Research Article GU J Sci 35(4): 1585-1595 (2022) DOI: 10.35378/gujs.838411

Gazi University
Journal of Science
http://dergipark.gov.tr/gujs

Cholesky Factorization of the Generalized Symmetric $k-$ Fibonacci Matrix

Cahit KOME*

Nevşehir Hacı Bektaş Veli University, Department of Information Technology, 50300, Nevşehir, Turkey.

Highlights

- We focus on factorizations and inverse factorizations of special lower triangular matrices.
- We propose several new identities of the $k-$Fibonacci sequence.
- We give Cholesky factorization of generalized some special symmetric matrices.

Article Info

Received: 10 Dec 2020
Accepted: 25 Oct 2021

Keywords

Generalized
$k-$Fibonacci matrix
Symmetric generalized
$k-$Fibonacci matrix
Cholesky factorization

Abstract

Matrix methods are a useful tool while dealing with many problems stemming from linear recurrence relations. In this paper, we discuss factorizations and inverse factorizations of two kinds of generalized $k-$Fibonacci matrices. We derive some useful identities of the $k-$Fibonacci sequence. We investigate the Cholesky factorization of the generalized symmetric $k-$Fibonacci matrix by using these identities.

1. INTRODUCTION

The Fibonacci and Lucas numbers arise in several fields such as mathematics, physics, computer science, and related fields. These numbers have attracted the attention of researchers for years. Until now, several studies have been conducted on the applications and generalization of these sequences. An interesting generalization of the Fibonacci sequence, $k-$Fibonacci sequence, $\{F_{k,n}\}_{n=0}^{\infty}$, was presented by Falcon and Plaza [1]. For $k \in \mathbb{R}^+$ and $n \in \mathbb{N}_0$, the $k-$Fibonacci numbers are defined by

$$F_{k,n+2} = kF_{k,n+1} + F_{k,n}, \quad F_{k,0} = 0, \quad F_{k,1} = 1. \tag{1}$$

In particular, for $k = 1$ and $k = 2$, we obtain the Fibonacci and Pell numbers respectively. Moreover, the ratio of the quotient of two successive terms of $k-$Fibonacci numbers converges to $r_1(k) = \frac{k+\sqrt{k^2+4}}{2}$, which is the positive root of the equation $r^2 - kr - 1 = 0$. Moreover, the $k-$Fibonacci numbers are generated by the powers of the following 2×2 companion matrix:

$$\begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{k,n+1} & F_{k,n} \\ F_{k,n} & F_{k,n-1} \end{bmatrix}. \tag{2}$$

Now, we denote the set of all $n \times n$ matrices by $\mathcal{M}_n$. For any lower triangular matrix, $E \in \mathcal{M}_n$, with positive diagonal entries, we may write $G = EE^*$, where $G \in \mathcal{M}_n$, $G = SS^*$, and $S \in \mathcal{M}_n$. This factorization, which is called as Cholesky factorization of G, is unique if S is nonsingular.

*e-mail: cahit@nevsehir.edu.tr, cahitkome@gmail.com

A block diagonal matrix $U \in \mathcal{M}_n$ can be defined by

$$U = \begin{bmatrix} U_{11} & 0 & & \\ 0 & U_{22} & & 0 \\ & & \ddots & \\ & 0 & & U_{ii} \end{bmatrix}, \quad (3)$$

where $U_{jj} \in \mathcal{M}_{n_j}$, $j = 1,2,\dots,i$, and $\sum_{j=1}^{i} n_j = n$. Notationally, this matrix can be denoted as $U = U_{11} \oplus U_{22} \oplus \dots \oplus U_{ii}$ or, in a short form, $\oplus \sum_{j=1}^{i} U_{jj}$. This sum is called as direct sum of the matrices $U_{11}, U_{22}, \dots, U_{ii}$.

Matrix factorization provides considerable convenience in engineering problems and large matrix computations. In recent years, several authors have studied the applications and factorizations of special matrices whose entries are well−known number sequences [2-9]. For example, Kılıç and Taşçı discussed the factorizations of the Pell and symmetric Pell matrices [2]. Lee et al. investigated the eigenvalues and factorizations of the $n \times n$ Fibonacci matrix [3]. Later, Lee and Kim examined the factorization of the generalized Fibonacci matrix and they found some bounds for the eigenvalues of the generalized symmetric Fibonacci matrices [4]. Zhang and Zhang derived some identities including Lucas numbers using the Pascal matrix and the Lucas matrix [5]. Stanica extended some results on the factorization of matrices associated with Lucas, Pascal, Stirling sequences by the Fibonacci matrix [6]. Irmak and Köme investigated the Cholesky factorization of the symmetric Lucas matrix and they obtain the upper and lower bounds for the eigenvalues of the symmetric Lucas matrix by using some majorization techniques [9].

