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Unbounded Vectorial Cauchy Completion of Vector Metric Spaces

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Highlights
• Unbounded vectorial convergence in vector metric spaces is studied.
• Relations between uo-convergence and $u_{d,E}$-convergence are examined.
• An unbounded Cauchy completion of any vector metric space is constructed.
• A more general completion of vector metric spaces is obtained.

Abstract

A sequence (a_n) in a Riesz space E is called uo-convergent (or unbounded order convergent) to $a \in E$ if $|a_n - a| \wedge u \overset{o}{\to} 0$ for all $u \in E_+$ and unbounded order Cauchy (uo-Cauchy) if $|a_n - a_{n+p}|$ is uo-convergent to 0. In the first part of this study we define $u_{d,E}$-convergence (or unbounded vectorial convergence) in vector metric spaces, which is more general than usual metric spaces, and examine relations between unbounded order convergence, unbounded vectorial convergence, vectorial convergence and order convergence. In the last part we construct the unbounded Cauchy completion of vector metric spaces by the motivation of the fact that every metric space has Cauchy completion. In this way, we have obtained a more general completion of vector metric spaces.

1. INTRODUCTION

Riesz spaces are partially ordered vector spaces in which structure of the space and partial order is compatible, and for any two elements their supremum exists. A Riesz space E is called Dedekind complete if every nonempty subset of E has a supremum whenever it is bounded from above. For any decreasing sequence (a_n) of E, we write $a_n \downarrow 0$ if $inf\, a_n = 0$. Besides, the Riesz space E is called Archimedean if $n^{-1}a \downarrow 0$ for every $a \in E_+$. One of the central tools in Riesz spaces is order convergence. A sequence (a_n) of E is called o-convergent (or order convergent) to $a \in E$ and written as $a_n \overset{o}{\to} a$ if there exists a sequence (b_n) in E such that $b_n \downarrow 0$ and $|a_n - a| \leq b_n$ for all *n*. Every order convergent sequence is order bounded, but this kind of convergence cannot be applied to sequences which are not order bounded. However, there is a more general convergence applicable to not only all bounded but also some unbounded sequences in a Riesz space. A sequence (a_n) in a Riesz space E is called uo-convergent (or unbounded order convergent) to $a \in E$ if $|a_n - a| \wedge u \overset{o}{\to} 0$ for all $u \in E_+$. Also, a sequence (a_n) in a Riesz space is called unbounded order Cauchy (or uo-Cauchy) if $|a_n - a_{n+p}|$ is uo-convergent to 0. It was first defined by H. Nakano in [1] as "individual convergence". Later, R. Demarr [2] used the name "unbounded order convergence" and he showed that one can embed a locally convex space E into a particular ordered vector space in which uo-convergence is equivalent to topological convergence in *E*. Demarr also gave examples indicate the resemblance and difference of o-convergence and uo-convergence. If E is Riesz space of real valued functions defined on a set *X*, o-convergence and uo-convergence coincide if and only if *X* is finite; moreover, these types of convergence coincide for sequence even if *X* is infinite. If *E* is Riesz space of real valued integrable functions defined on [0,1], o-convergence and uo-convergence do not coincide for

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sequences. After that, in Dedekind complete Riesz spaces which have weak units, S. Kaplan constituted two characterizations of uo-convergence [3]. Besides, uo-convergent and uo-Cauchy nets are used in characterization of Banach lattices which have the positive Schur property and KB spaces by N. Gao and F. Xanthos [4]. Later, more properties of uo-convergence are studied by N. Gao, V. Troitsky and F. Xanthos [5]. For other facts regarding Riesz spaces we point to references [6,7].

Vector metric spaces equipped with a Riesz space E valued distance map are defined in [8,9]. A vector metric d on a non-empty set X is a map $d : X \times X \to E$ satisfying the followings:

$$d(x,y) = 0 \text{ if and only if } x = y \quad \text{and} \quad d(x,y) \le d(x,z) + d(y,z)$$

for all $x, y, z \in X$. The set X or the triple (X, d, E) is called a vector metric space. Also, vectorial convergence is defined in this type of spaces and examined some properties of this convergence. A sequence (x_n) in X vectorially converges (simply E-converges) to $x \in X$, which is written as $x_n \xrightarrow{d,E} x$, if there exists a sequence a_n in E such that $d(x_n, x) \le a_n$ and $a_n \downarrow 0$ for all n. A nonempty subset A of X is called E-bounded if there is a nonzero $a \in E_+$ such that $d(x,y) \le a$ for all $x, y \in X$. As the order convergence is not applicable to unbounded order sequences in a Riesz space E, the E-convergence is also not applicable to sequences which are not E-bounded in a vector metric space X. In this study, we define unbounded vectorially convergent and unbounded vectorial Cauchy sequences, and examine some properties of them. In addition to that we construct the unbounded Cauchy completion of a given vector metric spaces.

