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Mathematical Programming for Estimation of Parameters in Random Blocks Model

(Review)

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ABSTRACT

Parameter estimation is quite important in Statistics. Statisticians are engaged in various studies on this problem. Use of optimization methods in the solution of this estimation problem have become common especially after 1970's. The present study has the objective of estimating parameters in a random blocks design, completed random block design, balanced-incomplete random block design, and random block design in the case of a missing observation model equation capitalizing on the significance of optimization methods in statistics. In this study, minimum mean absolute deviations (MINMAD) method is defined and suggests the goal programming (GP) model for estimation of parameters in the random blocks model equation and compares the results obtained with those given by least squares method (LSM)

Keywords: MINMAD, Goal Programming, Randomize Block Design, Completed Random Block Design, Balanced-incomplete Random Block Design, Random Block Design in Case of a Missing Observation

1. INTRODUCTION

Taking the norms L_1, L_2, L_∞ as the basis, the work accomplished can be summarized as follows.

The model incorporating the least squares estimator was tried to be solved by using the quadratic programming algorithm. Later, by using the norm L_∞, the algorithm was suggested under the non-negative constraints of β (1).

There have been many studies conducted for selecting the best subset of multiple linear regression models of norm L_1 and L_∞ Branch-boundary algorithm has been used for the norm L_∞ (1).

For the solution of the MINMAD regression problem modeled as a linear programming problem, an algorithm based upon simplex algorithm but with some changes is given. For ensuring that β estimations in this algorithm remain at the base, changes were introduced to the criteria used in the selection of variables entering in and leaving out of this base (2).

The norm L_1 may be preferred in linear regression because of disadvantages such as the assumption of normality of the least squares estimator and it's proneness to the impact of extreme values in the data set. Therefore, there are quite a few studies where the norm L_1 and optimization methods are used together. Following those studies conducted with the L_1 norm and simplex algorithm, a specific solution has been obtained for the dual linear programming program and a very effective algorithm has been defined for the solution of primal linear programming problem (1,3).

A different linear programming problem incorporating L_1 norm was formulated and discriminate analysis was applied (4).

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For a multivariate multiple regression model, the problem of estimation was formulated by using the L_1 criterion and was solved as a multi objective linear programming problem (5).

Eminkahyagil and Apaydın (1997) estimated β parameters in the multivariate multiple regression model through goal programming. In this model, the problem based on the criterion of the least mean absolute deviations was formulated as a goal programming problem. Algorithm was suggested for the estimation of β parameters in the multivariate multiple regression model (6).

Apaydın (1997) applied a branch-boundary algorithm in the selection of the best subset in multiple linear regression. The multiple linear regression problem was addressed as a linear programming problem considering the least absolute difference criterion or L_1 norm. For the selection of the best model, the regression problem was formulated as a mixed integer programming problem (7).

Section 2 of this paper is devoted to the explanation of some basic concepts and methods in relation to linear programming and simplex process for the regression problem modeled as MINMAD and to the MINMAD method used for the estimation of parameters in a random block model

Section 3 will give the definitions of completed block design, balanced and incomplete block design and random block design in case of missing observation as well as equation systems and other definitions for estimation of parameters.

Finally, Section 4 will engage in a MINMAD estimation of parameters in a random blocks model equation and in case of a missing observation model defined as a goal programming model. The result will then be compared to those of the least squares method.

2. MINIMUM MEAN ABSOLUTE DEVIATION METHOD

In this section, there will be definitions of the MINMAD method for linear and goal programming in the context of the regression problem modeled as MINMAD. A model will be suggested for estimating parameters in linear regression.

Arthanari and Dodge (1982) defined the linear programming problem as

P2.1. $\text{Minimum} \sum d_{1i} + \sum d_{2i}$

$$X\beta + d_1 - d_2 = Y$$

$$d_1, d_2 \geq 0$$

β unrestricted in sign

where X is matrix of known constants, Y is vector of dependent variable and β is vector of unknown parameters to be estimated. d_{1i} is negative deviation for i. observation and d_{2i} is positive deviation for i. observation (1,2,8).