Motivated by the above cited works, in this paper, we define generalized $n \times n$ k −Fibonacci matrix of the first kind and of the second kind, $\mathcal{H}_n[x,k] = [h_{ij}]$ and $\mathcal{R}_n[x,k] = [r_{ij}]$, as

$$h_{ij} = \begin{cases} F_{k,i-j+1}x^{i-j}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0 \end{cases}, \qquad r_{ij} = \begin{cases} F_{k,i-j+1}x^{i+j-2}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0 \end{cases}. \quad (4)$$

In particular, for $n = 4$, we get

$$\mathcal{H}_4[x,k] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ kx & 1 & 0 & 0 \\ (k^2+1)x^2 & kx & 1 & 0 \\ k(k^2+2)x^3 & (k^2+1)x^2 & kx & 1 \end{bmatrix}$$

and

$$\mathcal{R}_4[x,k] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ kx & x^2 & 0 & 0 \\ (k^2+1)x^2 & kx^3 & x^4 & 0 \\ k(k^2+2)x^3 & (k^2+1)x^4 & kx^5 & x^6 \end{bmatrix}.$$

Moreover, we define the generalized symmetric k −Fibonacci matrix, $\mathcal{Q}_n[x,k] = [q_{ij}]$, as

$$q_{ij} = q_{ji} = \begin{cases} \sum_{m=1}^{i} F_{k,m}^2 x^{2i-2}, & i = j, \\ q_{i,j-2}x^2 + kq_{i,j-1}x, & i+1 \leq j \end{cases}. \quad (5)$$

For example,

$$\mathcal{Q}_4[x,k] = \begin{bmatrix} 1 & kx & (k^2+1)x^2 & k(k^2+2)x^3 \\ kx & (k^2+1)x^2 & k(k^2+2)x^3 & (k^4+3k^2+1)x^4 \\ (k^2+1)x^2 & k(k^2+2)x^3 & (k^4+3k^2+2)x^4 & k(k^2+2)^2x^5 \\ k(k^2+2)x^3 & (k^4+3k^2+1)x^4 & k(k^2+2)^2x^5 & (k^6+5k^4+7k^2+2)x^6 \end{bmatrix}.$$

This study is organized as follows. In Section 2, we derive some useful identities which are used in the factorization process. In Section 3, we investigate factorizations and inverse factorizations of $\mathcal{H}_n[x,k]$ and $\mathcal{R}_n[x,k]$. Moreover, we give the Cholesky factorization of $\mathcal{Q}_n[x,k]$ for any nonzero real number x.

2. k −FIBONACCI IDENTITIES

In this section, we give some useful identities of the k −Fibonacci numbers.

Lemma 2. 1. Let $F_{k,n}$ be the k −Fibonacci number. Then, we have

$$F_{k,2n+1} = F_{k,n}^2 + F_{k,n+1}^2. \tag{6}$$

Proof. We will use induction method for proving the theorem. It's clear that Equation (6) holds for $n = 1$. We assume that Equation (6) holds for n. We will show that Equation (6) holds for $n + 1$. Thus, we get

$$\begin{aligned} F_{k,2n+3} &= kF_{k,2n+2} + F_{k,2n+1} \\ &= k\left(kF_{k,2n+1} + F_{k,2n}\right) + F_{k,2n+1} \\ &= (k^2+1)F_{k,2n+1} + kF_{k,2n} \\ &= (k^2+2)F_{k,2n+1} - F_{k,2n-1}. \end{aligned} \tag{7}$$

By induction hypothesis, we obtain

$$\begin{aligned} F_{k,2n+3} &= (k^2+2)F_{k,2n+1} - F_{k,2n-1} \\ &= (k^2+2)\left(F_{k,n}^2 + F_{k,n+1}^2\right) - \left(F_{k,n}^2 - F_{k,n-1}^2\right) \\ &= (k^2+1)F_{k,n}^2 + (k^2+2)F_{k,n+1}^2 - F_{k,n-1}^2. \end{aligned} \tag{8}$$

In addition, we have

$$\begin{aligned} F_{k,n+2}^2 + F_{k,n-1}^2 &= \left(kF_{k,n+1} + F_{k,n}\right)^2 + \left(F_{k,n+1} - kF_{k,n}\right)^2 \\ &= (k^2+1)F_{k,n+1}^2 + (k^2+1)F_{k,n}^2. \end{aligned} \tag{9}$$

By virtue of (8) and (9), we obtain

$$F_{k,2n+3} = F_{k,n+1}^2 + F_{k,n+2}^2. \tag{10}$$

Lemma 2. 2. Let $F_{k,n}$ be the k −Fibonacci number. Then we have

$$kF_{k,n}F_{k,n-1} + F_{k,n-1}^2 - F_{k,n}^2 = (-1)^n. \tag{11}$$

Lemma 2. 3. Let $F_{k,n}$ be the k −Fibonacci number. Then we have

$$kF_{k,n}F_{k,n-1} = F_{k,n+1}^2 - F_{k,n-1}^2 - kF_{k,n}F_{k,n+1}. \tag{12}$$

Proof. Lemma 2.2 and Lemma 2.3 can be proven similar to the proof of Lemma 2.1. So, we omit the proofs.

Lemma 2. 4. [10] Let $F_{k,n}$ be the n −th term of the sequence $\{F_{k,n}\}_{n\in\mathbb{N}}^{\infty}$. Then we have

$$\sum_{i=0}^{n} F_{k,i}^2 = \frac{F_{k,n}F_{k,n+1}}{k}. \tag{13}$$

Lemma 2. 5. Let $F_{k,n}$ be the k −Fibonacci number. Then we have

$$F_{k,1}F_{k,2} + F_{k,2}F_{k,3} + \cdots + F_{k,n-1}F_{k,n} = \frac{F_{k,2n+1} - kF_{k,n}F_{k,n+1} - 1}{2k}.$$