2. UNBOUNDED VECTORIAL CONVERGENCE

Throughout this section we assume X as a vector metric space (X, d, E).

Definition 1. Let (x_n) be a sequence in X and $x \in X$. A sequence (x_n) in X is said to be unbounded vectorially convergent (or $u_{d,E}$-convergent) to x if $d(x_n, x) \wedge u \xrightarrow{o} 0$ for all $u \in E_+$. We will denote this vectorial convergence by $x_n \xrightarrow{u_{d,E}} x$.

Definition 2. A sequence (x_n) in X is said to be unbounded E-Cauchy (or u_E-Cauchy) sequence if for every $u \in E_+$ there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \wedge u \le a_n$ for all p and n.

Definition 3. The vector metric space X is said to be u_E-complete if every u_E-Cauchy sequence in X is $u_{d,E}$-convergent.

Assume (X, d, E) is a vector metric space, $x \in X$ and $x_n \xrightarrow{u_{d,E}} x$. Then

1. The limit x is unique,

2. Every subsequence of (x_n) is $u_{d,E}$-convergent to x,

3. If $x_n \xrightarrow{u_{d,E}} x$ and $y_n \xrightarrow{u_{d,E}} y$ then $d(x_n, y_n) \xrightarrow{uo} d(x,y)$.

Moreover, the followings are satisfied.

4. Every $u_{d,E}$-convergent sequence is u_E-Cauchy sequence,

5. If a u_E-Cauchy sequence (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{u_{d,E}} x$, then $x_n \xrightarrow{u_{d,E}} x$.

6. If (x_n) and (y_n) are u_E-Cauchy sequences in X, then $d(x_n, y_n)$ is uo-Cauchy sequence in E.

Now we will give a simple example.

Example 1. Let $X = \mathbb{R}$ and $E = \mathbb{R}^2$. We know that $\mathbb{R}^2$ is an Archimedean Riesz space with coordinatwise ordering. Define a function $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ as $d(x,y) = \left(\frac{1}{2}|x-y|, \frac{1}{3}|x-y|\right)$. It is easy to see that (X,d,E) is a vector metric space. Let also (x_n) be a sequence whose general term is $x_n = \frac{1}{n}$. For any $u \in E_+$ and for any $n \in \mathbb{N}$, we get $d(x_n, 0) \wedge u \le \left(\frac{2}{3n}, \frac{2}{3n}\right)$. By taking $a_n = \left(\frac{2}{3n}, \frac{2}{3n}\right)$ we can observe that (a_n) is a sequence in E and $a_n \downarrow (0,0)$. Therefore $x_n \xrightarrow{u_{d,E}} 0$.

Now we will give a very important example which emphasize the relation between unbounded vectorial convergence and unbounded order convergence.

Example 2. By taking $X = E$ and $d(x,y) = |x-y|$, unbounded vectorial convergence become simply unbounded order convergence.

The following theorem presents us the relation between vectorial convergence and unbounded vectorial convergence.

Theorem 1. Every vectorial convergent sequence is unbounded vectorial convergent.

Proof. Let (X,d,E) be a vector metric space and (x_n) be a vectorial convergent sequence in X. Also let $x_n \xrightarrow{d,E} x$. Then there exists a sequence (a_n) in E such that $d(x_n, x) \le a_n$ and $a_n \downarrow 0$. Since given $u \in E_+$ $d(x_n, x) \wedge u \le d(x_n, x) \le a_n$ for all u in E_+, then we have $x_n \xrightarrow{u_{d,E}} x$.

However, the converse of previous theorem need not to be true.

Example 3. Let c_0 be the set of real sequences converging to zero, $X = E = c_0$ and $d(x,y) = |x-y|$ for all $x, y \in c_0$. It is easy to see that vectorial convergence become order convergence and as we have already mentioned unbounded vectorial convergence become unbounded convergence. Suppose that (e_n) is a sequence whose terms are standard unit vectors in c_0. Since the sequence (e_n) is not ordered bounded, $e_n \not\xrightarrow{o} 0$ in c_0. However we can observe that $e_n \xrightarrow{uo} 0$.

Definition 4. Let X be a vector metric space and Y is a nonempty subset of X. Then Y is said to be u_E-dense whenever for every $x \in X$, there exists a sequence (x_n) in Y satisfying $x_n \xrightarrow{u_{d,E}} x$.

