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Results on Bivariate Modified (p,q)-Bernstein Type Operators

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Highlights

- This paper is about the modification of bivariate (p,q)-Bernstein operators.
- We demonstrate some important approximation properties like a rate of convergence, moments etc.
- We establish a two-dimensional version of (p,q)-Bernstein operators can be used in applied fields.

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Abstract

Here, we construct a modification of the (p,q)-Bernstein operators for the two-dimensional case. We study some important properties of these new operators. We estimate the rate of convergence of these operators using modulus of continuity then we give these estimation for functions belonging to class $Lip_M(\alpha_1,\alpha_2)$.

1. INTRODUCTION

Bernstein described called Bernstein polynomials, which he used in the Weierstrass theorem in [1]. New generalizations and applications of Bernstein operators introduced by many authors. For example, q-Bernstein polynomials were defined [2] by Phillips in 1997. Some generalizations of Bernstein operators using q-integers can be found in [3-6]. Then q-calculus is extended to (p,q)-calculus in approximation theory. Firstly, Mursaleen et al. carried this concept to the approximation theory [7]. After then, (p,q)-operators have been studied by different authors for examples [8-17]. Karaisa [18] defined bivariate form of (p,q)-Bernstein operators. Karahan and Izgi gave generalized (p,q)-Bernstein operators and constructed their important properties in [19]. Cevik studied over a certain interval the approach properties of this operator given in [20].

The purpose of our paper is to investigate two dimensional generalized (p,q)-Bernstein operator given in [19]. This generalization is aimed to be used in applied areas such as computational methods in applied mathematics and numerical analysis.

The paper is designed as following: After giving the necessary basic information, in the second section some fundamental results are examined on $\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]$ for generalized (p,q)- Bernstein operators similar to that given in [20]. We introduce Korovkin type approximation properties of our operator for every $f\in C(I^2)$ where $I^2=\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]\times\left[0,\frac{[m+1]_{p,q}}{[m+3]_{p,q}}\right]$ in Section 3. We work out the rates of convergence using modulus of continuity and Lipschitz type function in fourth section. In the last section, results and references are given.
Let us recall some representation of (p,q)-analysis. (p,q)-integer $[n]_{p,q}$ is introduced by

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$[n]_{p,q} \coloneqq \frac{p^n-q^n}{p-q}, 0<q<p\le 1, n=0,1,2\cdots$.

For every $n \in \mathbb{N}$, (p,q)-factorial is defined as:

$$[n]_{p,q}! = \begin{cases} [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q}, & n\ge 1 \\ 1, & n=0 \end{cases}.$$

The (p,q)-binomial expansion is

$$(x+y)^n \coloneqq (x+y)(px+qy)(p^2x+q^2y)\dots(p^{n-1}x+q^{n-1}y)$$

and (p,q)-binomial coefficients are given as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q}.$$

Details related to (p,q)-calculus can be found in [21]. In [7], (p,q)-Bernstein operators were introduced by Mursaleen et al. Karaisa [18] defined bivariate (p,q)- Bernstein operator for $D \coloneqq [0,1]\times[0,1], f \in C(D)$. Karahan and Izgi gave generalized (p,q)-Bernstein operators in [19] for every $f \in C\left[0,\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$.

Motivated by these studies, we give a generalized (p,q)-Bernstein operators, We will use as follows for every $f \in C\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]$. Let $0 \le x \le \frac{[n+1]_{p,q}}{[n+3]_{p,q}}$ and

$$D_{n,k}^{(p,q)}(x) := \left(\frac{[n+3]_{p,q}}{[n+1]_{p,q}}\right)^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1}\left(p^s\frac{[n+1]_{p,q}}{[n+3]_{p,q}} - q^s x\right)$$

for every $f \in C\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]$,

$$\tilde{B}_n^{(p,q)}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n} f\left(\frac{[n+1]_{p,q}[k]_{p,q}}{[n+3]_{p,q}[n]_{p,q}p^{k-n}}\right) D_{n,k}^{(p,q)}(x). \quad (1)$$

2. SOME PRELIMINARY RESULTS

This section contains Lemma and Theorems as a preparation for the main section. Proofs can be easily done with the method in [19, 20]. Therefore, they will be given without proof.

Lemma 2.1. Let $f \in C\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right], n\ge 1$ and $0<q<p\le 1$. Then the operators $\tilde{B}_n^{(p,q)}(f;x)$ are linear and positive.

Lemma 2.2. Let $0<q<p\le 1$ and $x \in \left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]$. Then we get

i. $\tilde{B}_n^{(p,q)}(1;x) = 1$,

ii. $\tilde{B}_n^{(p,q)}(t;x) = x$,

iii. $\tilde{B}_n^{(p,q)}(t^2;x) = \frac{p^{n-1}}{[n]_{p,q}} \frac{[n+1]_{p,q}}{[n+3]_{p,q}} x + \frac{q[n-1]_{p,q}}{[n]_{p,q}} x^2,$

iv. $\tilde{B}_n^{(p,q)}(t^3;x) = \left(\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right)^2 \left(\frac{p^{n-1}}{[n]_{p,q}}\right)^2 x + \frac{[n+1]_{p,q}}{[n+3]_{p,q}} \frac{p^{n-2}q[n-1]_{p,q}(2p+q)}{[n]_{p,q}^2} x^2$
$+ \frac{q^3[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}^2} x^3,$

v. $\tilde{B}_n^{(p,q)}(t^4;x) = \left(\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right)^3 \left(\frac{p^{n-1}}{[n]_{p,q}}\right)^3 x$
$+ \left(\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right)^2 \frac{p^{2n-4}[n-1]_{p,q}q(3p^2+3pq+q^2)}{[n]_{p,q}^3} x^2$
$+ \left(\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right) \frac{p^{n-3}[n-1]_{p,q}[n-2]_{p,q}q^3(3p^2+2pq+q^2)}{[n]_{p,q}^3} x^3$
$+ \frac{q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}{[n]_{p,q}^3} x^4 .$

The approximation theorem for the space $C\left[0, \frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]$ given below and the case of $\lim_{n\to\infty} \frac{p_n^{n-1}}{[n]_{p_n,q_n}} = 0$ in the following rate of convergence of operators should be taken into account.

Theorem 2.1. Let $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 1$, with $0 < q_n < p_n \leq 1$. Then for each $f \in C\left[0, \frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]$, $\tilde{B}_n^{(p_n,q_n)}(f,x)$ converges uniformly to f on $\left[0, \frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]$.

