binomial removals", Journal of Probability and Statistics, Doi:10.1155/2013/183652, (2013).

[9] Oguntunde, P.E., Babatunde, O.S., Ogunmola, A.O., "Theoretical analysis of the kumaraswamy-inverse exponential distribution", International Journal of Statistics and Applications, 4(2): 113-116, (2014).

[10] Oguntunde, P., Adejumo, O., "The transmuted inverse exponential distribution", International Journal of Advanced Statistics and Probability, 3(1): 1-7, (2014).

[11] Oguntunde, P.E., "Generalisation of the inverse exponential distribution: statistical properties and applications", Phd. Thesis, Covenant University College of Science and Technology, Ota, Ogun State, 128-142 (2017).

[12] Mahdavi, A., Kundu, D., "A new method for generating distributions with an application to exponential distribution", Communications in Statistics - Theory and Methods, 46(13): 6543-6557, (2017).

[13] Lee, C., Famoye, F., Alzaatreh, A.Y., "Methods for generating families of univariate continuous distributions in the recent decades", Wiley Interdisciplinary Reviews: Computational Statistics, 5(3): 219-238, (2013).

[14] Dey, S., Alzaatreh, A., Zhang, C., Kumar, D., "A new extension of generalized exponential distribution with application to ozone data", Ozone: Science & Engineering, 39(4): 273-285, (2017).

[15] Dey, S., Sharma, V.K., Mesfioui, M., "A new extension of weibull distribution with application to lifetime data", Annals of Data Science, 4(1): 31-61, (2017).

[16] Isik, H., Turfan, D., Ozel, G., "The double truncated dagum distribution with applications", International Journal of Mathematics and Statistics, 18(2): 61-78, (2017).

[17] Kenney, J.F., Mathematics of Statistics Part 1., Chapman & Hall, London, (1939).

[18] Moors, J.J.A., "A quantile alternative for kurtosis", The Statistician, 37(1): 25-32. (1988).

[19] Shanker, R., Shukla, K.K., Fesshaye, H., "On weighted lindley distribution and its applications to model lifetime data", Jacobs Journal of Biostatistics, 1(1): 002, (2016).

[20] Shanker, R., Shukla, K.K., Mishra, A., "A three-parameter weighted lindley distribution and its applications to model survival time", Statistics, 18(2): 291-310, (2017).

[21] Singh, S.K., Singh, U., Yadav, A., Viswkarma, P.K., "On the estimation of stress strength reliability parameter of inverted exponential distribution", International Journal of Scientific World, 3(1): 98-112, (2015).

[22] Sharma, V.K., Singh, S.K., Singh, U., Agiwal, V., “The inverse lindley distribution: A stress-strength reliability model”, arXiv: 1405.6268, (2014).

[23] Efron, B., “Logistic regression, survival analysis and the kaplan-meier curve”, Journal of the American Statistical Association, 83(402): 414-425, (1988).