A map $f: X \to Y$ is said to be E-isometry if $d(x,y) = \rho(f(x), f(y))$ for all $x, y \in X$ whenever (X,d,E) and (Y,p,E) are vector metric spaces. Besides, X and Y are called E-isometric whenever f is onto [8].

3. UNBOUNDED CAUCHY COMPLETION OF VECTOR METRIC SPACES

Suppose (X,d,E) is a vector metric space but it is not u_E-complete. In this section, we are aiming to build a u_E-complete vector metric space $(\hat{X}, \hat{d}, E)$ of (X,d,E).

Definition 5. Let (X,d,E) be a vector metric space and $(\hat{X}, \hat{d}, E)$ be a u_E-complete vector metric space. If X is E-isometric to a u_E-dense subset of $\hat{X}$ then $(\hat{X}, \hat{d}, E)$ is called unbounded Cauchy completion (shortly u_E-Cauchy completion) of (X,d,E).

Theorem 2. Every vector metric space (X,d,E) has a u_E-completion $(\hat{X}, \hat{d}, E)$.

Proof. First of all, we need to define a map $\varphi: X \to \hat{X}$ which is an E-isometry and $\varphi(X)$ is u_E-dense in $\hat{X}$. Suppose E is a uo-Cauchy complete Riesz space and also suppose (x_n) and (y_n) are two u_E-Cauchy sequences in X. Now we define an equivalence relation over the set of all u_E-Cauchy sequences in X as

$$(x_n) \frown (y_n) \Leftrightarrow d(x_n, y_n) \overset{uo}{\to} 0.$$

We use the notation $\hat{x}, \hat{y}, \dots$ for the equivalence classes of these u_E-Cauchy sequences and $\hat{X}$ for the set of these equivalence classes. Let us define a map $\hat{d}: \hat{X} \times \hat{X} \to E$ as

$$\hat{d}(\hat{x}, \hat{y}) = \text{uo-lim } d(x_n, y_n).$$

Due to that $(d(x_n, y_n))$ is a uo-Cauchy sequence whenever (x_n) and (y_n) are u_E-Cauchy sequences in X, and E is uo-Cauchy complete, this map is meaningful. Also it is well defined because uo-limit is unique. Moreover, $(\hat{X}, \hat{d}, E)$ is a vector metric space. By using the fact that for all $x, y \in X$, the constant sequences $(x_n) = (x, x, \dots)$ and $(y_n) = (y, y, \dots)$ are u_E-Cauchy sequences in X. Besides, by $(x, x, \dots) \in \hat{x}$ and $(y, y, \dots) \in \hat{y}$ we define the map $\varphi: X \to \hat{X}$ as $\varphi(x) = \hat{x}$. Since

$$\hat{d}(\varphi(x), \varphi(y)) = \hat{d}(\hat{x}, \hat{y}) = \text{uo-lim } d(x_n, y_n) = d(x, y),$$

φ is an E-isometry, and hence (X, d, E) is E-isometric to $(\varphi(X), \hat{d}, E)$.

Now we will show that $\varphi(X)$ is u_E-dense in $\hat{X}$. Let $(x_n) \in \hat{x}$ and, for fixed n_0 we can consider $\varphi(x_{n_0})$ as an element of $\varphi(X)$. It is clear that $\varphi(x_{n_0})$ is the equivalence class generated by the constant sequence $(x_{n_0}, x_{n_0}, \dots)$. For every $n \geq n_0$ we can find an integer p such that $n = n_0 + p$. For every $u \in E_+$ there is a sequence (a_n) in E such that $(a_n) \downarrow 0$ and

$$\hat{d}(\varphi(x_{n_0}), \hat{x}) = \text{uo-lim } d(x_{n_0}, x_n) = \text{uo-lim } d(x_{n_0}, x_{n_0+p}) \leq a_{n_0}$$

holds for all $n \geq n_0$ with $n = n_0 + p$ since (x_n) is u_E-Cauchy. Therefore, we get $\hat{d}(\varphi(x_{n_k}), \hat{x}) \leq a_{n_k}$ for all k, in other words, $\varphi(x_{n_k}) \overset{u_{d,E}}{\longrightarrow} \hat{x}$.