Lemma 2.3. The following formulas for moments are easily obtained using Lemma 2.2

i. $\tilde{B}_n^{(p,q)}((t-x)^0;x) = 1,$

ii. $\tilde{B}_n^{(p,q)}((t-x)^1;x) = 0,$

iii. $\tilde{B}_n^{(p,q)}((t-x)^2;x) = \frac{[n+1]_{p,q}}{[n+3]_{p,q}} \frac{p^{n-1}}{[n]_{p,q}} x + \left(\frac{[n-1]_{p,q}}{[n]_{p,q}} q - 1\right) x^2,$

iv. $\tilde{B}_n^{(p,q)}((t-x)^4;x) = \left\{\frac{p^{n-3}[n]_{p,q}^2(-p^2+2pq-q^2) + p^{n-5}[n]_{p,q}(-p^3+3pq^2+q^3)}{[n]_{p,q}^3}\right.$
$\left. - \frac{p^{3n-6}(p^2+p^3+2pq^2+q^3)}{[n]_{p,q}^3}\right\} x^4 + \left(\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right) \left\{\frac{p^{n-3}[n]_{p,q}^2(p^2-2pq+q^2)}{[n]_{p,q}^3}\right.$
$\left. + \frac{p^{2n-5}[n]_{p,q}(-q^3-4pq^2-3p^2q+2q^3) - p^{3n-6}(3p^3+3pq^2+5p^2q+q^3)}{[n]_{p,q}^3}\right\} x^3$
$+ \left(\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right)^2 \left\{\frac{p^{2n-4}[n]_{p,q}(-p^2+3pq+q^2) - p^{3n-5}[n]_{p,q}(3p^2+3pq+q^2)}{[n]_{p,q}^3}\right\} x^2$
$- \left(\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right)^3 \frac{p^{3n-3}}{[n]_{p,q}^3} x.$

We illustrate an approximation of the operators $\tilde{B}_n^{(p_n,q_n)}(f,x)$ for a function $f(x)$ by Maple programs.

Example 2.1. Let $f(x) = x^2 + 1$. In Figure 1 a) and b) the graphs of the approximation of operator $\tilde{B}_n^{(p_n,q_n)}(f,x)$ to the function f is given. We demonstrate $n = 2$(green), $n = 4$ (red), $n = 6$ (black), $n = 8$ (cyan), $n = 10$ (magenta) in operators. In Figure 1 a) take $\mathrm{q} = 0.8, \mathrm{p} = 0.9$ and in Figure 1 b) take $\mathrm{q} = 0.5, \mathrm{p} = 0.9$.

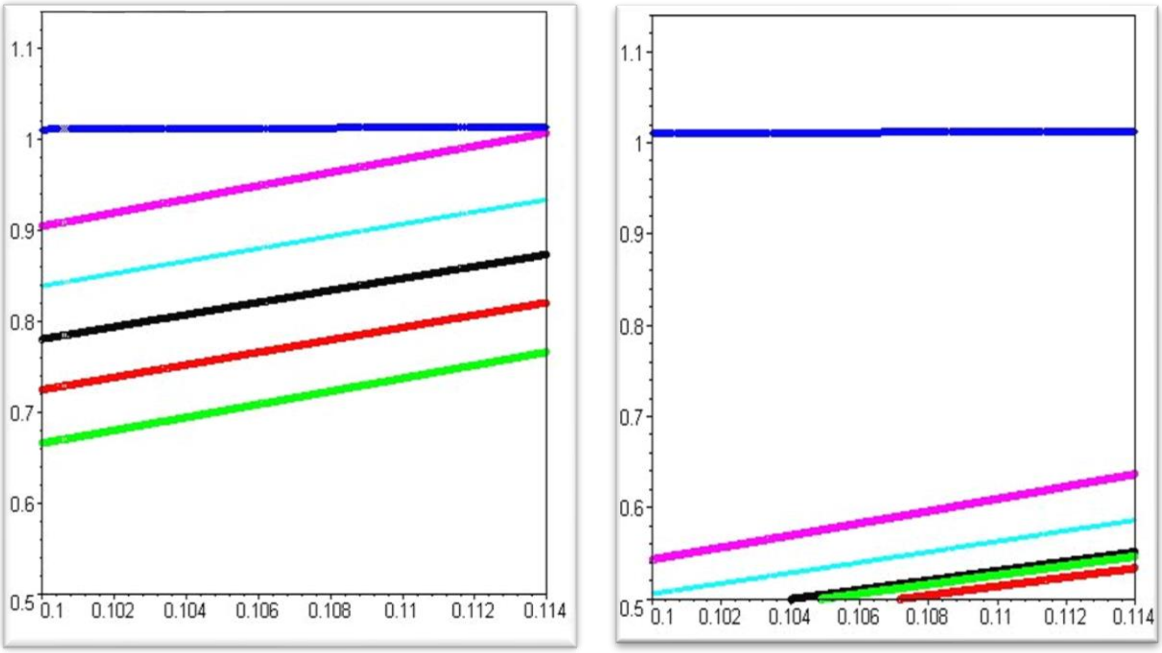

Figure 1. a) and b) *Approximation of* $\tilde{B}_n^{(p_n,q_n)}(f,x)$ *to* f *for different values of p and q*

Example 2.2. Let $g(x) = \frac{1}{10^3}\left(e^{(x^2+1)} + \cos(x)\right) + x^{3^{e^x}}1$. In Figure 2. a) and b) the graphs of the approximation of operator $\tilde{B}_n^{(p_n,q_n)}(g,x)$ to the function g is given. We demonstrate $n = 5$(green), $n = 6$ (red), $n = 7$ (black), $n = 9$ (cyan), $n = 10$ (magenta) in operators. In Figure 2. a) take $\mathrm{q} = 0.9, \mathrm{p} = 0.95$ and in Figure 2. b) take $\mathrm{q} = 0.85, \mathrm{p} = 0.88$.

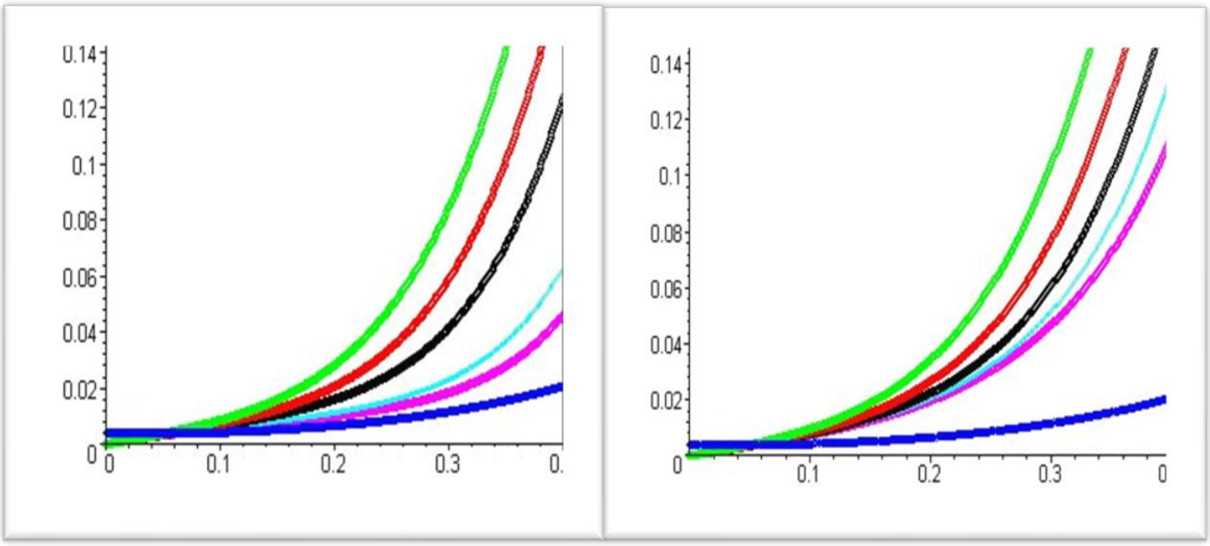

Figure 2. a) and b) *Approximation of* $\tilde{B}_n^{(p_n,q_n)}(g,x)$ *to* g *for different values of p and q*