2.1. Minmad Method for Goal Programming

In this case where the system goals are conflicting, a satisfactory solution can be found for decision makers by using multi-purpose decision making methods. Goal programming is one of the most widely used methods in decision making with respect to multiple criteria. It has been shown in earlier studies that in cases where the problem of estimating the parameters of a multivariate multiple regression is redefined as a goal programming model, the estimation of β parameters yield more satisfactory results than those obtained from classical regression analysis. This method is superior to the classical one particularly in cases where the number of observations is limited and multiple connections exist. The objective of goal programming used in regression analysis is the minimization of the difference between observed values and estimation values (6,9).

Considering the constraints on β, Charnes, Cooper and Sueyoshi (1986) defined the goal programming model to be the equivalent of

$$\underset{\beta}{Min} \sum_{j=1}^{n} \left| y_j - \sum_{i=1}^{m} \beta_i X_{ij} \right|, \qquad \beta \in B$$

as

$$Min \sum_{j=1}^{n} (d_j^+ + d_j^-)$$

$$\sum_{i=1}^{m} \beta_i X_{ij} + d_j^- - d_j^+ = y_j$$

$$d_j^- d_j^+ = 0$$

$$d_j^-, d_j^+ \geq 0$$

which is the non-linear goal programming model (10). Narula and Korhonen (1994) made this model linear by adding the constraint CBH=D. Eminkahyagil and Apaydın (1997) defined the goal programming model as

P2.2. $Min \sum_{j=1}^{n} (d_j^- + d_j^+)$

$$\sum_{i=1}^{m} \beta_i X_{ij} + d_j^- - d_j^+ = y_j$$

$$\sum_{i=1}^{m} W_i \beta_i \leq d$$

β_i unrestricted in sign $i=1, \ldots, m$

$d_j^-, d_j^+ \geq 0, \quad j=1,\ldots,n$

and where $\sum W_i = 1, \quad 0 \leq W \leq 1$. d^-, d^+ are negative deviation and positive deviation, respectively (6).

3. ESTIMATION OF PARAMETERS IN RANDOM BLOCK MODELS

In cases where experimental units are not fully homogenous, the design must be developed by dividing these units up into more homogenous sub-units. This will eliminate the heterogeneity of experimental units to a certain extent. These relatively more homogenous sub-unit are called "blocks". Elimination of excess variance in experimental units through the sum of squares amongst blocks will allow for a smaller error variance. Since data in the design of a random blocks experiment are designed with respect to two criteria as "block" and "treatment", the process is also called "double classification" (1, 11).

Let (a) signify the number of treatment and (b) the number of blocks. The statistical model used for this design is a follows:

$$Y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \qquad i=1,\ldots,a \quad j=1,\ldots,b \qquad [1]$$

In the model [1] above,

Y_{ij} : Any observation,

μ : real mean effect,

τ_i : i. treatment effect,

β_j : j. block effect, and

ε_{ij} : random error term with a normal distribution where the mean is 0 and variance is σ^2.

Treatment and blocks are considered as the effect of fixed factors. Treatment and block effects are defined as deviations from the real mean. Then $\sum_{i=1}^{a} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_j = 0$ (11,12).

3.1. Suggested Model

This section suggests models for estimating parameters in random block design.

Model 1.

$$\text{Minimum } e'd_1^- + e'd_2^+$$

$$X\theta + Id_1^- - Id_2^+ = Y$$

$$\sum_{i=1}^{a} \tau_i = 0, \ \sum_{j=1}^{b} \beta_j = 0$$

$$d_1^- \geq 0, \quad d_2^+ \geq 0$$

θ is unrestricted in sign

$\theta = (\mu, \tau_1, \ldots, \tau_a; \beta_1, \ldots, \beta_b)$

where e' is a unit vector, θ is random block design parameters.

In Model 1 above, parameters are given the weight *W* in which case the model turns into the following.

Model 2 has been formed considering Model 1.

