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Gumbel-geometric Distribution: Properties and Applications

Bamidele Mustapha OSENI[1,*], Hassan M. OKASHA[2,3]

[1]*Federal University of Technology Akure, Department of Statistics, PMB 704, Akure, Nigeria*
[2]*Al- Azhar University, Faculty of Science, Department of Mathematics, 11884, Nasr City, Cairo, Egypt*
[3]*King Abdul Aziz University, Faculty of Science, Department of Statistics, 21589, Jeddah, Saudi Arabia*

Highlights

- A three-parameter generalization of the Gumbel distribution, called Gumbel-geometric distribution.
- Explicit expressions for the properties of the distribution.
- Maximum likelihood estimation of the parameters of the distribution.
- Asymptotic properties examined through simulation experiment.
- Flexibility of the distribution illustrated by comparing with some existing distributions.

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Abstract

A three-parameter generalization of the Gumbel distribution, which we call Gumbel-geometric distribution, is defined and investigated. The shape of the density and hazard function is examined and discussed. Explicit expressions for the moment generating function, the characteristics function and the *r*th order statistic are obtained. Other properties of the distribution are also discussed. The method of maximum likelihood is proposed for the estimation of the parameter of the model and discussed. A simulation experiment is carried out to examine the asymptotic properties of the distribution. The result shows that the MSE decreases to zero as $n \to \infty$ while the bias either increases or decreases (depending on the sign) for each of the parameters. The new distribution is applied to two datasets and compared to some existing generalization to illustrate its flexibility.

1. INTRODUCTION

The Gumbel distribution sometimes referred to as the log-Weibull distribution is an extreme value distribution of type I, which is particularly useful when the underlying sample is distributed as exponential or normal. It is mostly used in modelling of maximum (or minimum) values of random variables, especially when the maximum (or minimum) values are collected as a list of observations. This distribution which finds usage in engineering and hydrology can be traced back to a discussion on mean largest distance, from the origin, of *n* random points on a straight line having a fixed length, [1, 2]. The distribution function (cdf) and the density function (pdf) of the distribution are respectively given by,

$$F_G(x;\mu,\sigma) = \exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}, \qquad x \in (-\infty,\infty) \text{ and} \tag{1}$$

$$f_G(x;\mu,\sigma) = \frac{1}{\sigma}\exp\left\{-\frac{x-\mu}{\sigma} - \exp\left(-\frac{x-\mu}{\sigma}\right)\right\}, \quad x \in (-\infty,\infty) \tag{2}$$

where $-\infty < \mu < \infty$ is the location parameter and $\sigma > 0$ is the scale parameter.
The Gumbel pdf is unimodal with mode μ and skewed to the right. The distribution may be perceived as an extension of the exponential distribution with an increasing or decreasing hazard function [3]. Several

*Corresponding author, e-mail: bmoseni@futa.edu.ng

generalizations and modifications have been carried out on the distribution to increase its flexibility. One of such generalization is the exponentiated Gumbel (EG) distribution introduced by Nadarajah [4] with cdf,

$$F_{EG}(x) = 1 - \left[1 - \exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{\alpha}. \tag{3}$$

The distribution (3) possesses some interesting physical interpretation similar to the exponentiated exponential distribution, in the sense that it gives the distribution of the lifetime of a system comprising *n*-components in series, each independently and identically distributed (iid) according to (1) [4]. Following the work of Eugene, Lee and Famoye [5], Nadarajah and Kotz [6] proposed the beta-Gumbel (BG) distribution. The BG distribution was found to be a generalization of the arcsine distribution, which arises frequently in statistical communication theory. Another interesting modification was constructed by Cordeiro, Nadarajah and Ortega [7] using the Kum-G distribution which was obtained from the works of Kumaraswamy [8] and Jones [9]. The resulting distribution, called the Kumaraswamy-Gumbel (KG), is the time to failure of a system with *n* independent components. Each component is assumed to have *m* independent sub-components and the failure of sub-component results into the failure of the entire system.

This work introduces another generalization of the Gumbel distribution, which we call Gumbel-geometric distribution. In the sense of reliability as expressed by Gupta and Kundu [10], Nadarajah [4] and Cordeiro, Nadarajah and Ortega [7], the distribution has a nice physical interpretation and may be constructed from reliability studies. The lifetime distribution of a system of '*n*' iid components arranged in series, each component having a cdf $G(x)$ is given by the conditional distribution function,

$$P(X \le x \mid N = n) = 1 - \left(1 - G(x)\right)^{n} \qquad x \in \mathrm{R},\ n \in \mathrm{N} \tag{4}$$

where R is a real number set and N is a set of natural numbers. Since a system with series components fails if a component fail, it is assumed that the number of components *N* at a specific time is a random variable having geometric distribution. That is, *N* is the number of components functioning before the last failure, thus the joint probability distribution is given by

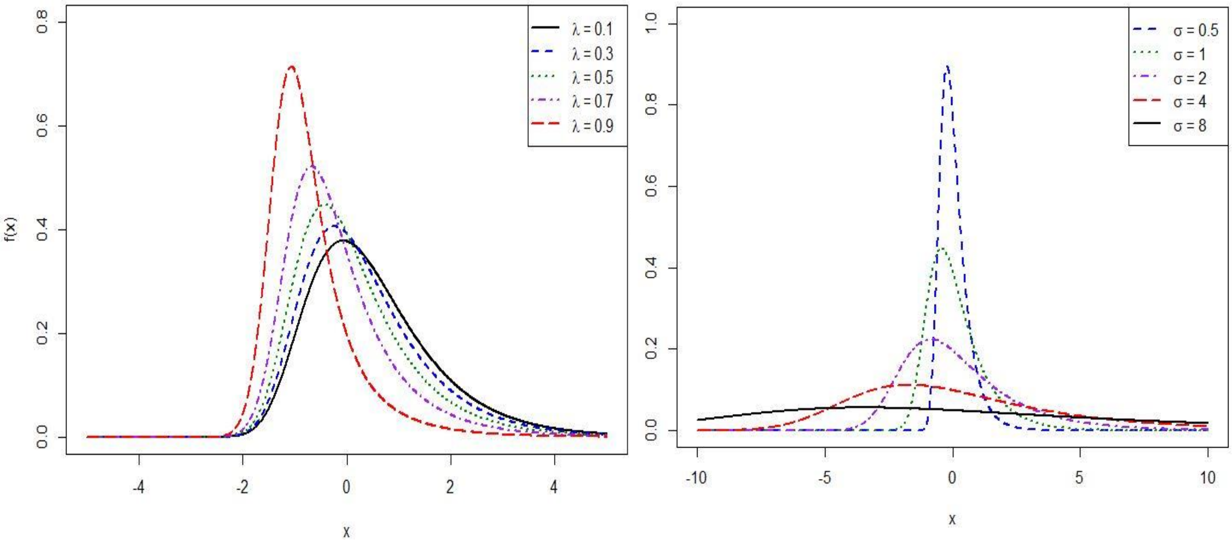
$$P(X \le x, N = n) = \lambda^{n-1}(1-\lambda)\left[1 - \left(1 - G(x)\right)^{n}\right]. \tag{5}$$