The modulus of continuity given with $w(f;\delta)$, it is given by the following relation in litearature

$$w(f,\delta)=\sup_{|x_1-x_2|\le\delta}|f(x_1)-f(x_2)|\,,x_1,x_2\in I.$$

Theorem 2.2. Let $f\in C\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]$, then

$$\left|\tilde{B}_n^{(p,q)}(f;x)-f(x)\right|\le\left(1+\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right)\omega(f;\delta_n),$$

where $\delta_n=\sqrt{\left(\left(\frac{p^{n-1}}{[n]_{p,q}}\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right)^2\left(2\frac{q[n-1]_{p,q}}{[n]_{p,q}}-1\right)\right)}$.

Theorem 2.3. Let $C^2\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]:=\left\{f\in C\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]:\ f',f''\in C\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]\right\}$, the sequences (p_n), (q_n) satisfying $0<q_n<p_n\le 1$ such that as $n\to\infty$ $p_n\to 1$, $q_n\to 1$ and $p_n^n\to\alpha$, $q_n^n\to\beta$ where $0\le\alpha,\beta<1$ and $0<\gamma\le 1$. For every $f\in C^2\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]$, we get

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(\tilde{B}_n^{(p_n,q_n)}(f;x)-f(x)\right)=\frac{x(\gamma-\alpha x)}{2}f''(x).$$

We recall a function belongs to $Lip_M(\alpha)$ if $|f(t)-f(x)|\le M|t-x|^\alpha$, for $M>0$, $t,x\in\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]$ and $\alpha\in(0,1]$.

Theorem 2.4. Let $f\in Lip_M(\alpha)$, $0<q_n<p_n\le 1$ then the following inequality holds

$$\left|\tilde{B}_n^{(p_n,q_n)}(f;x)-f(x)\right|\le M\left\{\frac{p_n^{n-1}\left(\frac{[n+1]_{p_n,q_n}}{[n+3]_{p_n,q_n}}\right)^2}{[n]_{p_n,q_n}}\right\}^{\frac{\alpha}{2}}.$$

3. MAIN RESULTS

Let $I^2=\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]\times\left[0,\frac{[m+1]_{p,q}}{[m+3]_{p,q}}\right]$, $f:I^2\to\mathbb{R}^2$ and for $i\in\{1,2\}$ $0<q_i<p_i\le 1$. Then a modification of the two-dimensional (p,q)-Bernstein operator be described as follows:

$$\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)=\frac{1}{p_1^{\frac{n(n-1)}{2}}p_2^{\frac{m(m-1)}{2}}}\sum_{k=0}^{n}\sum_{j=0}^{m}D_{n,k}(p_1,q_1;x)D_{m,j}(p_2,q_2;y)$$
$$\times f\left(\frac{[n+1]_{p_1,q_1}[k]_{p_1,q_1}}{[n+3]_{p_1,q_1}[n]_{p_1,q_1}p_1^{k-n}},\frac{[m+1]_{p_2,q_2}[j]_{p_2,q_2}}{[m+3]_{p_2,q_2}[m]_{p_2,q_2}p_2^{j-m}}\right),\qquad (2)$$

where

$$D_{n,k}(p_1,q_1;x)=\left(\frac{[n+3]_{p_1,q_1}}{[n+1]_{p_1,q_1}}\right)^n\begin{bmatrix}n\\k\end{bmatrix}_{p_1,q_1}p_1^{\frac{k(k-1)}{2}}x^k\prod_{s=0}^{n-k-1}\left(p_1^s\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}-q_1^sx\right),$$
$$D_{m,j}(p_2,q_2;y)=\left(\frac{[m+3]_{p_2,q_2}}{[m+1]_{p_2,q_2}}\right)^m\begin{bmatrix}m\\j\end{bmatrix}_{p_2,q_2}p_2^{\frac{j(j-1)}{2}}y^j\prod_{r=0}^{m-j-1}\left(p_2^r\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}-q_2^ry\right).$$

Let $I^2=\left[0,\frac{[n+1]_{p,q}}{[n+3]_{p,q}}\right]\times\left[0,\frac{[m+1]_{p,q}}{[m+3]_{p,q}}\right]$, $0<q_i<p_i\leq 1$, where $i\in\{1,2\}$ and $f:I^2\to\mathbb{R}^2$. For $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)$, we can do the following demonstrations:

$$ {}^{x}\tilde{B}_n^{(p_1,q_1)}(f;x,y)=\frac{1}{p_1^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}f\left(\frac{[n+1]_{p_1,q_1}[k]_{p_1,q_1}}{[n+3]_{p_1,q_1}[n]_{p_1,q_1}p_1^{k-n}},y\right)D_{n,k}(p_1,q_1;x) \tag{3}$$

and

$$ {}^{y}\tilde{B}_m^{(p_2,q_2)}(f;x,y)=\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{j=0}^{m}f\left(x,\frac{[m+1]_{p_2,q_2}[j]_{p_2,q_2}}{[m+3]_{p_2,q_2}[m]_{p_2,q_2}p_2^{j-m}}\right)D_{m,j}(p_2,q_2;y). \tag{4}$$

Theorem 3.1. The operators ${}^{x}\tilde{B}_n^{(p_1,q_1)}$, ${}^{y}\tilde{B}_m^{(p_2,q_2)}$ define on $C(I^2)$. Then the following results holds.

$\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)= {}^{x}\tilde{B}_n^{(p_1,q_1)}(f;x,y)\,{}^{y}\tilde{B}_m^{(p_2,q_2)}(f;x,y)$

$= {}^{y}\tilde{B}_m^{(p_2,q_2)}(f;x,y)\,{}^{x}\tilde{B}_n^{(p_1,q_1)}(f;x,y)$.