Proof. By virtue of Lemma 2.3, we have

$$kF_{k,1}F_{k,2} = F_{k,3}^2 - F_{k,1}^2 - kF_{k,2}F_{k,3}$$

$$kF_{k,2}F_{k,3} = F_{k,4}^2 - F_{k,2}^2 - kF_{k,3}F_{k,4}$$

$$kF_{k,3}F_{k,4} = F_{k,5}^2 - F_{k,3}^2 - kF_{k,4}F_{k,5}$$

$$\vdots$$

$$kF_{k,n-2}F_{k,n-1} = F_{k,n}^2 - F_{k,n-2}^2 - kF_{k,n-1}F_{k,n}$$

$$kF_{k,n-1}F_{k,n} = F_{k,n+1}^2 - F_{k,n-1}^2 - kF_{k,n}F_{k,n+1}.$$

By considering $F_{k,1} = 1$ and $F_{k,2} = k$ and arranging the above equations, we have

$$2k\left(F_{k,1}F_{k,2} + F_{k,2}F_{k,3} + \cdots + F_{k,n-1}F_{k,n}\right) = F_{k,n}^2 + F_{k,n+1}^2 - kF_{k,n}F_{k,n+1} - F_{k,1}^2 - F_{k,2}^2 + kF_{k,1}F_{k,2}$$

and

$$F_{k,1}F_{k,2} + F_{k,2}F_{k,3} + \cdots + F_{k,n-1}F_{k,n} = \frac{F_{k,2n+1} - kF_{k,n}F_{k,n+1} - 1}{2k}.$$

Therefore the proof is complete.

For more identities of the Fibonacci and Lucas numbers, we refer to the book [11].

3. FACTORIZATIONS

In this section, for any nonzero real number x, we investigate the factorizations of $\mathcal{H}_n[x,k]$, $\mathcal{R}_n[x,k]$ and $\mathcal{Q}_n[x,k]$. Let $\mathcal{I}_n$ be an $n \times n$ identity matrix. Moreover, we define the matrices $\mathcal{L}_n[x,k]$, $\overline{\mathcal{H}}_n[x,k]$ and $\mathcal{A}_i[x,k]$ by

$$\mathcal{L}_0[x,k] = \begin{bmatrix} 1 & 0 & 0 \\ kx & 1 & 0 \\ x^2 & 0 & 1 \end{bmatrix}, \quad \mathcal{L}_{-1}[x,k] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & kx & 1 \end{bmatrix} \tag{14}$$

and $\mathcal{L}_i[x,k] = \mathcal{L}_0[x,k] \oplus \mathcal{I}_i$, $i = ,1,2,\ldots$, $\overline{\mathcal{H}}_n[x,k] = [1] \oplus \mathcal{H}_{n-1}[x,k]$, $\mathcal{A}_1[x,k] = \mathcal{I}_n$, $\mathcal{A}_2[x,k] = \mathcal{I}_{n-3} \oplus \mathcal{L}_{-1}[x,k]$, and, for $i \geq 3$, $\mathcal{A}_i[x,k] = \mathcal{I}_{n-i} \oplus \mathcal{L}_{i-3}[x,k]$.

Now, by definition of the matrix product and using $k-$Fibonacci sequence, we consider a factorization of the generalized $k-$Fibonacci matrix of the first kind.

Lemma 3. 1. For $i \geq 3$,

$$\overline{\mathcal{H}}_i[x,k].\mathcal{L}_{i-3}[x,k] = \mathcal{H}_i[x,k]. \tag{15}$$

From the definition of $\mathcal{A}_i[x,k]$, we know that $\mathcal{A}_n[x,k] = \mathcal{L}_{n-3}[x,k]$, $\mathcal{A}_1[x,k] = \mathcal{I}_n$ and $\mathcal{A}_2[x,k] = \mathcal{I}_{n-3} \oplus \mathcal{L}_{-1}[x,k]$. So, the following theorem are the consequence of Lemma 3.1.

Theorem 3. 2. The generalized $k-$Fibonacci matrix of the first kind, $\mathcal{H}_n[x,k]$, can be factorized by $\mathcal{A}_i[x,k]$'s as follows:

$$\mathcal{H}_n[x,k] = \mathcal{A}_1[x,k]\mathcal{A}_2[x,k] \ldots \mathcal{A}_n[x,k]. \tag{16}$$

For example,

$$\mathcal{H}_5[x,k] = \mathcal{A}_1[x,k]\mathcal{A}_2[x,k]\mathcal{A}_3[x,k]\mathcal{A}_4[x,k]\mathcal{A}_5[x,k]$$

$$= \mathcal{I}_5(\mathcal{I}_2 \oplus \mathcal{L}_{-1}[x,k])(\mathcal{I}_2 \oplus \mathcal{L}_0[x,k])([1] \oplus \mathcal{L}_1[x,k])\mathcal{L}_2[x,k]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & kx & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & kx & 1 & 0 \\ 0 & 0 & x^2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & kx & 1 & 0 & 0 \\ 0 & x^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ kx & 1 & 0 & 0 & 0 \\ x^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ kx & 1 & 0 & 0 & 0 \\ (k^2+1)x^2 & kx & 1 & 0 & 0 \\ k(k^2+2)x^3 & (k^2+1)x^2 & kx & 1 & 0 \\ (k^4+3k^2+1)x^4 & k(k^2+2)x^3 & (k^2+1)x^2 & kx & 1 \end{bmatrix}. \quad (17)$$