The lifetime distribution of the system is therefore given by the marginal of (5) as

$$P(X \le x) = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1}\left[1 - \left(1 - G(x)\right)^{n}\right] = \frac{G(x)}{1-\lambda+\lambda G(x)}. \tag{6}$$

The marginal distribution function in (6) can be used as a generator of "G-geometric" distribution where "G" is the baseline distribution. The generator given in (6) provides a way of generating distributions that are geometric extreme stable [11]. The baseline distribution function may be interpreted as the distribution of each independent identically distributed component in the system. In this work, we assume that the function $G(x)$ is the distribution function of the Gumbel distribution. This is reasonable, especially in reliability sense, since the Gumbel distribution models the extremes, and the biggest value of a parallel (or smallest value of a series) sub-component determines the strength of the component. Despite several emphases on reliability, the resulting distribution function can be applied as an alternative to other generalizations of the Gumbel distribution; such as Kumaraswamy-Gumbel, beta-Gumbel and exponentiated exponential Gumbel distributions; with greater flexibility.

2. THE GUMBEL-GEOMETRIC DISTRIBUTION



(a) $\sigma=1$ (b) $\lambda=0.5$

Figure 1. *The probability density function of GG for* $\mu=0$ *and some selected values of* σ *and* λ

A random variable *X*, with values $x\in(-\infty,\infty)$, is a Gumbel-geometric (GG) random variable with parameters $\mu\in(-\infty,\infty)$, $\sigma\in(0,\infty)$ and $\lambda\in[0,1)$, if the cdf is given by

$$F(x)=\exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right)\left(1-\lambda+\lambda\exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right)\right)^{-1}. \tag{7}$$

The probability density function (pdf) of the GG random variable is given by

$$f(x)=\frac{(1-\lambda)}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}-\exp\left(-\frac{x-\mu}{\sigma}\right)\right)\left(1-\lambda+\lambda\exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right)\right)^{-2}, \tag{8}$$

where μ is the location parameter and σ is the scale parameter inherited from the Gumbel distribution. Further flexibility is added by the parameter λ as shown in Figure 1.

The distribution (8) reduces to Gumbel distribution when $\lambda=0$. The cdf and the pdf may be written in the series form using the well-known binomial theorems

$$(1+y)^{-n}=\sum_{k=0}^{\infty}\frac{(n+k-1)!}{(n-1)!k!}(-1)^k y^k \quad \text{for } |y|<1 \tag{9}$$

$$\text{and } (a+by)^n=\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}a^{n-k}(by)^k \ . \tag{10}$$

Clearly, the expression, $\lambda\left[\exp\left(-\exp\left(-(x-\mu)/\sigma\right)\right)-1\right]$, has an absolute value smaller than unity, when $\lambda\in[0,1)$. Therefore, the cdf and pdf of GG distribution can be written as

$$F(x)=\sum_{j=0}^{\infty}\lambda^j S_j(x)F_G(x) \text{ and}$$

$$f(x)=(1-\lambda)\sum_{j=0}^{\infty}(j+1)\lambda^{j}S_j(x)f_G(x) \tag{11}$$

where $F_G(x)$ and $f_G(x)$ are as defined in (1) and (2) respectively and S_j denotes

$$S_j(x)=\sum_{k=0}^{j}\frac{j!}{(j-k)!k!}(-1)^k\exp\left(-k\exp(-(x-\mu)/\sigma)\right).$$

3. LIMITS AND SHAPE OF THE DISTRIBUTION

The limits of $f(x)$, the density function of GG as $x\to-\infty$ and as $x\to\infty$ are both zero. The behaviour of $f(x)$ between the two limit points can be examined by determining the nature of the turning points. Since both $f(x)$ and $\log\left(f(x)\right)$ have the same shape, $\log\left(f(x)\right)$ is examined for its simpler mathematical tractability.

The derivative of $\log\left(f(x)\right)$, where $f(x)$ is the pdf of the GG distribution, is given by,

$$\frac{d\log\left(f(x)\right)}{dx}=\frac{1}{\sigma}\left(\nu-1-\frac{2\lambda\nu\exp(-\nu)}{1-\lambda+\lambda\exp(-\nu)}\right) \tag{12}$$

where $\nu=\exp(-(x-\mu)/\sigma)$. The turning points of the curve $f(x)$ are at the points where

$$(\nu-1)\exp(\nu)/(\nu+1)=\lambda/(1-\lambda). \tag{13}$$

These points correspond to the modes of $f(x)$ and the nature of the points are determined by $d^2\log\left(f(x)\right)/dx^2=u(x)$, where $u(x)$ is given by

$$u(x)=\frac{2\lambda\nu}{\sigma^2\left(1-\lambda+\lambda\exp(-\nu)\right)^2}\left(\nu\exp(-\nu)-\lambda\nu\exp(-2\nu)-\exp(-\nu)\right)-\frac{\nu}{\sigma^2}.$$

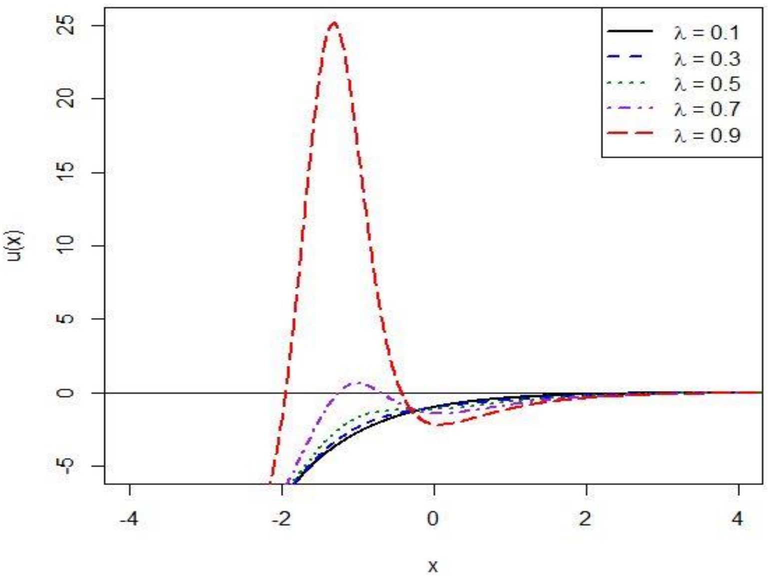


Figure 2. *Shapes of $u(x)$ for $\mu=0$, $\sigma=1$ and some selected values of λ*