Proof

$${}^{x}\tilde{B}_n^{(p_1,q_1)}(f;x,y)\,{}^{y}\tilde{B}_m^{(p_2,q_2)}(f;x,y)= {}^{x}\tilde{B}_n^{(p_1,q_1)}\left({}^{y}\tilde{B}_m^{(p_2,q_2)};x,y\right)$$

$$= {}^{x}\tilde{B}_n^{(p_1,q_1)}\left(\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{j=0}^{m}f\left(x,\frac{[m+1]_{p_2,q_2}[j]_{p_2,q_2}}{[m+3]_{p_2,q_2}[m]_{p_2,q_2}p_2^{j-m}}\right)D_{m,j}(p_2,q_2;y)\,;x,y\right)$$

$$=\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{j=0}^{m}D_{m,j}(p_2,q_2;y)\;{}^{x}\tilde{B}_n^{(p_1,q_1)}\left(f\left(x,\frac{[m+1]_{p_2,q_2}[j]_{p_2,q_2}}{[m+3]_{p_2,q_2}[m]_{p_2,q_2}p_2^{j-m}}\right);x,y\right)$$

$$=\frac{1}{p_1^{\frac{n(n-1)}{2}}\,p_2^{\frac{m(m-1)}{2}}}\sum_{j=0}^{m}\sum_{k=0}^{n}D_{n,k}(p_1,q_1;x)\,D_{m,j}(p_2,q_2;y)$$

$$\times f\left(\frac{[n+1]_{p_1,q_1}[k]_{p_1,q_1}}{[n+3]_{p_1,q_1}[n]_{p_1,q_1}p_1^{k-n}},\frac{[m+1]_{p_2,q_2}[j]_{p_2,q_2}}{[m+3]_{p_2,q_2}[m]_{p_2,q_2}p_2^{j-m}}\right).$$

Similarly,

$${}^{y}\tilde{B}_m^{(p_2,q_2)}(f;x,y)\,{}^{x}\tilde{B}_n^{(p_1,q_1)}(f;x,y)= {}^{y}\tilde{B}_m^{(p_2,q_2)}\left({}^{x}\tilde{B}_n^{(p_1,q_1)};x,y\right)$$

$$= {}^{y}\tilde{B}_m^{(p_2,q_2)}\left(\frac{1}{p_1^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}f\left(\frac{[n+1]_{p_1,q_1}[k]_{p_1,q_1}}{[n+3]_{p_1,q_1}[n]_{p_1,q_1}p_1^{k-n}},y\right)D_{n,k}(p_1,q_1;x)\,;x,y\right)$$

$$=\frac{1}{p_1^{\frac{n(n-1)}{2}}p_2^{\frac{m(m-1)}{2}}}\sum_{k=0}^{n}\sum_{j=0}^{m}D_{n,k}(p_1,q_1;x)\,D_{m,j}(p_2,q_2;y)$$

$$\times f\left(\frac{[n+1]_{p_1,q_1}[k]_{p_1,q_1}}{[n+3]_{p_1,q_1}[n]_{p_1,q_1}p_1^{k-n}},\frac{[m+1]_{p_2,q_2}[j]_{p_2,q_2}}{[m+3]_{p_2,q_2}[m]_{p_2,q_2}p_2^{j-m}}\right).$$

Therefore, we have the desired result.

Lemma 3.1. Let $f: I^2 \to \mathbb{R}^2$ and for $i \in \{1,2\}$, $0 < q_i < p_i \leq 1$. Then we have next equality,

i. $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;x,y) = 1,$

ii. $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\theta;x,y) = x,$

iii. $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\tau;x,y) = y,$

iv. $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\theta^2;x,y) = \frac{p_1^{n-1}}{[n]_{p_1,q_1}}\left(\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}\right)x + \frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}q_1x^2,$

v. $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\tau^2;x,y) = \frac{p_2^{m-1}}{[m]_{p_2,q_2}}\left(\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}\right)y + \frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}}q_2y^2,$

vi. $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\theta^2+\tau^2;x,y) = \frac{p_1^{n-1}}{[n]_{p_1,q_1}}\left(\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}\right)x + \frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}q_1x^2$
$+\left(\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}\right)\frac{p_2^{m-1}}{[m]_{p_2,q_2}}y + \frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}}q_2y^2.$

Proof Actually, by (2) we get

i. $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;x,y) = \frac{1}{p_1^{\frac{n(n-1)}{2}}}\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{k=0}^{n}\sum_{j=0}^{m}D_{n,k}(p_1,q_1;x)\,D_{m,j}(p_2,q_2;y)$

$$=\frac{1}{p_1^{\frac{n(n-1)}{2}}p_2^{\frac{m(m-1)}{2}}}\left(\frac{[n+3]_{p_1,q_1}}{[n+1]_{p_1,q_1}}\right)^n\left(\frac{[m+3]_{p_2,q_2}}{[m+1]_{p_2,q_2}}\right)^m\left[\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p_1,q_1}p_1^{\frac{k(k-1)}{2}}x^k\right.$$

$$\left.\times\prod_{s=0}^{n-k-1}\left(p_1^s\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}-q_1^sx\right)\right]\left[\sum_{j=0}^{m}\begin{bmatrix}m\\j\end{bmatrix}_{p_2,q_2}p_2^{\frac{j(j-1)}{2}}y^j\prod_{r=0}^{m-j-1}\left(p_2^r\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}-q_2^ry\right)\right]=1.$$

ii. From the (3), (4) we get

$$\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\theta;x,y)=\frac{1}{p_1^{\frac{n(n-1)}{2}}p_2^{\frac{m(m-1)}{2}}}\sum_{k=0}^{n}\sum_{j=0}^{m}f\left(\frac{[n+1]_{p_1,q_1}[k]_{p_1,q_1}}{[n+3]_{p_1,q_1}[n]_{p_1,q_1}p_1^{k-n}},y\right)$$

$$\times D_{n,k}(p_1,q_1;x)D_{m,j}(p_2,q_2;y)$$

$$= \frac{1}{p_1^{\frac{n(n-3)}{2}} p_2^{\frac{m(m-1)}{2}}} \left(\frac{[n+3]_{p_1,q_1}}{[n+1]_{p_1,q_1}}\right)^{n-1} \left(\frac{[m+3]_{p_2,q_2}}{[m+1]_{p_2,q_2}}\right)^{m} p_2^{\frac{m(m-1)}{2}} \left(\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}\right)^{m}$$

$$\times \left[\sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p_1,q_1} p_1^{\frac{k(k-3)}{2}} x^k \prod_{s=0}^{n-k-1} \left(p_1^s \frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}} - q_1^s x\right)\right] \quad k \to k+1$$

$$= \frac{1}{p_1^{\frac{n(n-3)}{2}}} \left(\frac{[n+3]_{p_1,q_1}}{[n+1]_{p_1,q_1}}\right)^{n-1} \left[\sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p_1,q_1} p_1^{\frac{(k+1)(k-2)}{2}} x^{k+1} \prod_{s=0}^{n-k-2} \left(p_1^s \frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}} - q_1^s x\right)\right]$$

$$= \frac{x}{p_1^{\frac{(n-1)(n-2)}{2}}} \left(\frac{[n+3]_{p_1,q_1}}{[n+1]_{p_1,q_1}}\right)^{n-1} \left(\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}\right)^{n-1} p_1^{\frac{(n-2)(n-1)}{2}} = x.$$

Similarly, $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\tau; x, y) = y$ is obtained.