We consider another factorization of $\mathcal{H}_n[x,k]$. Then, $n \times n$ matrix $\mathcal{T}_n[x,k] = [t_{ij}]$ is defined as:

$$t_{ij} = \begin{cases} F_{k,i}x^{i-j}, & j = 1, \\ 1, & i = j, \\ 0, & \text{otherwise}, \end{cases} \quad \text{i.e.,} \quad \mathcal{T}_n[x,k] = \begin{bmatrix} F_{k,1} & 0 & \cdots & 0 \\ F_{k,2}x & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,n}x^{n-1} & 0 & \cdots & 1 \end{bmatrix}. \quad (18)$$

Theorem 3.3. For $n \geq 2$,

$$\mathcal{H}_n[x,k] = \mathcal{T}_n[x,k](\mathcal{I}_1 \oplus \mathcal{T}_{n-1}[x,k])(\mathcal{I}_2 \oplus \mathcal{T}_{n-2}[x,k]) \ldots (\mathcal{I}_{n-2} \oplus \mathcal{T}_2[x,k]). \quad (19)$$

We know that

$$\mathcal{L}_0[x,k]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -kx & 1 & 0 \\ -x^2 & 0 & 1 \end{bmatrix}, \mathcal{L}_{-1}[x,k]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -kx & 1 \end{bmatrix} \text{ and } \mathcal{L}_i[x,k]^{-1} = \mathcal{L}_0[x,k]^{-1} \oplus \mathcal{I}_i. \quad (20)$$

Now, we define the matrix $\mathcal{J}_i[x,k] = \mathcal{A}_i[x,k]^{-1}$. Thus , we obtain

$$\mathcal{J}_1[x,k] = \mathcal{A}_1[x,k]^{-1} = \mathcal{I}_n, \mathcal{J}_2[x,k] = \mathcal{A}_2[x,k]^{-1} = \mathcal{I}_{n-3} \oplus \mathcal{L}_{-1}[x,k]^{-1} = \mathcal{I}_{n-2} \oplus \begin{bmatrix} 1 & 0 \\ -kx & 1 \end{bmatrix}$$

and

$$\mathcal{J}_i[x,k] = \mathcal{A}_i[x,k]^{-1}.$$

Moreover, we know that

$$\mathcal{T}_n[x,k]^{-1} = \begin{bmatrix} -F_{k,1} & 0 & \cdots & 0 \\ -F_{k,2}x & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -F_{k,n}x^{n-1} & 0 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad (\mathcal{I}_i \oplus \mathcal{T}_{n-i}[x,k])^{-1} = \mathcal{I}_i \oplus \mathcal{T}_{n-i}[x,k]^{-1}. \tag{21}$$

Therefore, the following corollary holds.

Corollary 3. 4. For $n \geq 2$,

$$\begin{aligned} \mathcal{H}_n[x,k]^{-1} &= \mathcal{A}_n[x,k]^{-1}\mathcal{A}_{n-1}[x,k]^{-1} \dots \mathcal{A}_2[x,k]^{-1}\mathcal{A}_1[x,k]^{-1} \\ &= \mathcal{J}_n[x,k]\mathcal{J}_{n-1}[x,k] \dots \mathcal{J}_2[x,k]\mathcal{J}_1[x,k] \\ &= (\mathcal{I}_{n-2} \oplus \mathcal{T}_2[x,k])^{-1} \dots (\mathcal{I}_1 \oplus \mathcal{T}_{n-1}[x,k])^{-1}\mathcal{T}_n[x,k]^{-1}. \end{aligned} \tag{22}$$

From Corollary 3.4, we have

$$\mathcal{H}_n[x,k]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ -kx & 1 & 0 & 0 & \dots & 0 \\ -x^2 & -kx & 1 & 0 & \dots & 0 \\ 0 & -x^2 & -kx & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -x^2 & -kx & 1 \end{bmatrix}. \tag{23}$$

For a factorization of generalized k −Fibonacci matrix of the second kind, $\mathcal{R}_n[x,k]$, we define the matrices $\mathcal{M}_n[x,k]$, $\overline{\mathcal{R}}_n[x,k]$ and $\mathcal{N}_n[x,k]$ by

$$\mathcal{M}_0[x,k] = \begin{bmatrix} 1 & 0 & 0 \\ kx & x^2 & 0 \\ 1 & 0 & x^2 \end{bmatrix}, \quad \mathcal{M}_{-1}[x,k] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & kx & x^2 \end{bmatrix} \tag{24}$$

and $\mathcal{M}_i[x,k] = \mathcal{M}_0[x,k] \oplus x^2\mathcal{I}_i,\ i = ,1,2,\dots,$ $\overline{\mathcal{R}}_n[x,k] = [1] \oplus \mathcal{R}_{n-1}[x,k]$, $\mathcal{N}_1[x,k] = \mathcal{I}_n$, $\mathcal{N}_2[x,k] = \mathcal{I}_{n-3} \oplus \mathcal{M}_{-1}[x,k]$, and, for $i \geq 3$, $\mathcal{N}_i[x,k] = \mathcal{I}_{n-i} \oplus \mathcal{M}_{i-3}[x,k]$. Thus, we can give the following Lemma.

Lemma 3. 5. For $i \geq 3$,

$$\mathcal{R}_i[x,k] = \overline{\mathcal{R}}_i[x,k]\mathcal{M}_{i-3}[x,k]. \tag{25}$$

Proof. For $i = 3$, we have $\mathcal{R}_3[x,k] = \overline{\mathcal{R}}_3[x,k]\mathcal{M}_0[x,k]$.