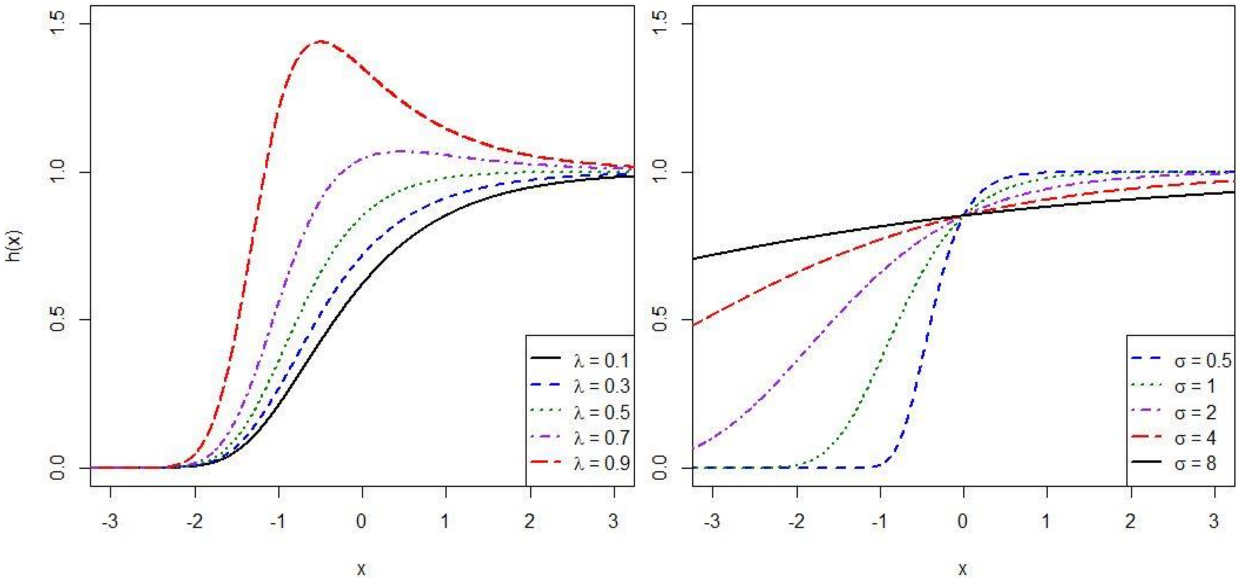
Depending on whether $u(x_0)<0$, $u(x_0)>0$ $u(x_0)=0$, where $x=x_0$ is a solution of (13), the turning points can be a local maximum, a local minimum or a point of inflexion. It is worthy of note to mention that when $\lambda=0$, $x=\mu$ is a root of (13) and $u(\mu)<0$. This conforms to the behaviour of Gumbel distribution. Figure 2 shows the shapes of $u(x)$ for $\mu=0$, $\sigma=1$ and some selected values of λ.

4. HAZARD AND QUANTILE FUNCTIONS

One of the most important quantities for characterizing life phenomena is the hazard rate function. The hazard rate function of the GG distribution is given by,

$$h(x)=\frac{\nu\exp(-\nu)}{\sigma\left(1-\exp(-\nu)\right)\left(1-\lambda+\lambda\exp(-\nu)\right)} \tag{14}$$

where $\nu=\exp\left(-(x-\mu)/\sigma\right)$.



(a) $\sigma=1$ (b) $\lambda=0.5$

Figure 3. *Hazard rate function curves of GG for* $\mu=0$ *and some selected values of* σ *and* λ

The shape of the hazard rate function is sometimes considered when considering the suitability of a distribution function in describing a dataset. This shape which can be increasing, decreasing, or "bath-tube" is determined for the GG distribution in a similar manner as that of $f(x)$ in section (3). Clearly, $h(x)\to 0$ as $x\to-\infty$ and $h(x)\to 1$ as $x\to\infty$. The shape between the two limit points is determined by examining the nature of the roots of

$$\left(2\lambda-2\exp(-\nu)-\lambda\nu\exp(-\nu)-1\right)\left(\nu+1\right)^{-1}\exp(-\nu)=1-\lambda\ . \tag{15}$$

The plot of hazard function for various values of the parameters is shown in Figure 3. Increase in the parameter λ, while other parameters are kept constant, increases the mode of the hazard function. The shape parameter σ also changes the shape of the hazard function.

Another important quantity of a distribution is the quantile function which is generally defined as the inverse distribution function and may be used in place of distributions for modelling datasets [9, 12]. The quantile function for the GG distribution is defined by

$$F^{-1}(p) = \mu - \sigma \ln\left(-\ln\left(\frac{p(\lambda-1)}{\lambda p - 1} \right)\right). \tag{16}$$

The quantile (16) can be used in the simulation of the GG random variable, if the variable p in (16) is assumed to be uniformly distributed and all parameters are fixed as desired.

5. OTHER PROPERTIES

5.1. Moments and Generating Functions

The moments of random variables are very important ways of summarizing the random variables in terms of their parameters.

Proposition 5.1. Suppose X is a random variable with GG density function defined in (8), then the nth moment of X can be written as,

$$\mathrm{E}(X^n) = (1-\lambda)\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{n} \frac{(-1)^{k+m}\lambda^j \mu^{n-m}\sigma^m n!(k+1)!}{m!k!(n-m)!(j-k)!} I(m,k) \tag{17}$$

where $I(m,k) = \left(\frac{\partial}{\partial \alpha}\right)^m \left[(k+1)^{-\alpha}\Gamma(\alpha) \right]\Big|_{\alpha=1}$.

Proof. The proof of this proposition follows a similar pattern to section 4, of Nadarajah [4]. Since the density function of *X* is as defined in (8), then the *n*th moment of *X* can be written as

$$\mathrm{E}(X^n) = \frac{(1-\lambda)}{\sigma}\int_{-\infty}^{\infty} x^n \exp\left(-\frac{x-\mu}{\sigma} - \exp\left(-\frac{x-\mu}{\sigma}\right)\right)\left(1-\lambda+\lambda\exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right)\right)^{-2} dx. \tag{18}$$

Setting $v = \exp(-(x-\mu)/\sigma)$, (18) can be re-expressed as

$$\mathrm{E}(X^n) = (1-\lambda)\int_0^{\infty} (\mu - \sigma \log v)^n \exp(-v)\left(1-\lambda+\lambda\exp(-v)\right)^{-2} dv \ . \tag{19}$$

Using the binomial expansion (10) to expand the first bracket in the integral, (19) can be written as

$$\mathrm{E}(X^n) = (1-\lambda)\sum_{m=0}^{n}\binom{n}{m}\mu^{n-m}(-\sigma)^m I(m) \tag{20}$$

where the integral $I(m)$ is defined as

$$I(m) = \int_0^{\infty} (\log v)^m \exp(-v)\left(1-\lambda+\lambda\exp(-v)\right)^{-2} dv. \tag{21}$$

Further expansion of (21) using the representation (9) and (10), yields

$$I(m)=\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(j+1)!}{(j-k)!k!}\lambda^{j}(-1)^{k}I(m,k) \tag{22}$$

where the integral . $I(m,k)=\int_{0}^{\infty}(\log\nu)^{m}\exp\left(-(k+1)\nu\right)d\nu$.