Model 2.

$$\text{Minimum } e'd_1^- + e'd_2^+$$

$$X\theta + Id_1^- - Id_2^+ = Y$$

$$W\theta \leq d$$

$$\sum_{i=1}^{a} \tau_i = 0, \ \sum_{j=1}^{b} \beta_j = 0$$

$$d_1^- \geq 0, \quad d_2^+ \geq 0$$

$$d = 1$$

$$\sum W = 1 \quad 0 \leq W \leq 1$$

θ is unrestricted in sign.

3.2. Estimation of Parameters in Completed Random Block Model

The data used for estimating parameters in a completed random block design are given in Appendix A. For the Problems 1, 4 and 5 a=4 and b=4 in the equation [1] are chosen and for the Problems 2 and 3 a=3 and b=4 are chosen. Five problems are solved for Model 1 (M1) and Model 2 (M2). In Table 1 parameter estimation values obtained by computing LSM, Model 1 (M1) and Model 2 (M2) for five problems are given.

Table 1. Estimated parameter values for problems related to completed random block design

	$\hat{\mu}$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{\tau}_3$	$\hat{\tau}_4$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
Problem 1									
LSM	0.028	0.0095	-0.0055	-0.003	-0.0005	0.002	0.0045	0.0045	-0.0105
M1	0.04	-0.01	0	0.01	0	0.02	0	0	-0.02
M2	0.04	-0.01	0	0.01	0	0.02	0	0	-0.02
Problem 2									
LSM	0.067	-0.002	0.003	0		-0.0013	0.0037	0.001	-0.002
M1	0.067	-0.0037	0.003	0.003		-0.0005	0.0045	0.0015	-0.0055
M2	0.067	-0.0037	0.003	0.003		-0.0005	0.0045	0.0015	-0.0055
Problem 3									
LSM	19.6	-0.1	-1.8	1.9		-1.9	0.8	-0.3	1.4
M1	19.6	-0.3	-1.3	1.7		-2.3	1.8	-0.3	0.8
M2	19.2	0.3	-1.6	1.3		-1.5	0.6	-0.5	1.5
Problem 4									
LSM	9.6	-0.1	0	-0.2	0.3	-0.2	-0.2	0.1	0.3
M1	9.6	-0.1	0	-0.2	0.3	-0.2	-0.2	0.1	0.3
M2	9.6	-0.1	0	-0.2	0.3	-0.2	-0.2	0.1	0.3
Problem 5									
LSM	9.3	-0.03	0.09	-0.06	0	-0.06	0.04	0	0.02
M1	9.3	-0.05	0.05	-0.05	0.05	-0.15	0.15	-0.05	0.05
M2	9.3	-0.05	0.05	-0.05	0.05	-0.15	0.15	-0.05	0.05

3.3. Estimation of Parameters in Balanced-incomplete Random Block Design

In case all possible treatment combinations could not be obtained in each block, then the incomplete random block design is used. Such cases arise as a result of such reasons as shortages in experimental tools and equipment or the large size of blocks. Since each and every treatment cannot be realized in each and every block, it is possible to use incomplete random block design. In cases where the number of unrealized observations is equal for all groups, this incomplete random block design is called a "balanced" one. The statistical model used in balanced-incomplete random block design is identical to that given in [1].

It is possible to use randomized block design in which every treatment is not present in every block. These designs are known as randomized incomplete block designs. Such situations generally arise from imperfections in experimental apparatus, equipment and devices or from the large size of blocks. Random block design is used since all treatment combinations are not satisfied for each and every block. When the number of missing observations is equal for each group the completed random block design is named as an incomplete-balanced random block design (13,14). Data used for estimating parameters in a balanced-incomplete random block design are given in Appendix (B). Table 2 below gives the results of parameter estimations for problems relating to the balanced-incomplete random block design. Parameters have been estimated through LSM, Model 1 (M1) and Model 2 (M2).