The next theorem describes the factorization of $\mathcal{R}_n[x,k]$ for $i > 3$.

Theorem 3. 6. The generalized k −Fibonacci matrix of the second kind, $\mathcal{R}_n[x,k]$ can be factorized by $\mathcal{N}_i[x,k]$'s as follows:

$$\mathcal{R}_n[x,k] = \mathcal{N}_1[x,k]\mathcal{N}_2[x,k] \dots \mathcal{N}_n[x,k]. \tag{26}$$

Now, we consider another factorization of $\mathcal{R}_n[x,k]$. Let $\mathcal{K}_n[x,k]$ be $n \times n$ matrix as:

$$k_{ij}=\begin{cases}F_{k,i}x^{i-j}, & j=1,\\ x^2, & i=j,\\ 0, & \text{otherwise,}\end{cases} \quad \text{i.e.,} \quad \mathcal{K}_n[x,k]=\begin{bmatrix}F_{k,1} & 0 & \cdots & 0\\ F_{k,2}x & x^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ F_{k,n}x^{n-1} & 0 & \cdots & x^2\end{bmatrix}. \tag{27}$$

By the definition of the matrix $\mathcal{K}_n[x,k]$, we can give the factorization of $\mathcal{R}_n[x,k]$ in the following theorem.

Theorem 3. 7. For $n\geq 2$,

$$\mathcal{R}_n[x,k]=\mathcal{K}_n[x,k](\mathcal{I}_1\oplus\mathcal{K}_{n-1}[x,k])(\mathcal{I}_2\oplus\mathcal{K}_{n-2}[x,k])\ldots(\mathcal{I}_{n-2}\oplus\mathcal{K}_2[x,k]). \tag{28}$$

We know that

$$\mathcal{M}_0[x,k]^{-1}=\begin{bmatrix}1 & 0 & 0\\ -\frac{k}{x} & \frac{1}{x^2} & 0\\ -\frac{1}{x^2} & 0 & \frac{1}{x^2}\end{bmatrix}, \quad \mathcal{M}_{-1}[x,k]^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -\frac{k}{x} & \frac{1}{x^2}\end{bmatrix} \tag{29}$$

and $\mathcal{M}_i[x,k]^{-1}=\mathcal{M}_0[x,k]^{-1}\oplus\frac{1}{x^2}\mathcal{I}_i$. Define $\mathcal{U}_i[x,k]=\mathcal{N}_i[x,k]^{-1}$. Then,

$$\mathcal{U}_1[x,k]=\mathcal{I}_n, \mathcal{U}_2[x,k]=\mathcal{N}_2[x,k]^{-1}=\mathcal{I}_{n-3}\oplus\mathcal{M}_{-1}[x,k]^{-1}=\mathcal{I}_{n-2}\oplus\begin{bmatrix}1 & 0\\ -kx & 1\end{bmatrix}$$

and

$$\mathcal{U}_i[x,k]=\mathcal{N}_i[x,k]^{-1}=\mathcal{I}_{n-i}\oplus\mathcal{M}_{i-3}[x,k]^{-1}.$$

Furthermore, we know that

$$\mathcal{K}_n[x,k]^{-1}=\begin{bmatrix}-F_{k,1} & 0 & \cdots & 0\\ -\frac{F_{k,2}}{x} & \frac{1}{x^2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -F_{k,n}x^{n-3} & 0 & \cdots & \frac{1}{x^2}\end{bmatrix} \quad \text{and} \quad (\mathcal{I}_i\oplus\mathcal{K}_{n-i}[x,k])^{-1}=\mathcal{I}_i\oplus\mathcal{K}_{n-i}[x,k]^{-1}. \tag{30}$$

Now, we can give the following corollary.

Corollary 3. 8. For $n\geq 2$,

$$\begin{aligned}\mathcal{R}_n[x,k]^{-1}&=\mathcal{N}_n[x,k]^{-1}\mathcal{N}_{n-1}[x,k]^{-1}\ldots\mathcal{N}_2[x,k]^{-1}\mathcal{N}_1[x,k]^{-1}\\ &=\mathcal{U}_n[x,k]\mathcal{U}_{n-1}[x,k]\ldots\mathcal{U}_2[x,k]\mathcal{U}_1[x,k]\\ &=(\mathcal{I}_{n-2}\oplus\mathcal{K}_2[x,k])^{-1}\ldots(\mathcal{I}_1\oplus\mathcal{K}_{n-1}[x,k])^{-1}\mathcal{K}_n[x,k]^{-1}.\end{aligned} \tag{31}$$

From Corollary 3.8, we have

$$\mathcal{R}_n[x,k]^{-1}=\begin{bmatrix}1 & 0 & 0 & 0 & \dots & 0\\ -\frac{k}{x} & \frac{1}{x^2} & 0 & 0 & \dots & 0\\ -\frac{1}{x^2} & -\frac{k}{x^3} & \frac{1}{x^4} & 0 & \dots & 0\\ 0 & -\frac{1}{x^4} & -\frac{k}{x^5} & \frac{1}{x^6} & \dots & 0\\ \vdots & & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & -\frac{1}{x^{2n-4}} & -\frac{k}{x^{2n-3}} & \frac{1}{x^{2n-2}}\end{bmatrix}. \quad (32)$$

Now we define a generalized k −Fibonacci symmetric matrix $\mathcal{Q}_n[x,k]=[q_{ij}]$ as, for $i,j=1,2,\dots,n$,

$$q_{ij}=q_{ji}=\begin{cases}\sum_{m=1}^{i}F_{k,m}^2x^{2i-2}, & i=j,\\ q_{i,j-2}x^2+kq_{i,j-1}x, & i+1\le j,\end{cases} \quad (33)$$

where $q_{1,0}=0$. Then we know that for $j\ge 1$, $q_{1j}=q_{j1}=F_{k,j}x^{j-1}$ and $q_{2j}=q_{j2}=F_{k,j+1}x^j$.