Using (2.6.21.1) of Prudnikov, Brychkov and Marichev [13], the integral $I(m,i)$ can be evaluated as

$$I(m,k)=\left(\frac{\partial}{\partial\alpha}\right)^{m}\left[(k+1)^{-\alpha}\Gamma(\alpha)\right]\Big|_{\alpha=1} \tag{23}$$

Combining (20), (22) and (23) completes the proof.

The first few evaluations of (23) for $m=0,1,2,3$ and 4 which are useful in calculating the first four moments are

$$I(0,k)=\frac{1}{k+1}\ , \qquad I(1,k)=\frac{\Psi(1)-\log(k+1)}{k+1},$$

$$I(2,k)=\frac{1}{k+1}\left(\log(k+1)^{2}-2\log(k+1)\Psi(1)+\Psi(1,1)+\Psi(1)^{2}\right),$$

$$I(3,k)=\frac{1}{k+1}\left(-\log(k+1)^{3}+3\log(k+1)^{2}\Psi(1)-3\log(k+1)\Psi(1,1)-3\log(k+1)\Psi(1)^{2}\right.$$
$$\left.+\Psi(2,1)+\Psi(1,1)\Psi(1)+\Psi(1)^{3}\right) \text{ and}$$

$$I(4,k)=\frac{1}{k+1}\left(\log(k+1)^{4}-4\log(k+1)^{3}\Psi(1)+6\log(k+1)^{2}\Psi(1,1)+6\log(k+1)^{2}\Psi(1)^{2}\right.$$
$$-4\log(k+1)\Psi(2,1)-12\log(k+1)\Psi(1,1)\Psi(1)-4\log(k+1)\Psi(1)^{3}+\Psi(3,1)$$
$$\left.+4\Psi(2,1)\Psi(1)+3\Psi(1,1)^{2}+6\Psi(1,1)\Psi(1)^{2}+\Psi(1)^{4}\right)$$

where $\Psi(z)=\Gamma'(z)/\Gamma(z)$ is the digamma function.

Corollary 5.1. Suppose X is a random variable which follows GG distribution defined in (7), then the mean and second moment are given by

$$\mathrm{E}(X)=(1-\lambda)\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(j+1)!(-1)^{k}\lambda^{j}}{(1+k)!(j-k)!}(\sigma\log(1+k)+\mu-\sigma\Psi(1)) \quad \text{and} \tag{24}$$

$$\mathrm{E}(X^{2})=\frac{(1-\lambda)}{6}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(j+1)!(-1)^{k}\lambda^{j}}{(1+k)!(j-k)!}F_{2}(k) \tag{25}$$

$$F_2(k)=(6\sigma^2\Psi(1)^2-12\log(k+1)\sigma^2\Psi(1)+6\sigma^2\log(k+1)+\pi^2\sigma^2-12\mu\sigma\Psi(1)$$
$$+12\mu\sigma\log(k+1)+6\mu^2).$$

Proof. It is easy to establish the proof from (17).

Other characteristics of the distribution can be calculated from the moments. For example, the variance, skewness, and kurtosis can be calculated, respectively, using the relations

$$\mathrm{Var}(X)=\mathrm{E}(X^2)-\mathrm{E}^2(X), \qquad \kappa_3(X)=\frac{\mathrm{E}(X^3)-3\mathrm{E}(X)\mathrm{E}(X^2)+2\mathrm{E}^3(X)}{\mathrm{Var}^{3/2}(X)} \text{ and}$$

$$\kappa_4(X)=\frac{\mathrm{E}(X^4)-4\mathrm{E}(X)\mathrm{E}(X^3)+6\mathrm{E}(X^2)\mathrm{E}^2(X)-3\mathrm{E}^4(X)}{\mathrm{Var}^2(X)}.$$

Table 1. *Some descriptive statistic for some values of the parameters*

μ	σ	λ	$\mathrm{E}(X)$	$\mathrm{Var}(X)$	$\kappa_3(X)$	$\kappa_4(X)$
0	1	0.1	0.50484	1.59439	0.23255	14.01036
		0.3	0.33757	1.47490	0.20425	13.55952
		0.5	0.12563	1.31895	0.13373	13.11875
	5	0.1	2.52421	39.85983	0.23255	14.01036
		0.3	1.68785	36.87247	0.20425	13.55952
		0.5	0.62816	32.97373	0.13373	13.11876
	10	0.1	5.04842	159.43932	0.23255	14.01036
		0.3	3.37570	147.48988	0.20425	13.55952
		0.5	1.25633	131.89489	0.13373	13.11875
5	1	0.1	5.04842	1.59439	0.23255	127.08818
		0.3	5.33757	1.47490	0.20425	128.99401
		0.5	5.12563	1.31895	0.13373	132.56086
	5	0.1	7.52421	39.85983	0.23255	21.57318
		0.3	6.68785	36.87247	0.20425	20.23255
		0.5	5.62816	32.97373	0.13373	18.81087
	10	0.1	10.04842	159.43932	0.23255	16.85097
		0.3	8.37570	147.48988	0.20425	15.94980
		0.5	6.25633	131.89485	0.13373	14.82757
10	1	0.1	10.50484	1.59439	0.23255	484.32528
		0.3	10.33757	1.47490	0.20425	447.83202
		0.5	10.12563	1.31895	0.13370	479.45847
	5	0.1	12.52421	39.85983	0.23255	36.66236
		0.3	11.68785	36.87247	0.20425	35.32486
		0.5	10.62816	32.97373	0.13373	33.60114
	10	0.1	15.04842	159.43932	0.23255	21.57318
		0.3	13.37570	147.48988	0.20425	20.37412
		0.5	11.25633	131.89482	0.13372	18.81087

The values of mean, variance, skewness and kurtosis for some selected values of the parameters are shown in Table 1. It is observed from the Table that increasing the value of μ increases the value of mean and kurtosis but variance and skewness remain unaffected. Also, increasing the value of σ, increases the value of mean and variance while skewness remains unaffected. Kurtosis decreases with increase in σ for values of $\mu>0$ but is unaffected by σ when $\mu=0$. Finally, increasing the value of the parameter λ results in a decrease in the values of all four properties, when $\lambda \le 0.5$. Routine calculations show that the properties are unstable for all values of $\lambda > 0.5$.