Table 2. Estimated parameter values for problems related to balanced-incomplete random block design

	$\hat{\mu}$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{\tau}_3$	$\hat{\tau}_4$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
Problem 1									
LSM	3.65	-0.056	0.476	-0.499	0.083	-0.105	-0.3	0.27	0.13
M1	3.7	0.087	0.538	-0.46	-0.163	-0.2	-0.2	0.3	0.1
M2	3.7	0.087	0.538	-0.46	-0.163	-0.2	-0.2	0.3	0.1
Problem 2									
LSM	2.5	-0.263	0.176	-0.026	0.113	1.06	0.85	-0.949	-0.964
M1	2.5	-0.4	0.2	0.2	0	0.8	0.9	-0.7	-1
M2	2.5	-0.4	0.2	0.2	0	0.8	0.9	-0.7	-1
Problem 3									
LSM	72.5	-1.1	-0.9	-0.5	2.5	0.9	3	-3.9	0
M1	72.5	-1.5	-0.5	-0.5	2.5	1	3	-4	0
M2	72.5	-1.5	-0.5	-0.5	2.5	1	3	-4	0
Problem 4									
LSM	8.04	-1.12	-0.405	0.848	0.679	-1.27	0.03	0.536	0.698
M1	8.15	-0.5	-0.5	0.6	0.33	-0.5	-0.1	0.1	0.4
M2	8.15	-0.5	-0.5	0.6	0.33	-0.5	-0.1	0.1	0.4
Problem 5									
LSM	73.25	-1	-0.25	0.25	1	0	1.75	-1.24	-0.5
M1	73	-1	0	0	1	0	2	-1	-1
M2	73	-1	-0.2	0.1	1.1	-0.1	1.9	-0.9	-0.9

3.4. Estimation of Parameters in Random Block Design in Case of A Missing Observation

In the research carried out, trials are set up in equation and in balance with the observation numbers at the beginning. However, sometimes some observations may be missing as a result of external factors within the period lasting till the end of the treatment. In such cases, analysis of the data becomes harder.

In spite of the fact that considerable efforts are spent in a treatment most of the time, the researcher may be faced with undesirable situations in significant applications. Among these, the most common is the problem of missing observations. Missing observations result from several reasons: errors or the reasons out of control, such as death of an animal, the experiment area being exposed to flood, illness of a worker and his inability to work, and loss of recorded data. Then, what are their effects in analysis method? Most of the experiments are designed as balanced or symmetric and missing observations disturb this balance. Thus, the original problem is disturbed and some alterations need to be made within the process.

If the experiment is being carried out with random block design, absence of an observation disturbs the orthogonality between the treatment and blocks. In such cases, it becomes obligatory to apply very complex analysis methods for the observations in different numbers. However, when especially missing observations are one or two, estimating an approximate value for the unknown observation is preferred, rather than applying complex analysis methods (13,14).

The data used for estimating parameters in the case of a missing observation in a random block design are given in Appendix (C). In Table 3, parameter estimation results of 5 Problems for random block design when an observation is a missing are shown. Estimation values of the missing observation have been found using the result given in Table 3 and they are given in Table 4.

Table 3. Parameter estimations values of the problems related to random block design in case of a single missing observation

		$\hat{\mu}$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{\tau}_3$	$\hat{\tau}_4$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
Problem 1										
	LSM	0.025	0.0125	-0.0025	0	-0.01	-0.0075	0.0075	0.0075	-0.0075
	M1	0.0275	0	0	0.02	-0.02	-0.028	0.003	0.033	-0.0075
	M2	0.0275	0	0	0.02	-0.02	-0.028	0.003	0.033	-0.0075
Problem 2										
	LSM	9.625	-0.05	-0.025	-0.175	0.25	-0.225	-0.2	0.1	0.325
	M1	9.625	-0.1	0	-0.2	0.3	-0.2	-0.3	0.08	0.47
	M2	9.625	-0.1	0	-0.2	0.3	-0.2	-0.3	0.08	0.47
Problem 3										
	LSM	9.225	0	0.125	-0.15	0.025	-0.025	0.075	0.025	-0.075
	M1	9.27	0.05	0.15	-0.15	-0.05	-0.025	0.075	0.075	-0.13
	M2	9.27	0.05	0.15	-0.15	-0.05	-0.025	0.075	0.075	-0.13
Problem 4										
	LSM	-0.89	2.14	0.14	-1.11	-1.17	1.89	0.64	-0.17	-2.36
	M1	-0.85	2.5	0.5	-1.5	-1.5	1	1	0	-2
	M2	-0.87	2.4	-0.1	-1.1	-1.1	2	0	0	-2
Problem 5										
	LSM	1.1	-0.39	-0.58	-1.64	2.61	-2.14	-1.89	0.67	3.36
	M1	0.8	-1	0	-2	3	-1.8	-2.8	1.3	3.3
	M2	1.1	-0.83	0.17	-1.83	2.5	-2.3	-1.3	0.7	2.9