iv. $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\theta^2; x, y)$

$$= \frac{1}{p_1^{\frac{n(n-1)}{2}} p_2^{\frac{m(m-1)}{2}}} \sum_{k=0}^{n} \sum_{j=0}^{m} f\left(\frac{[n+1]_{p_1,q_1}^2 [k]_{p_1,q_1}^2}{[n+3]_{p_1,q_1}^2 [n]_{p_1,q_1}^2 p_1^{2k-2n}}, y\right) D_{n,k}(p_1, q_1; x)\, D_{m,j}(p_2, q_2; y)$$

$$= \frac{x}{[n]_{p_1,q_1} p_1^{\frac{(n-1)(n-4)}{2}}} \left(\frac{[n+3]_{p_1,q_1}}{[n+1]_{p_1,q_1}}\right)^{n-2} \left[\sum_{k=0}^{n-1} [k+1]_{p_1,q_1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p_1,q_1} p_1^{\frac{(k-3)k}{2}} x^k\right.$$

$$\left.\times \prod_{s=0}^{n-k-2} \left(p_1^s \frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}} - q_1^s x\right)\right].$$

Using the fact that $[k+1]_{p_1,q_1} = p_1^k + q_1 [k]_{p_1,q_1}$, we get

$$\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\theta^2; x, y) = \frac{p_1^{n-1}}{[n]_{p_1,q_1}} \frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}} x + \frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}} q_1 x^2.$$

In a similar way we get $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\tau^2; x, y)$. Hence we can write

$$\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\theta^2 + \tau^2; x, y) = \left(\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}\right) \frac{p_1^{n-1}}{[n]_{p_1,q_1}} x + \frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}} q_1 x^2$$

$$+ \left(\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}\right) \frac{p_2^{m-1}}{[m]_{p_2,q_2}} y + \frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}} q_2 y^2.$$

Remark 3.1. Let $0 < q_{1,n} < p_{1,n} \leq 1$, $0 < q_{2,m} < p_{2,m} \leq 1$ and

$$\lim_{n\to\infty} p_{1,n} = \lim_{n\to\infty} q_{1,n} = 1\,, \lim_{m\to\infty} p_{2,m} = \lim_{m\to\infty} q_{2,m} = 1$$

then following equalities are hold

$$\lim_{n\to\infty} \frac{p_{1,n}^{n-1}}{[n]_{p_{1,n};q_{1,n}}} = \lim_{m\to\infty} \frac{p_{2,m}^{m-1}}{[m]_{p_{2,m};q_{2,m}}} = 0,$$

$$\lim_{n\to\infty} \frac{[n-1]_{p_{1,n};q_{1,n}}}{[n]_{p_{1,n};q_{1,n}}} q_{1,n} = \lim_{m\to\infty} \frac{[m-1]_{p_{2,m};q_{2,m}}}{[m]_{p_{2,m};q_{2,m}}} q_{2,m} = 1.$$

Theorem 3.2. Let $0 < q_{1,n} < p_{1,n} \leq 1,\ 0 < q_{2,m} < p_{2,m} \leq 1$ and $\lim_{n\to\infty} p_{1,n} = \lim_{n\to\infty} q_{1,n} = 1$, $\lim_{m\to\infty} p_{2,m} = \lim_{m\to\infty} q_{2,m} = 1$. Then for every $f \in C(I^2)$ we have

$$\lim_{n,m\to\infty} \left\| \tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y) \right\|_{C(I^2)} = 0.$$

Proof In accordance to Volkov's theorem, since it is easy to show the case i-iii in Lemma 3.1, it is sufficient to show only following equality

$$\lim_{n,m\to\infty} \left\| \tilde{B}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(\theta^2+\tau^2;x,y) - (x^2+y^2) \right\|_{C(I^2)} = 0.$$

By definition of the norm, we get

$$\max_{(x,y)\in I^2} \left| \tilde{B}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(\theta^2+\tau^2;x,y) - (x^2+y^2) \right|$$

$$\leq \left| \left(\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}} \right)^2 \frac{p_1^{n-1}}{[n]_{p_1,q_1}} + \left(\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}} \right)^2 \frac{p_2^{m-1}}{[m]_{p_2,q_2}} \right|$$

$$+ \left| \left[\frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}} q_{1,n} - 1 \right] \left(\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}} \right)^2 + \left[\frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}} q_{2,m} - 1 \right] \left(\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}} \right)^2 \right|.$$

Here, using the equations

$$[n-1]_{p_{1,n},q_{1,n}} q_{1,n} = [n]_{p_{1,n},q_{1,n}} - p_{1,n}^{n-1}$$

and

$$[m-1]_{p_{2,m},q_{2,m}} q_{2,m} = [m]_{p_{2,m},q_{2,m}} - p_{2,m}^{m-1}$$

we write

$$\max_{(x,y)\in I^2} \left| \tilde{B}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(\theta^2+\tau^2;x,y) - (x^2+y^2) \right|$$

$$\leq \frac{2p_{1,n}^{n-1}}{[n]_{p_{1,n},q_{1,n}}} \left(\frac{[n+1]_{p_{1,n},q_{1,n}}}{[n+3]_{p_{1,n},q_{1,n}}} \right)^2 + \frac{2p_{2,m}^{m-1}}{[m]_{p_{2,m},q_{2,m}}} \left(\frac{[m+1]_{p_{2,m},q_{2,m}}}{[m+3]_{p_{2,m},q_{2,m}}} \right)^2.$$

For each $f \in C(I^2)$ according to definition of sequences of $p_{1,n}, q_{1,n}$ and $p_{2,m}, q_{2,m}$ it can be get

$$\lim_{n,m\to\infty} \left\| \tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y) \right\|_{C(I^2)} = 0.$$

Hence, the proof of theorem is complete.

Lemma 3.2. For $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)$ the following equations are true.

$$\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\theta-x)^2;x,y) = \frac{p_1^{n-1}[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}[n]_{p_1,q_1}}x + \left(\frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}q_1 - 1\right)x^2,$$

$$\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\tau-y)^2;x,y) = \frac{p_2^{m-1}[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}[m]_{p_2,q_2}}y + \left(\frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}}q_2 - 1\right)y^2.$$

Proof

$$\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\theta-x)^2;x,y) = \tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\theta^2;x,y) - 2xx + x^2\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;x,y)$$

$$= \frac{p_1^{n-1}[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}[n]_{p_1,q_1}}x + \left(\frac{q_1[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}} - 1\right)x^2.$$

Similarly,

$$\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\tau-y)^2;x,y) = \frac{[m+1]_{p_2,q_2}p_2^{m-1}}{[m+3]_{p_2,q_2}[m]_{p_2,q_2}}y + \left(\frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}}q_2 - 1\right)y^2.$$

Example 3.1. In Figure 3. a) for $f(x,y) = \frac{3}{xy+1}$. and in Figure 3.b) for $f(x,y) = sin^2(x^2-3) - cos^2(y^2-2) - 4$ the graphs of the approximation of operator $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f,x,y)$ is given. We demonstrate in Figure 3. a) f(blue), $(m,n) = (3,2)$(green), $(m,n) = (4,4)$(red), $(m,n) = (4,5)$(cyan) and in Figure 3. b) $(m,n) = (1,2)$(violet), $(m,n) = (2,2)$(plum), $(m,n) = (3,4)$(magenta), $(m,n) = (4,4)$(aquamarine). In Figure 3. a) and b) take $(p_1,q_1) = (0.65, 0.57) = (p_2,q_2)$.