Now we will show that $(\hat{X}, \hat{d}, E)$ is u_E-complete vector metric space generated by (X, d, E). Suppose $(\hat{x}_n)$ is a u_E-Cauchy sequence in $\hat{X}$. Since $\varphi(X)$ is u_E-dense in $\hat{X}$, we can find y_n in X and (a_n) in E with $\hat{d}(\varphi(y_n), \hat{x}_n) \leq a_n$ for each n. For all $u \in E_+$ and all p and n we have

$$d(y_n, y_{n+p}) \wedge u = \hat{d}(\varphi(y_n), \varphi(y_{n+p})) \wedge u$$
$$\leq \hat{d}(\varphi(y_n), \hat{x}_n) + \hat{d}(\hat{x}_n, \hat{x}_{n+p}) + \hat{d}(\hat{x}_{n+p}, \varphi(y_{n+p})) \leq 3a_n.$$

Therefore (y_n) is a u_E-Cauchy in X. Let $\hat{y}$ is equivalence class generated by (y_n), since $\varphi(x_n) \overset{u_{d,E}}{\longrightarrow} \hat{y}$ we get

$$\hat{d}(\hat{y}, \hat{x}_n) \leq \hat{d}(\hat{y}, \varphi(y_n)) + \hat{d}(\varphi(y_n), \hat{x}_n) \leq 2a_n$$

which means $x_n \overset{u_{d,E}}{\longrightarrow} \hat{y}$. Hence $(\hat{X}, \hat{d}, E)$ is u_E-complete.

Theorem 3. Completion of a vector metric space is unique.

Proof. We will follow standard procedure, taking two u_E-completion of X and showing that they are isometrically equivalent to each other. Suppose $(\hat{X}, \hat{d}, E)$ is the completion of X as previously mentioned and $(\tilde{X}, \tilde{d}, E)$ is also u_E-complete vector metric spaces such that there exists an E-isometry $\psi: X \to \tilde{X}$ with $\psi(X)$ is u_E-dense in $\tilde{X}$.

First of all we define a map $f_0 : \varphi(X) \to \psi(X)$ as $f_0(\varphi(x)) = \psi(X)$ for all $x \in X$. It can be easily seen that $\varphi(X)$ and $\psi(X)$ are E-isometric. We know that there is a u_E-Cauchy sequence $\varphi(x_n)$ in $\varphi(X)$ such

that $\varphi(x_n) \xrightarrow{u_{d,E}} \hat{x}$ if $\hat{x} \in \hat{X}$. Due to that f_0 is an E-isometry, $\psi(x_n) = (f_0 \circ \varphi(x_n))$ is also an u_E-Cauchy sequence in $\tilde{X}$. By using u_E-completeness of $\tilde{X}$, we can say that there exists $\tilde{x}$ in $\tilde{X}$ such that $\psi(x_n) \xrightarrow{u_{d,E}} \tilde{x}$. Now we can define a map $f : \hat{X} \to \tilde{X}$ with $f(\hat{x}) = \tilde{x}$. As we know, for any $\hat{x}, \hat{y} \in \hat{X}$ there exist sequences $\varphi(x_n)$ and $\varphi(y_n)$ in $\varphi(X)$ such that $\varphi(x_n) \xrightarrow{u_{d,E}} \hat{x}$ and $\varphi(y_n) \xrightarrow{u_{d,E}} \hat{y}$. Besides, we can observe that $\psi(x_n) \xrightarrow{u_{d,E}} f(\hat{x})$ and $\psi(y_n) \xrightarrow{u_{d,E}} f(\hat{y})$. By using completeness part we get

$$\tilde{d}(f(\hat{x}), f(\hat{y})) = \text{uo-lim } \tilde{d}(\psi(x_n), \psi(y_n)) = \text{uo-lim } \hat{d}(\varphi(x_n), \varphi(y_n)) = \hat{d}(\hat{x}, \hat{y})$$

because f_0 is an E-isometry. Finally, we are showing that f is surjective. For any $\tilde{x} \in \tilde{X}$ there exists a u_E-Cauchy sequence $\psi(x_n)$ in $\psi(X)$ such that $\psi(x_n) \xrightarrow{u_{d,E}} \tilde{x}$. Due to that $\varphi(X)$ is E-isometric to $\psi(X)$, it is deduced that $\varphi(x_n)$ is a u_E-Cauchy sequence in $\hat{X}$. Since f is an E-isometry, $f(\varphi(x_n)) \xrightarrow{u_{d,E}} f(\hat{x})$ whenever $\hat{x}$ is $\hat{d}$-limit of $\varphi(x_n)$. For each n we have

$$\tilde{d}(\tilde{x}, f(\hat{x})) \leq \tilde{d}(\tilde{x}, \psi(x_n)) + \tilde{d}(f(\hat{x}), f(\varphi(x_n)))$$

because $f(\varphi(x_n)) = \psi(x_n)$. Hence $f(\hat{x}) = \tilde{x}$ is satisfied in $\tilde{X}$.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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