For example,

$$Q_4[x,k]=\begin{bmatrix}1 & kx & (k^2+1)x^2 & k(k^2+2)x^3\\ kx & (k^2+1)x^2 & k(k^2+2)x^3 & (k^4+3k^2+1)x^4\\ (k^2+1)x^2 & k(k^2+2)x^3 & (k^4+3k^2+2)x^4 & k(k^2+2)^2x^5\\ k(k^2+2)x^3 & (k^4+3k^2+1)x^4 & k(k^2+2)^2x^5 & (k^6+5k^4+7k^2+2)x^6\end{bmatrix}.$$

From the definition of $\mathcal{Q}_n[x,k]$, we can give the following lemmas.

Lemma 3. 9. For $j\ge 3$, $q_{3j}=F_{k,4}\left(F_{k,j-3}+\frac{F_{k,j-2}F_{k,3}}{k}\right)x^{j+1}$.

Proof. From Lemma 2.4, we know that $q_{3,3}=\sum_{i=1}^{3}F_{k,i}^2x^4=\left(F_{k,1}^2+F_{k,2}^2+F_{k,3}^2\right)x^4=\frac{F_{k,3}F_{k,4}}{k}x^4$. Therefore, for $F_{k,0}=0$, $q_{3,3}=F_{k,4}\left(F_{k,0}+\frac{F_{k,1}F_{k,3}}{k}\right)x^4$.

By induction, for $j\ge 3$, we find that $q_{3,j}=F_{k,4}\left(F_{k,j-3}+\frac{F_{k,j-2}F_{k,3}}{k}\right)x^{j+1}$.

We know that $q_{1,3}=q_{3,1}=F_{k,3}x^2$ and $q_{2,3}=q_{3,2}=F_{k,4}x^3$. Also, we know that $q_{1,4}=q_{4,1}=F_{k,4}x^3$, $q_{2,4}=q_{4,2}=F_{k,5}x^4$ and $q_{3,4}=q_{4,3}=F_{k,4}\left(F_{k,1}+\frac{F_{k,2}F_{k,3}}{k}\right)x^5$.

Lemma 3. 10. For $j\ge 4$, $q_{4j}=F_{k,4}\left(F_{k,j-4}+F_{k,j-4}F_{k,3}+\frac{F_{k,j-3}F_{k,5}}{k}\right)x^{j+2}$.

Using Lemmas 3.9 and 3.10, we can obtain $q_{5,1},q_{5,2},q_{5,3}$ and $q_{5,4}$. So, we can give the next lemma.

Lemma 3. 11. For $j\ge 5$, $q_{5j}=\left(F_{k,j-5}F_{k,4}\left(1+F_{k,3}+F_{k,5}\right)+\frac{F_{k,j-4}F_{k,5}F_{k,6}}{k}\right)x^{j+3}$.

Proof. As $q_{5,5}=\frac{F_{k,5}F_{k,6}}{k}x^8$, we have the desired lemma by induction.

Lemma 3. 12. For $j\ge i\ge 6$, we have

$$q_{i,j}=\left(F_{k,j-i}F_{k,4}\left(1+F_{k,3}+F_{k,5}\right)+F_{k,j-i}F_{k,5}F_{k,6}+\cdots+F_{k,j-i}F_{k,i-1}F_{k,i}+\frac{F_{k,j-i+1}F_{k,i}F_{k,i+1}}{k}\right)x^{i+j-2}. \quad (34)$$

Now, we can give the following theorem as a consequence of Lemmas 3.9 – 3.12.

Theorem 3. 13. For $n \geq 1$ a positive integer, we have

$$\mathcal{U}_n[x,k]\mathcal{U}_{n-1}[x,k]\ldots\mathcal{U}_1[x,k]\mathcal{Q}_n[x,k] = \mathcal{H}_n[x,k]^T \quad (35)$$

as well as the Cholesky factorization of $\mathcal{Q}_n[x,k]$ can be given by

$$\mathcal{Q}_n[x,k] = \mathcal{R}_n[x,k]\mathcal{H}_n[x,k]^T. \quad (36)$$

Proof. From Corollary 3.8, we have $\mathcal{R}_n[x,k]^{-1}\mathcal{Q}_n[x,k] = \mathcal{H}_n[x,k]^T$. Then the theorem holds.

Let $\mathcal{V}[x,k] = [v_{ij}] = \mathcal{R}_n[x,k]^{-1}\mathcal{Q}_n[x,k]$. From the definition of $\mathcal{Q}_n[x,k]$ and (32), $v_{ij} = 0$ for $i+1 \leq j$. Now, we take into account the case $j \geq i$. From Lemmas 3.9 – 3.12 and (32), we know that $v_{ij} = h_{ji}$ for $i \leq 5$. We consider $j \geq i \geq 6$.