Proposition 5.2. Suppose X is a random variable with GG density function defined in (8), then the moment generating function of X can be written as,

$$M_X(t) = (1-\lambda)\exp(\mu t)\Gamma(1-\sigma t)\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k \lambda^j (j+1)!}{k!(j-k)!}(k+1)^{(\sigma t-1)} . \quad (26)$$

Proof. By definition, the moment generating function of a random variable is defined by

$$M_X(t) = \int \exp(tx) f(x) dx .$$

Since the density function of *X* is defined by (8), using the series representation in (11), the moment generating function of *X* can be expressed as

$$M_X(t) = (1-\lambda)\exp(\mu t)\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k \lambda^j (j+1)!}{k!(j-k)!} I(k,v) \quad (27)$$

where $v = \exp(-(x-\mu)/\sigma)$ and $I(k,v)$ denotes the integral

$$I(k,v) = \int_0^{\infty} v^{-\sigma t} \exp\left(-(k+1)v\right) dv .$$

Using (2.3.3.1) of Prudnikov, Brychkov and Marichev [13], the integral $I(k,v)$ can be expressed as

$$I(k,v) = (k+1)^{(\sigma t-1)}\Gamma(1-\sigma t) . \quad (28)$$

Substituting (28) into (27) completes the proof.

Proposition 5.3. Let X be a random variable with GG density function defined in (8), then the characteristics function of X is defined by,

$$\Phi_X(t) = (1-\lambda)\exp(i\mu t)\Gamma(1-i\sigma t)\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k \lambda^j (j+1)!}{k!(j-k)!}(k+1)^{(i\sigma t-1)} \quad (29)$$

where $i = \sqrt{-1}$.

Proof. The proof follows a similar procedure as that of Proposition 2.

5.2. Order Statistics

One of the oldest models for ordered random variables, which naturally arises whenever observations are listed in increasing order of magnitude, is the order statistics [14]. Order statistics is also quite useful in modelling extremes.

Proposition 5.4. Suppose the random variables $X_1, X_2, ..., X_n$ are independent and identically distributed GG random variables. The density function $f_{r:n}(x)$ of the rth order statistics, for $r = 1,...,n$ is given by,

$$f_{r:n}(x)=\frac{(1-\lambda)v}{\sigma B(r,n-r+1)}\sum_{k=0}^{n-1}(-1)^k\binom{n-r}{k}\exp(-(k+r)v)\left(1-\lambda+\lambda\exp\left(-v\right)\right)^{-(1+k+r)} \tag{30}$$

where $v=\exp(-(x-\mu)/\sigma)$.

Proof. The density function of the rth order statistics with pdf $f(x)$ and cdf $F(x)$ is generally defined by

$$f_{r:n}(x)=\frac{f(x)}{B(r,n-r+1)}\sum_{k=0}^{n-1}(-1)^k\binom{n-r}{k}F(x)^{k+r-1}.$$

Since $X_r; r=1,...,n$ are distributed as GG, replacing $f(x)$ and $F(x)$ with the density and distribution functions in (7) and (8) and simplifying completes the proof.

Corollary 5.2. Suppose $X_1, X_2,..., X_n$ are random samples from GG distribution. $v=\exp(-(x-\mu)/\sigma)$, then the density function;

a) $f_{1:n}(x)$ of the 1st order statistics is,

$$f_{1:n}(x)=\frac{(1-\lambda)v}{\sigma B(1,n)}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\exp(-(k+1)v)\left(1-\lambda+\lambda\exp\left(-v\right)\right)^{-(k+2)},$$

b) $f_{n:n}(x)$ of the nth order statistics is.

$$f_{n:n}(x)=\frac{(1-\lambda)v}{\sigma B(1,n)}\sum_{k=0}^{n-1}\frac{1}{k!k!}\exp(-(k+n)v)\left(1-\lambda+\lambda\exp\left(-v\right)\right)^{-(1+k+n)}.$$

Proof. The proof of part (a) easily follows by substituting $r=1$ into (30). Replacing "*r*" with "*n*" in (30) and using the relation $(-k)!=(-1)^k k!$ (see, Thukral [15]) completes the proof of part (b).

5.3. Entropy

An important measure of the variation of the uncertainty of a random variable is entropy [16, 17]. Several measures of entropy exist and the most popular is the Renyi entropy defined as

$$R(\eta)=(1-\eta)^{-1}\log\left(\int f^{\eta}(x)dx\right) \text{ for all } \eta>0,\ \eta\neq 1.$$

Proposition 5.5. Suppose $f(x)$ is the density function of a random variable X which has GG distribution as defined in (7). The entropy of X is defined by,

$$R(\eta)=\log\sigma+\frac{1}{(1-\eta)}\left(\eta\log(1-\lambda)+\log\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^j\Gamma(2\eta+j)}{(j-1)!k!}\left((1+k)\eta\right)^{-\eta}\right) \tag{31}$$

for all $\eta>0,\ \eta\neq 1$.

Proof. From the definition, the entropy of the random variable X with a distribution defined in (7) can be expressed as

$$R(\eta)=\log\sigma+\frac{1}{(1-\eta)}\left(\eta\log(1-\lambda)+\log I(\eta)\right) \tag{32}$$

where $I(\eta)=\int_0^\infty v^{\eta-1}\exp(-\eta v)\left(1-\lambda+\lambda\exp(-\eta v)\right)^{-2\eta}dv$.

Using (9) and (10), the integral $I(\eta)$ can be re-expressed as

$$I(\eta)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^j\Gamma(2\eta+j)}{(j-1)!k!\Gamma(\eta)}I(k,\eta) \tag{33}$$

where $I(k,\eta)$ denotes the integral $I(k,\eta)=\int_0^\infty v^{\eta-1}\exp(-(1+k)\eta v)dv$.

Finally, by (2.3.3.1) in Prudnikov, Brychkov and Marichev [13], $I(k,\eta)$ can be written as

$$I(k,\eta)=\left((1+k)\eta\right)^{-\eta}\Gamma(\eta). \tag{34}$$

Combining (32), (33) and (34) completes the proof.