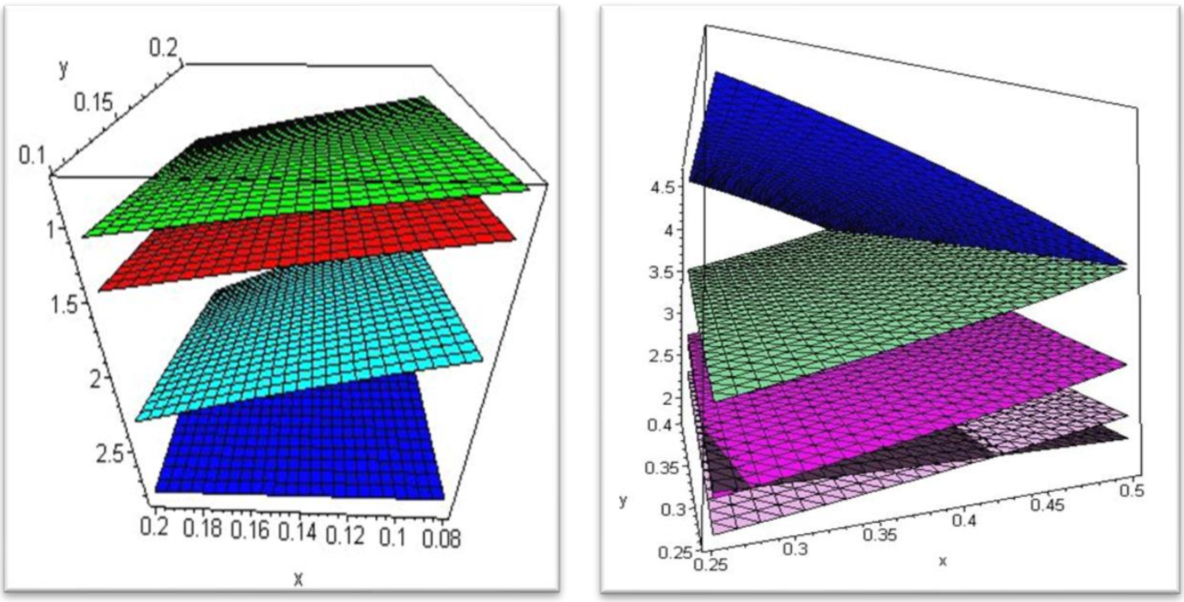

Figure 3. a) and b) *Approximation to* $\frac{3}{xy+1}$ *and* $sin^2(x^2-3) - cos^2(y^2-2) - 4$

Example 3.2. In Figure 4. a) and Figure 4. b) the graphs of the approximation of operator $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f,x,y)$ to $f(x,y)=e^{xy+1}$is given. We demonstrate in Figure 4. a) $(m,n)=(3,2)$(khaki), $(m,n)=(4,4)$(violet), $(m,n)=(4,5)$(cyan) and in Figure 4. b) $(m,n)=(3,2)$(red), $(m,n)=(4,4)$(plum), $(m,n)=(4,5)$(cyan). In Figure 4. a) take $(p_1,q_1)=(0.65,0.57)=(p_2,q_2)$.and b) take $(p_1,q_1)=(0.52,0.51)$ and $(p_2,q_2)=(0.51,0.50)$.

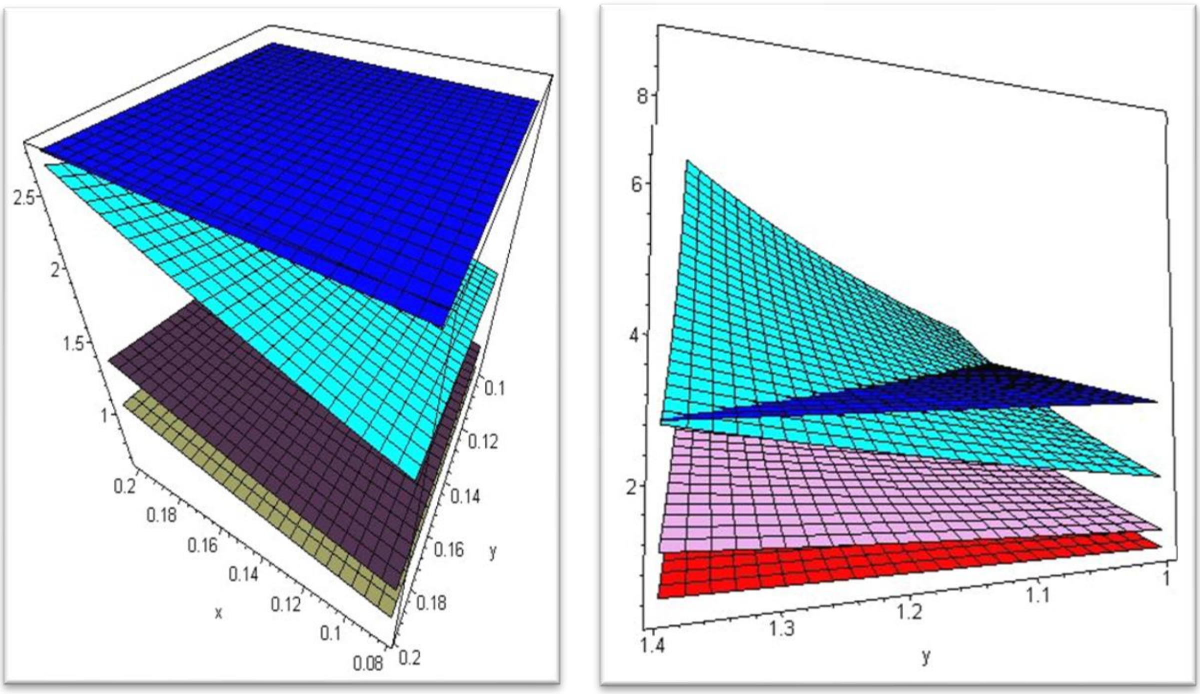

Figure 4. a) and b) *Approximation of $\tilde{B}_n^{(p_n,q_n)}(f,x)$ to f for different values of (p_1,q_1) and (p_2,q_2)*

4. RATES OF CONVERGENCES

Now, we give some convergence properties of $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)$ by the following well known definitions of complete and first, second modulus of continuity.