Then, using (32), we get

$$\begin{aligned} v_{ij} &= -\frac{1}{x^{2i-4}}q_{i-2,j} - \frac{k}{x^{2i-3}}q_{i-1,j} + \frac{1}{x^{2i-2}}q_{i,j} \\ &= \frac{x^{i+j-2}}{x^{2i-2}}\left(F_{k,j-i}F_{k,4}\left(1+F_{k,3}+F_{k,5}\right) + F_{k,j-i}F_{k,5}F_{k,6} + \cdots + F_{k,j-i}F_{k,i-1}F_{k,i} + \frac{F_{k,j-i+1}F_{k,i}F_{k,i+1}}{k}\right) \\ &\quad -\frac{kx^{i+j-3}}{x^{2i-3}}\left(F_{k,j-i+1}F_{k,4}\left(1+F_{k,3}+F_{k,5}\right) + F_{k,j-i+1}F_{k,5}F_{k,6}\right. \\ &\quad \left. + \cdots + F_{k,j-i+1}F_{k,i-2}F_{k,i-1} + \frac{F_{k,j-i+2}F_{k,i-1}F_{k,i}}{k}\right) \\ &\quad -\frac{x^{i+j-4}}{x^{2i-4}}\left(F_{k,j-i+2}F_{k,4}\left(1+F_{k,3}+F_{k,5}\right) + F_{k,j-i+2}F_{k,5}F_{k,6}\right. \\ &\quad \left. + \cdots + F_{k,j-i+2}F_{k,i-3}F_{k,i-2} + \frac{F_{k,j-i+3}F_{k,i-2}F_{k,i-1}}{k}\right) \\ &= x^{j-i}\left(\left(F_{k,j-i} - kF_{k,j-i+1} - F_{k,j-i+2}\right)F_{k,4}\left(1+F_{k,3}+F_{k,5}\right)\right. \\ &\quad + \left(F_{k,j-i} - kF_{k,j-i+1} - F_{k,j-i+2}\right)F_{k,5}F_{k,6} + \cdots + \left(F_{k,j-i} - kF_{k,j-i+1} - F_{k,j-i+2}\right)F_{k,i-3}F_{k,i-2} \\ &\quad + \left(F_{k,j-i} - kF_{k,j-i+1} - \frac{F_{k,j-i+3}}{k}\right)F_{k,i-2}F_{k,i-1} + \left(F_{k,j-i} - F_{k,j-i+2}\right)F_{k,i-1}F_{k,i} + \\ &\quad \left. F_{k,j-i+1}\frac{F_{k,i}F_{k,i+1}}{k}\right). \end{aligned}$$

Since $\left(F_{k,j-i} - kF_{k,j-i+1} - F_{k,j-i+2}\right) = -2kF_{k,j-i+1}$, $F_{k,j-i} - kF_{k,j-i+1} - \frac{F_{k,j-i+3}}{k} = -\frac{(2k^2+1)F_{k,j-i+1}}{k}$ and $F_{k,j-i} - F_{k,j-i+2} = -kF_{k,j-i+1}$, we get

$$\begin{aligned} v_{ij} &= F_{k,j-i+1}\left(-2kF_{k,4} - 2k\left(F_{k,3}F_{k,4} + F_{k,4}F_{k,5} + \cdots + F_{k,i-3}F_{k,i-2} + F_{k,i-2}F_{k,i-1}\right)\right. \\ &\quad \left. -\frac{1}{k}F_{k,i-2}F_{k,i-1} - kF_{k,i-1}F_{k,i} + \frac{F_{k,i}F_{k,i+1}}{k}\right)x^{j-i}. \end{aligned} \quad (37)$$

Since $F_{k,4} = k(k^2+2)$ and using Lemma 2.5, we have

$$v_{ij} = F_{k,j-i+1}\left(-2k^2(k^2+2) - 2k\left(\frac{F_{k,2(i-1)+1} - kF_{k,i-1}F_{k,i} - 1}{2k} - k(k^2+2)\right) - \frac{1}{k}F_{k,i-2}F_{k,i-1}\right.$$

$$-kF_{k,i-1}F_{k,i}+\frac{F_{k,i}F_{k,i+1}}{k})x^{j-i}$$
$$=F_{k,j-i+1}\left(1-F_{k,2i-1}-\frac{F_{k,i-2}F_{k,i-1}}{k}+\frac{F_{k,i}F_{k,i+1}}{k}\right)x^{j-i}.$$

By virtue of Lemma 2.1 and after some basic calculations, we get

$$v_{ij}=F_{k,j-i+1}\left(1-F_{k,i}^2-F_{k,i-1}^2+F_{k,i}^2+F_{k,i-1}^2\right)x^{j-i}$$
$$=F_{k,j-i+1}x^{j-i}. \tag{38}$$

Hence $\mathcal{V}_n[x,k]=\mathcal{H}_n[x,k]^T$ for $1\le i,j\le n$.

Thus, $\mathcal{R}_n[x,k]^{-1}\mathcal{Q}_n[x,k]=\mathcal{H}_n[x,k]^T$, that is, the Cholesky factorization of $\mathcal{Q}_n[x,k]$ is given by $\mathcal{Q}_n[x,k]=\mathcal{R}_n[x,k]\mathcal{H}_n[x,k]^T$.