6. PARAMETER ESTIMATION

6.1. Maximum Likelihood Estimation

The parameter estimation by the method of maximum likelihood is considered. Let $x_i(i=1,2,...,n)$ be a random sample of size *n* drawn from the GG distribution. The logarithm of the likelihood function (log-likelihood) is given by

$$L(x;\lambda,\mu,\sigma)=n\log(1-\lambda)-n\log(\sigma)-2\sum_{i=1}^{n}\log\left(1-\lambda+\lambda\exp\left(-v_i\right)\right)+\sum_{i=1}^{n}\log v_i-\sum_{i=1}^{n}v_i \tag{35}$$

where $v_i=\exp\left(-(x_i-\mu)/\sigma\right)$. Taking the derivatives with respect to the parameters yields

$$\frac{\partial L}{\partial\lambda}=-\frac{n}{1-\lambda}-2\sum_{i=1}^{n}\left(\exp(-v_i)-1\right)\left(1-\lambda+\lambda\exp\left(-v_i\right)\right)^{-1}, \tag{36}$$

$$\frac{\partial L}{\partial\mu}=\frac{n}{\sigma}-\frac{1}{\sigma}\sum_{i=1}^{n}v_i+\frac{2\lambda}{\sigma}\sum_{i=1}^{n}v_i\exp\left(-v_i\right)\left(1-\lambda+\lambda\exp\left(-v_i\right)\right)^{-1}, \tag{37}$$

$$\frac{\partial L}{\partial\sigma}=-\frac{n}{\sigma}-\frac{1}{\sigma}\sum_{i=1}^{n}\log(v_i)+\frac{1}{\sigma}\sum_{i=1}^{n}v_i\log(v_i)-\frac{2\lambda}{\sigma}\sum_{i=1}^{n}v_i\log(v_i)\exp\left(-v_i\right)\left(1-\lambda+\lambda\exp\left(-v_i\right)\right)^{-1}. \tag{38}$$

The maximum likelihood estimates of the parameters are obtained by equating (36) - (38) to zero and solving the resulting equations simultaneously. The solution of (36) – (38) is best computed iteratively since it does not exist in closed form. Suitable initial estimates of the parameters are obtained by transforming the data to Gumbel density and using appropriate values of μ and σ. The variance of the estimates may be computed iteratively through the Fisher's information matrix, which is obtained from (36)-(38) by taking expectations of the second partial derivatives.

6.2. Simulation

Table 2. *Biases and MSE of the GG distribution from the simulation experiment*

GG (λ,μ,σ)	N	λ		μ		σ	
		Bias	MSE	Bias	MSE	Bias	MSE
GG(0.5,2.0,1.0)	25	-0.096743	0.153059	-0.047019	0.915716	-0.578120	0.481725
	50	-0.088382	0.136800	0.006319	0.649567	-0.543854	0.459821
	100	-0.067663	0.122915	0.055731	0.487212	-0.561520	0.493270
	150	-0.063915	0.109565	0.071267	0.372915	-0.548206	0.487465
GG (0.5,-0.7,0.3)	25	-0.115592	0.156073	-0.008089	0.092022	-0.006297	0.013442
	50	-0.100965	0.139438	-0.000421	0.066713	0.002255	0.010156
	100	-0.086876	0.127351	0.010349	0.049512	0.007734	0.007503
	150	-0.080416	0.112944	0.015528	0.037216	0.007771	0.005910
GG(0.9,0.8,1.0)	25	-0.203154	0.160342	-0.170790	0.594653	-0.561002	0.452162
	50	-0.152186	0.107085	-0.104249	0.464068	-0.549168	0.452693
	100	-0.120414	0.078398	-0.081997	0.367149	-0.549168	0.452693
	150	-0.004411	0.016748	-0.057431	0.169030	0.042555	0.023141

The behaviour of the maximum likelihood estimates (MLEs) of the parameters of GG distribution is investigated through a simulation experiment. Different values are assigned to the parameters and samples of sizes $n = 25,$ 50, 100 and 150 are generated using the quantile function in (16). The MLEs of the parameters are computed for each of the sample sizes and the variances are obtained iteratively through the information matrices. This experiment is replicated 1,000 times. The mean squared errors (MSE) and the biases which are used in the evaluation of the parameters are computed using

$$\text{Bias}(\hat{\theta}) = \frac{1}{r}\sum_{i=1}^{r}\left(\hat{\theta}_i - \theta\right) \quad \text{and} \quad \text{MSE}(\hat{\theta}) = \text{Var}(\theta) + \text{Bias}(\theta)^2$$

where r, θ and $\hat{\theta}$ are the number of replications, the parameter being investigated and the estimate of the parameter.

The values of the MSE and biases of GG distribution, with different pre-assigned values of parameters λ, μ and σ, from the Monte Carlo experiment are shown in Table 2. It is observed that the MSE decrease to zero as $n \to \infty$ for each of the parameters, while the biases either increase or decrease depending on the sign.

7. APPLICATION

The flexibility of the distribution model in relation to the Gumbel distribution and other existing generalization are illustrated using datasets from Hinkley [18] and Changery [19]. In the first instance, the appropriateness of the GG distribution in fitting the data is established using the Kolmogorov-Smirnov (*K*-S) test. Then, a comparison of the fit of the GG distribution to the Gumbel distribution is carried out using the likelihood ratio test.

The three other generalizations of Gumbel distribution, namely (a) the beta-Gumbel (BG), (b) the Kumaraswamy-Gumbel (KG) and (c) the exponentiated generalized Gumbel (EGG), are also considered and compared with the GG distribution. The density functions of the distributions, respectively, may be written as

$$f_{BG}(x)=\frac{1}{\sigma B(\alpha,\beta)}u\exp(-\alpha u)(1-u)^{\beta-1}; \qquad f_{KG}(x)=\frac{\alpha\beta}{\sigma}u\exp(-\alpha u)(1-\exp(-\alpha u))^{\beta-1}$$

$$f_{EGG}(x)=\frac{\alpha\beta}{\sigma}u\exp(-u)\left(1-\exp(-u)\right)^{\alpha-1}\left(1-\left[1-\exp(-\text{u})\right]^{\alpha-1}\right)^{\beta-1}$$

where $u=\exp(-(x-\mu)/\sigma)$, $\alpha>0,\ \ \beta>0,\ \ \sigma>0$ and $-\infty<x,\mu<\infty$.

The maximum likelihoods of the parameters and other statistics are computed to illustrate the flexibility of the distribution in relation to these generalizations.

7.1. The Precipitation Data

Table 3. *Precipitations in March (inches) for Minneapolis/St Paul by Hinkley [18]*

0.77	1.74	0.81	1.20	1.95	1.20	0.47	1.43	3.37	2.20	3.00	3.09	1.51	2.10	0.52
1.62	1.31	0.32	0.59	0.81	2.81	1.87	1.18	1.35	4.75	2.48	0.96	1.89	0.90	2.05

The data presented in Table 3 consist of 30 successive values of precipitation (inches) in March for the twin cities of Minneapolis-St Paul . The data is obtained from [20] but originally appeared in Hinkley [18].