For every $f\in C(I^2)$ and $(\theta,\tau),(x,y)\in I^2$ complete and partial modulus of continuity are defined as following respectively in for examples [11]

$$\omega(f,\delta_{n,m})=\sup\left\{|f(\theta,\tau)-f(x,y)|:\sqrt{(\theta-x)^2+(\tau-y)^2}\le\delta_{n,m}\right\},$$

$$\omega^1(f;\delta)=\sup\{|f(x_1,y)-f(x_2,y)|:y\in I\ ve\ |x_1-x_2|\le\delta\},$$

$$\omega^2(f;\delta)=\sup\{|f(x,y_1)-f(x,y_2)|:x\in I\ ve\ |y_1-y_2|\le\delta\}.$$

Theorem 4.1. For sufficiently large n,m and every $f\in C(I^2)$, rate of convergence of $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)$ is calculated by the following inequality using the modulus of continuity

$$\left|\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)\right|\le 2\omega(f;\delta_{n,m}),$$

where

$$\delta_{n,m}=\left[\left(\frac{p_1^{n-1}}{[n]_{p_1,q_1}}\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}\right)^2\left(2\frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}q_1-1\right)\right.$$
$$\left.+\left(\frac{p_2^{m-1}}{[m]_{p_2,q_2}}\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}\right)^2\left(2\frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}}q_2-1\right)\right]^{1/2}.$$

Proof Definition of complete modulus of continuity we get

$$\left|\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)\right|\leq\frac{1}{p_1^{\frac{n(n-1)}{2}}}\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{k=0}^{n}\sum_{j=0}^{m}D_{n,k}(p_1,q_1;x)D_{m,j}(p_2,q_2;y)$$

$$\times\left|f\left(\frac{[n+1]_{p_1,q_1}[k]_{p_1,q_1}}{[n+3]_{p_1,q_1}[n]_{p_1,q_1}p_1^{k-n}},\frac{[m+1]_{p_2,q_2}[j]_{p_2,q_2}}{[m+3]_{p_2,q_2}[m]_{p_2,q_2}p_2^{j-m}}\right)-f(x,y)\right|$$

$$\leq\frac{1}{p_1^{\frac{n(n-1)}{2}}}\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{k=0}^{n}\sum_{j=0}^{m}D_{n,k}(p_1,q_1;x)D_{m,j}(p_2,q_2;y)\,\omega(f;\delta_{n,m})\left(\frac{\sqrt{(\theta-x)^2+(\tau-y)^2}}{\delta_{n,m}}+1\right)$$

$$\leq\omega(f;\delta_{n,m})\left\{1+\frac{1}{\delta_{n,m}}\left\{\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\theta-x)^2+(\tau-y)^2;x,y)\right\}^{1/2}\right\}$$

$$\leq\omega(f;\delta_{n,m})\left\{1+\frac{1}{\delta_{n,m}}\left[\left(\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}\right)^2\left(\frac{p_1^{n-1}}{[n]_{p_1,q_1}}\right)^2\left(2\frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}q_1-1\right)\right.\right.$$

$$\left.\left.+\left(\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}\right)^2\left(\frac{p_2^{m-1}}{[m]_{p_2,q_2}}\right)^2\left(2\frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}}q_2-1\right)\right]^{1/2}\right\}.$$

Using Remark 3.1. and choosing

$$\delta_{n,m}=\left[\left(\frac{p_1^{n-1}}{[n]_{p_1,q_1}}\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}\right)^2\left(2\frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}q_1-1\right)\right.$$
$$\left.+\left(\frac{p_2^{m-1}}{[m]_{p_2,q_2}}\frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}\right)^2\left(2\frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}}q_2-1\right)\right]^{1/2},$$

we get our desired result.

Theorem 4.2. For $f\in C(I^2)$, the following inequality is true

$$\left|\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)\right|\leq 2\left(\omega^1(f;\delta_n)+\omega^2(f;\delta_m)\right),$$

where

$$\delta_n=\sqrt{\left(\frac{p_1^{n-1}}{[n]_{p_1,q_1}}\frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}\right)^2\left(2\frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}q_1-1\right)},$$

$$\delta_m = \sqrt{\left(\frac{p_2^{m-1}}{[m]_{p_2,q_2}} \frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}\right)^2 \left(2\frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}} q_2 - 1\right)}.$$

Proof Applying the Cauchy-Schwarz inequality

$$\left|\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)\right| \leq \tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(\theta,\tau) - f(x,y)|;x,y)$$

$$\leq \omega^1(f;\delta_n)\left[1 + \frac{1}{\delta_n}\left(\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\theta-x)^2;x,y)\right)^{1/2}\right]$$
$$+\omega^2(f;\delta_m)\left[1 + \frac{1}{\delta_m}\left(\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\tau-y)^2;x,y)\right)^{1/2}\right]$$
$$\leq \omega^1(f;\delta_n)\left[1 + \frac{1}{\delta_n}\sqrt{\left(\frac{p_1^{n-1}}{[n]_{p_1,q_1}} \frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}\right)^2 \left(2\frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}} q_1 - 1\right)}\right]$$
$$+\omega^2(f;\delta_m)\left[1 + \frac{1}{\delta_m}\sqrt{\left(\frac{p_2^{m-1}}{[m]_{p_2,q_2}} \frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}\right)^2 \left(2\frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}} q_2 - 1\right)}\right].$$

Choosing

$$\delta_n = \sqrt{\left(\frac{p_1^{n-1}}{[n]_{p_1,q_1}} \frac{[n+1]_{p_1,q_1}}{[n+3]_{p_1,q_1}}\right)^2 \left(2\frac{[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}} q_1 - 1\right)},$$

$$\delta_m = \sqrt{\left(\frac{p_2^{m-1}}{[m]_{p_2,q_2}} \frac{[m+1]_{p_2,q_2}}{[m+3]_{p_2,q_2}}\right)^2 \left(2\frac{[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}} q_2 - 1\right)},$$

then ends the proof of theorem.

Let f be a continuous real valued function $\alpha_1, \alpha_2 \in (0,1]$ and $(\theta,\tau),(x,y) \in I^2$. There exists $M > 0$:

$$|f(\theta,\tau) - f(x,y)| \leq M|\theta - x|^{\alpha_1}|\tau - y|^{\alpha_2},$$

then f is named Lipschitz continuous function. The set of Lipschitz continuous functions demonstrated by $Lip_M(\alpha_1,\alpha_2)$ [18, 22].

Theorem 4.3. Let $(x,y) \in I^2$ and $f \in Lip_M(\alpha_1,\alpha_2)$. (δ_n) and (δ_m) are the sequences defined in the Theorem 4.2., then we get the following inequalities for the operators $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)$

$$\left|\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)\right| \leq M\left(\sqrt{\delta_n}\right)^{\alpha_1}\left(\sqrt{\delta_m}\right)^{\alpha_2}.$$

Proof Let $f \in Lip_M(\alpha_1,\alpha_2)$. Using linearity and positivity of $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)$, we have

$$\left|\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)\right| \leq \tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(\theta,\tau) - f(x,y)|, q_n;x,y)$$

$$\leq M\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(|\theta - x|^{\alpha_1};x)\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(|\tau - y|^{\alpha_2};y).$$

Taking

$p' = \frac{2}{\alpha_1}$, $q' = \frac{2}{2-\alpha_1}$ and $p'' = \frac{2}{\alpha_2}$, $q'' = \frac{2}{2-\alpha_2}$ then applying Hölder's inequality and using $\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(1; x, y) = 1$, we have

$$\left|\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f; x, y) - f(x, y)\right|$$

$$\leq M\left\{\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(|\theta - x|^2; x)\right\}^{\alpha_1/2}\left\{\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(1; x)\right\}^{\alpha_1/2}$$
$$\times\left\{\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(|\tau - y|^2; y)\right\}^{\alpha_2/2}\left\{\tilde{B}_{n,m}^{(p_1,q_1),(p_2,q_2)}(1; y)\right\}^{\alpha_2/2}$$

$$\leq M\left(\sqrt{\delta_n}\right)^{\alpha_1}\left(\sqrt{\delta_m}\right)^{\alpha_2}.$$

Thus, Theorem 4. 3. is proved.