For example,

$$\mathcal{Q}_4[x,k]=\begin{bmatrix}1 & kx & (k^2+1)x^2 & k(k^2+2)x^3\\ kx & (k^2+1)x^2 & k(k^2+2)x^3 & (k^4+3k^2+1)x^4\\ (k^2+1)x^2 & k(k^2+2)x^3 & (k^4+3k^2+2)x^4 & k(k^2+2)^2x^5\\ k(k^2+2)x^3 & (k^4+3k^2+1)x^4 & k(k^2+2)^2x^5 & (k^6+5k^4+7k^2+2)x^6\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0 & 0\\ kx & x^2 & 0 & 0\\ (k^2+1)x^2 & kx^3 & x^4 & 0\\ k(k^2+2)x^3 & (k^2+1)x^4 & kx^5 & x^6\end{bmatrix}\begin{bmatrix}1 & kx & (k^2+1)x^2 & k(k^2+2)x^3\\ 0 & 1 & kx & (k^2+1)x^2\\ 0 & 0 & 1 & kx\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$=\mathcal{R}_4[x,k]\mathcal{H}_4[x,k]^T. \tag{39}$$

Since $\mathcal{Q}_n[x,k]^{-1}=(\mathcal{H}_n[x,k]^T)^{-1}\mathcal{R}_n[x,k]^{-1}$, we have

$$\mathcal{Q}_n[x,k]^{-1}=\begin{bmatrix}k^2+2 & 0 & -\frac{1}{x^2} & 0 & 0 & 0 & \cdots & 0\\ 0 & \frac{k^2+2}{x^2} & 0 & -\frac{1}{x^4} & 0 & 0 & \cdots & 0\\ -\frac{1}{x^2} & 0 & \frac{k^2+2}{x^4} & 0 & -\frac{1}{x^6} & 0 & \cdots & 0\\ 0 & -\frac{1}{x^4} & 0 & \frac{k^2+2}{x^6} & 0 & -\frac{1}{x^8} & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & -\frac{1}{x^{2n-8}} & 0 & \frac{k^2+2}{x^{2n-6}} & 0 & -\frac{1}{x^{2n-4}}\\ 0 & \cdots & \cdots & 0 & -\frac{1}{x^{2n-6}} & 0 & \frac{k^2+1}{x^{2n-4}} & -\frac{k}{x^{2n-3}}\\ 0 & \cdots & \cdots & \cdots & 0 & -\frac{1}{x^{2n-4}} & -\frac{k}{x^{2n-3}} & \frac{1}{x^{2n-2}}\end{bmatrix} \tag{40}$$

By virtue of Theorem 3.13, we give the following identity.

Corollary 3. 14. Let $F_{k,n}$ be the k −Fibonacci number. Then

$$\left(F_{k,n}F_{k,n-m}+\cdots+F_{k,m+1}F_{k,1}\right)x^{2n-m-2}=\begin{cases}\left(\frac{F_{k,n}F_{k,n-(m-1)}-\xi(m)F_{k,m}}{k}\right)x^{2n-m-2}, & \text{if } n \text{ is odd}\\ \left(\frac{F_{k,n}F_{k,n-(m-1)}-\xi(m+1)F_{k,m}}{k}\right)x^{2n-m-2}, & \text{if } n \text{ is even,}\end{cases} \tag{41}$$

where $\xi(m) = m - 2\left\lfloor\frac{m}{2}\right\rfloor$ is a parity function, i.e., $\xi(m) = 0$ if m is even and $\xi(m) = 1$ if m is odd. In particular, if we multiply the i-th row of $\mathcal{H}_n[x,k]$ and the i −th column of $\mathcal{H}_n[x,k]^T$, we obtain Lemma 2.4. Moreover, Lemma 2.4 is the special case of Corollary 3.14 for $m = 0$.

CONFLICT OF INTEREST

We have no conflict of interest to declare by the author.

REFERENCES

[1] Falcon, S., Plaza, Á., "The $k -$ Fibonacci sequence and the Pascal $2 -$ triangle", Chaos, Solitons & Fractals, 33(1): 38-49, (2007).

[2] Kilic, E., Tasci, D., "The linear algebra of the Pell matrix", Boletín de la Sociedad Matemática Mexicana, 11(3): 163–174, (2005).

[3] Lee, G. Y., Kim, J. S., Lee, S. G., "Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices", Fibonacci Quarterly, 40(3): 203–211, (2002).

[4] Lee, G.Y., Kim, J.S., "The linear algebra of the $k -$ Fibonacci matrix", Linear Algebra and its Applications, 373: 75–87, (2003).

[5] Zhang, Z., Zhang, Y., "The Lucas matrix and some combinatorial identities", Indian Journal of Pure and Applied Mathematics, 38(5): 457–465, (2007).

[6] Stanica, P., "Cholesky factorizations of matrices associated with r −order recurrent sequences", Integers: Electronic Journal of Combinatorial Number Theory, 5(2): A16, (2005).

[7] Kocer, E.G., Mansour, T., Tuglu, N., "Norms of Circulant and Semicirculant Matrices with the Horadam Numbers", ARS Combinatoria, 85, 353–359, (2007).

[8] Kızılates, C., Tuglu, N., "On the bounds for the spectral norms of geometric circulant matrices", Journal of Inequalities and Applications, 312, (2016).

[9] Irmak, N., Köme, C., "Linear algebra of the Lucas matrix", Hacettepe Journal of Mathematics and Statistics, 50(2), 549-558, (2021).

[10] Kilic, E., "Sums of the squares of terms of sequence u_n", Proceedings Mathematical Sciences, 118 (1): 27–41, (2008).

[11] Koshy, T., "Fibonacci and Lucas Numbers with applications", John Wiley & Sons, (2001).