The MLEs of the parameters of GG distribution obtained from the data are λ = 0.3338130, μ = 1.4135194 and σ = 0.800269. The observed information matrix of the data $I_0(\hat{\lambda},\hat{\mu},\hat{\sigma})$ and the variance-covariance matrix $I_0^{-1}(\hat{\lambda},\hat{\mu},\hat{\sigma})$ are respectively given by

$$I_0(\hat{\lambda},\hat{\mu},\hat{\sigma})=\begin{pmatrix} 91.67922 & -36.59910 & 0.64240 \\ -36.59910 & 57.75100 & -30.13512 \\ 0.64240 & -30.13512 & 21.72661 \end{pmatrix} \text{ and } I_0^{-1}(\hat{\lambda},\hat{\mu},\hat{\sigma})=\begin{pmatrix} 0.08522 & 0.19075 & 0.26205 \\ 0.19075 & 0.48963 & 0.67349 \\ 0.26205 & 0.67349 & 0.97241 \end{pmatrix}.$$

The 99% confidence intervals are [-1.599348, 2.266207], [0.041795, 2.784765] and [0.227983, 1.372319] respectively, for each of the parameters λ, μ and σ. The hypotheses,

$$H_0: F=F_{GG} \text{ versus } H_1: F\neq F_{GG}$$

are formulated for a test on how well the GG distribution appropriately fits the data. The appropriateness of the model is determined using the *K-S* distances between the empirical and fitted distribution functions. The value of the *K-S* statistic and corresponding *p*-value are 0.064322 and 0.9997 respectively. The small *K-S* statistic and the large *p*-value is an indication that the GG distribution appropriately fits the data. A comparison of how well the GG model fits the data in relation to the Gumbel distribution is carried out using the likelihood ratio test (LRT). Since the model reduces to the Gumbel distribution when $\lambda=0$, the hypotheses, $H_0:\lambda=0\ (G)$ versus $H_1:\lambda\neq 0\ (GG)$ are formulated. The likelihood value for the GG distribution is given by 38.65111, while the likelihood ratio statistic and the corresponding *p*-value are 3.4890 and 0.06174 respectively. The chi-square critical value (3.8415) for this test is greater than the calculated LRT statistic. A *p*-value of 0.06174 is an indication that the null hypothesis cannot be rejected. Accordingly, it is concluded that the λ is insignificant to the fit of the data at 0.05 level of significance. This is an instance where it may be desirous to fits the data with the Gumbel distribution which is a sub-model of the GG distribution.

The parameter estimates and the goodness of fit statistic for the precipitation data in Table 2 using the GG distribution and other generalizations are shown in Table 4. The log-likelihood for the three other generalizations (BG, KG and EGG) considered, are respectively 38.34992, 38.35189 and 37.72457. The K-S distance between the empirical and the fitted distribution functions are respectively 0.075806 with a *p*-value of 0.9953, 0.07550 with a *p*-value of 0.9955 and 0.079419 with a *p*-value of 0.9915. All distributions fit the data appropriately based on the small *K-S* distance and large *p*-values. Using other criteria such as the AIC, CAIC, and BIC to select the best model, the GG distribution provides the best fits for the data, since it has the lowest value of all statistic but the other three distributions compete favourably. The plot of the empirical and fitted distributions is given in Figure 4.

Table 4. *Estimates and goodness of fit statistic for the Precipitation data*

Dist.	BG	KG	EGG	GG
MLE	$\sigma=$ 0.43020	$\sigma=$ 0.42058	$\sigma=$ 0.12707	$\sigma=$ 0.80027
	$\mu=$ 0.91299	$\mu=$ 1.46377	$\mu=$ 0.13323	$\mu=$ 1.41352
	$\alpha=$ 0.83006	$\alpha=$ 0.21064	$\alpha=$ 0.14004	$\lambda=$ 0.33381
	$\beta=$ 0.44498	$\beta=$ 0.43489	$\beta=$ 2.55665	
K-S	0.075806	0.07550	0.079419	0.064322
P-value	0.9953	0.9955	0.9915	0.9997
AIC	84.69983	84.70377	83.44915	83.30222
CAIC	86.29983	86.30377	85.04915	84.22530
BIC	90.30462	90.30856	89.05394	87.50582

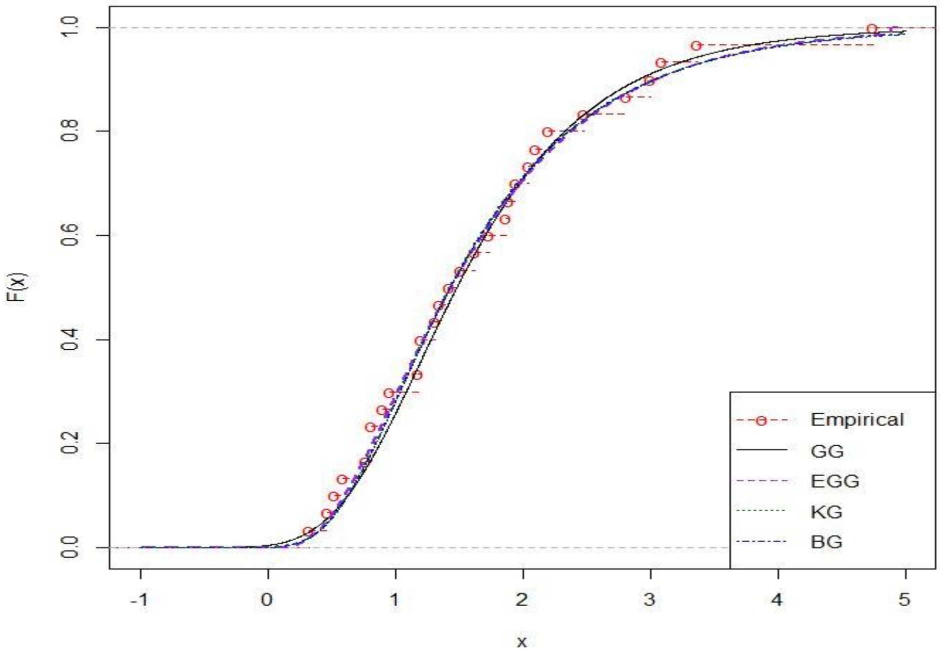


Figure 4. *The Empirical and fitted distributions for the Precipitation data*

7.2. The Maximum Annual Wind Speed Data

The maximum annual wind speed (mph) for Hartford, Connecticut between 1940 and 1979 is taken from Changery [19]. The data is historic and has appeared in several articles including Kinnison [21]. This data is presented in Table 5.