Example 4.1. We give the error estimates for $f(x,y) = \frac{(x+y+\sqrt{2})}{50}$ when $(p_1, q_1) = (0.92, 0.59)$ and $(p_2, q_2) = (0.96, 0.65)$ in Table 1.

Table 1. *Error estimates for different values of* (n, m)

n,m	error estimates with complete modulus of continuity	error estimates with partial modulus of continuity
10, 10	0.1683982664	0.1683469057
$\mathbf{10^2, 10^2}$	0.1695933260	0.1695451434
$\mathbf{10^3, 10^3}$	0.1695933262	0.1695451436
$\mathbf{10^4, 10^4}$	0.1695933261	0.1695451435
$\mathbf{10^5, 10^5}$	0.1695933262	0.1695451436
$\mathbf{10^5, 10^5}$	0.1695933261	0.1695451435
$\mathbf{10^6, 10^6}$	0.1695933262	0.1695451436
$\mathbf{10^7, 10^7}$	0.1695933260	0.1695451434
$\mathbf{10^8, 10^8}$	0.1695933262	0.1695451436

Example 4.2. We give the error estimates for $g(x,y) = \frac{1}{3x^2+5y^2+16}$ when $(p_1, q_1) = (0.90, 0.54)$ and $(p_2, q_2) = (0.95, 0.64)$ in Table 2.

Table 2. *Error estimates for different values of* (n, m)

n,m	error estimates with complete modulus of continuity	error estimates with partial modulus of continuity
10, 10	0.08422241720	0.1334602607
$\mathbf{10^2, 10^2}$	0.08458555582	0.1341653560
$\mathbf{10^3, 10^3}$	0.08458555582	0.1341653560
$\mathbf{10^4, 10^4}$	0.08458555580	0.1341653559
$\mathbf{10^5, 10^5}$	0.08458555578	0.1341653558
$\mathbf{10^5, 10^5}$	0.08458555582	0.1341653559
$\mathbf{10^6, 10^6}$	0.08458555572	0.1341653556
$\mathbf{10^7, 10^7}$	0.08458555582	0.1341653559
$\mathbf{10^8, 10^8}$	0.08458555582	0.1341653559

5. RESULTS

In this paper, a generalized of bivariate (p,q)-Bernstein operators are constructed. Then, rate of convergence of our operators was given. With these results, an example is given about the studies for approximation of bivariate (p,q) analogs of operators.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Bernstein, S.N., "Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités", Communications of the Kharkov Mathematical Society, 13(2): 1-2, (1912).

[2] Phillips, G.M., "Bernstein polynomials based on the q-integers", Annals of Numerical Mathematics, 4: 511-518, (1997).

[3] Ostrovska, S., "q-Bernstein polynomials and their iterates", Journal of Approximation Theory, 123(2): 232–255, (2003).

[4] Wang, H., "Voronovskaya-type formulas and saturation of convergence for q-Bernstein polynomials for $0 < q < 1$", Journal of Approximation Theory, 145:182–195, (2007).

[5] Buyukyazici, I., "On the approximation properties of two-dimensional q-Bernstein-Chlodowsky polynomials", Mathematical Communications, 14(2): 255-269, (2009).

[6] Gonul Bilgin, N., and Cetinkaya, M., "Approximation by three-dimensional q-Bernstein-Chlodowsky polynomials", Sakarya University Journal of Science, 22(6): 1774-1786, (2018).

[7] Mursaleen, M., Ansari, J.A., and Khan, A., "On (p,q)-analogue of Bernstein operators", Applied Mathematics and Computation, 278: 70–71, (2016).

[8] Kanat, K., and Sofyalioglu, M., "Some approximation results for Stancu type Lupaş-Schurer operators based on $(p,)$-integers", Applied Mathematics and Computation, 317: 129-142, (2018).

[9] Kanat, K., and Sofyalıoğlu, M., "Approximation by (p,q)-Lupaş–Schurer–Kantorovich operators", Journal of Inequalities and Applications, 2018(1): 217-229, (2018).

[10] Kanat, K., and Sofyalıoğlu, M., "On Stancu type generalization of (p,q)-Baskakov-Kantorovich operators", Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68(2): 1995-2013, (2019).

[11] Cai, Q. B., and Zhou, G., "On (p, q)-analogue of Kantorovich type Bernstein–Stancu–Schurer operators", Applied Mathematics and Computation, 276: 12-20, (2016).

[12] Kanat, K., and Sofyalıoğlu, M., "Some approximation results for (p, q)-Lupaş-Schurer operators", Filomat, 32(1), 217-229, (2018).

[13] Acar, T., "(p, q) -Generalization of Szász–Mirakyan operators", Mathematical Methods in the Applied Sciences, 39(10): 2685-2695, (2016).

[14] Kanat, K., and Sofyalıoğlu, M., "Approximation Properties of Stancu-Type (p,q)-Baskakov Operators", Bitlis Eren University Journal of Science, 8(3): 889-902, (2019).

[15] Bilgin N. G., and Eren M., “A Generalization of Two Dimensional Bernstein-Stancu Operators”, Sinop University Journal of Science, 6(2): 130-142, (2021).

[16] Mishra V.N. and Pandey, S., “On (p, q) Baskakov–Durrmeyer–Stancu operators”, Advances in Applied Clifford Algebras, 27: 1633-1646, (2017).

[17] Acar, T., Aral, A. and Mohiuddine, S.A., “Approximation by bivariate (p,q)-Bernstein-Kantorovich operators”, Iranian Journal of Science and Technology, Transactions A: Science, 42: 655–662, (2018).

[18] Karaisa, A., “On the approximation properties of bivariate (p,q)-Bernstein operators”, https://arxiv.org/abs/1601.05250v2, Access date: 28.01.2021

[19] Izgi, A., and Karahan, D., “On approximation properties of generalised (p,q)-Bernstein operators”, European Journal of Pure And Applied Mathematics, 11(2): 457-467, (2018).

[20] Cevik, E., “Approximation properties of modified (p,q)-Bernstein type operators”, MSc Thesis, Harran University Institute of Natural and Applied Sciences, Sanliurfa, 16-34, (2019).

[21] Chakrabarti, R., and Jagannathan, R., “A (p,q)-oscillator realization of two-parameter quantum algebras”, Journal of Physics A: Mathematical and General, 24: L711-L718, (1991).

[22] Cao, F., Ding, C., and Xu, Z., “On multivariate Baskakov operator”, Journal of Mathematical Analysis and Applications, 307(1): 274-291, (2005).