Table 5. *Maximum Annual Wind Speed for Hartford, Connecticut from 1940 - 1979*

31	22	12	25	32	55	27	21	31	34	57	54	48	46	49	40
44	40	34	45	22	16	20	18	32	21	42	19	26	29	43	

The MLEs of the parameters of the GG model that fits the data are $\lambda = 8.0784187$, $\mu = 49.3806330$ and $\sigma = 0.8126432$. The observed information matrix of the data $I_0(\hat{\lambda},\hat{\mu},\hat{\sigma})$ and the variance-covariance matrix $I_0^{-1}(\hat{\lambda},\hat{\mu},\hat{\sigma})$ are respectively given by

$$I_0(\hat{\lambda},\hat{\mu},\hat{\sigma}) = \begin{pmatrix} 2.57574 & -1.62437 & 14.38917 \\ -1.62437 & 1.47604 & -20.54154 \\ 14.38917 & -20.54154 & 380.28214 \end{pmatrix} \text{ and } I_0^{-1}(\hat{\lambda},\hat{\mu},\hat{\sigma}) = \begin{pmatrix} 13.68708 & 31.63956 & 1.19117 \\ 31.63956 & 75.86799 & 2.90095 \\ 1.19117 & 2.90095 & 0.11426 \end{pmatrix}.$$

The 99% confidence intervals are[0.140483, 1.486047], [31.996970, 66.799017] and [0.698382, 15.468667] respectively, for each of the parameters λ, μ and σ. The hypotheses,

$$H_0 : F = F_{GG} \text{ versus } H_1 : F \neq F_{GG}$$

are formulated for a test on how well the GG distribution appropriately fits the data. Similar to the precipitation data, the appropriateness of the model is determined using the *K-S* distances between the empirical and fitted distribution functions. The value of the *K-S* statistic and corresponding *p*-value are 0.12657 and 0.5434 respectively. The small *K-S* statistic and the large *p*-value is an indication that the GG distribution fits the data appropriately. Similar to the precipitation data, the hypotheses $H_0 : \lambda = 0\ (G)$ versus $H_1 : \lambda \neq 0\ (GG)$ are formulated for comparing how well the GG model fits the data in relation to the Gumbel distribution. The likelihood value for the GG distribution is given by 132.1358, while the likelihood ratio statistic and the corresponding *p*-value are 349.79 and 2.2×10^{-16} respectively. It is noted that the chi-square critical value (3.8415) for this test is lesser than the calculated LRT statistic. A *p*-value of 2.2×10^{-16} indicates that the null hypothesis should be rejected. Accordingly, it is concluded that the λ significantly contributes to the appropriateness of the fit of the data at 0.05 level of significance. In this case, the GG distribution provides a better fit to the data than the Gumbel distribution. The parameter estimates and the goodness of fit statistics using the GG distribution and other generalizations are presented in Table 6.

Table 6. *Estimates and goodness of fit statistic for the maximum Annual wind speed data*

Dist.	BG	KG	EGG	GG
MLE	$\sigma=$ 4.75179	$\sigma=$ 4.30621	$\sigma=$ 14.83043	$\sigma=$ 8.078419
	$\mu=$ 52.85498	$\mu=$ 41.75345	$\mu=$ 17.32013	$\mu=$ 49.380633
	$\alpha=$ 0.10331	$\alpha=$ 0.83304	$\alpha=$ 2.13788	$\lambda=$ 0.812643
	$\beta=$ 0.62021	$\beta=$ 0.65629	$\beta=$ 45.14888	
K-S	0.12069	0.12916	0.13212	0.12657
P-value	0.60480	0.51690	0.48740	0.5434
AIC	272.4618	272.8937	279.10270	270.2716
CAIC	273.6046	274.0366	280.24550	270.9382
BIC	279.2173	279.6493	285.85820	275.3382

The log-likelihood for the three generalizations; BG, KG and EGG, are 132.2309, 132.4469 and 135.5513 respectively. The K-S distance between the empirical and the fitted distribution functions are respectively 0.12069, 0.12916 and 0.13212 while the *p*-values are 0.60480, 0.51690 and 0.48740 respectively. All distributions are likely fit for the data since the *p*-values are all greater than 0.05. The best model for the data based on the three criteria of AIC, CAIC and BIC is the GG distribution since it has the lowest value of all criteria while the EGG can be said to be the least appropriate for the data. The plot of the empirical and fitted distributions is given in Figure 5.

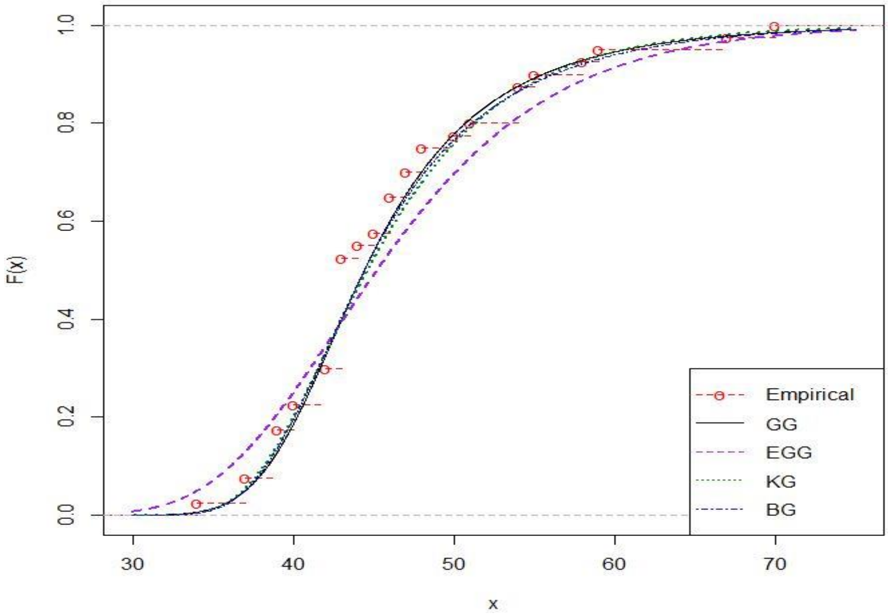


Figure 5. *The Empirical and fitted distributions for the maximum Annual wind speeds data*

8. CONCLUSION

A generalization of the Gumbel distribution, called the Gumbel-geometric, is defined and investigated. Several properties of the distribution such as the moments, hazard function, quantile function and the characteristics function are derived and studied. The shapes of the function are also investigated. This distribution which has a nice physical interpretation is quite flexible and has only three parameters unlike several other generalizations such as the beta-Gumbel, Kumaraswamy-Gumbel and exponentiated exponential Gumbel with four parameters. A simulation experiment conducted to examine the asymptotic properties of the distribution shows that the MSE decreases to zero as $n \to \infty$ while the bias either increases or decreases (depending on the sign) for each of the parameters. The model is applied to two real datasets and compared to these other generalizations to illustrate its flexibility.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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