Table 4. Estimation values of missing observation

	Problem 1			**Problem 2**			**Problem 3**			**Problem 4**			**Problem 5**		
	LSM	**M1**	**M2**	**LSM**	**M1**	**M2**	**LSM**	**M1**	**M2**	**LSM**	**M1**	**M2**	**LSM**	**M1**	**M2**
$\hat{Y}$	0.001	-0.02	-0.02	10.2	10.4	10.4	9	8.99	8.99	-2.17	-2.35	-2	1.2	1.97	1.97
Y	0.06			10.2			9.5			-3			3		

The data used for estimating parameters in the case of two missing observations in a random block design are given in Appendix (D). In the case of two missing observations, parameter estimation values found with LSM, Model 1 (M1) and Model 2 (M2) have been given in Table 5. Estimation values of missing observations have been found using the results given in Table 5 and they are given in Table 6. Estimation values of the missing observation are shown by $\hat{Y}$, and the real values are shown by *Y*.

Table 5. Parameter estimation values of the problems related to random block design in the case of two missing observations

	$\hat{\mu}$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{\tau}_3$	$\hat{\tau}_4$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
Problem 1									
LSM	9.58	-0.08	0.02	-0.13	0.25	-0.18	-0.153	0.147	0.27
M1	9.625	-0.1	0	-0.2	0.3	-0.2	-0.125	0.075	0.275
M2	9.625	-0.1	0	-0.2	0.3	-0.2	-0.125	0.075	0.275
Problem 2									
LSM	0.023	0.015	-0.001	-0.006	-0.008	-0.006	0.002	0.0095	-0.0055
M1	0.025	0.015	0.015	-0.025	-0.005	0.01	-0.01	0.02	-0.02
M2	0.025	0.015	0.015	-0.025	-0.005	0.01	-0.01	0.02	-0.02

Table 6. Estimation values of the missing observations

	Problem 1			**Problem 2**		
	LSM	**M1**	**M2**	**LSM**	**M1**	**M2**
$\hat{Y}_1$	9.8	9.8	9.8	0.019	-0.01	-0.01
$\hat{Y}_2$	10.1	10.2	10.2	0.09	0.03	0.03
Y_1		10			0.05	
Y_2		10.2			0.06	

4. CONCLUSION AND DISCUSSION

Considering the importance of parameter estimations in statistics, the present study has suggested a goal programming model for the estimation of parameters in the equations of balanced-incomplete random block design and completed random block design as specific states of random block design. And parameter estimations for various problems have been made through this model.

In conclusion, problems of completed random block design and balance-incomplete random block design have been solved though the methods of LMS and Model 1 (M1), Model 2 (M2). Table 1 and Table 2 shows that two methods LSM and Model 1 (M1), Model 2 (M2) give close or even equal values in parameter estimation.

When Table 3 is analyzed, it will be seen that the result generated for the parameter estimation values with the two methods are approximate or equal. W weights given to parameters have been effective. When Problem 4 in the Table 4 is analyzed, the real value of the missing value is (-3). The Parameter estimation value found with LSM is (2.17), the parameter estimation value generated with M1 model is (-2.35), parameter estimation value found with M2 model is (-2). The Parameter estimation value found with

M1 model is more approximate to the real value. A similar comment may be brought to the order problems.

When the Problem 1 in Table 5 is analyzed, the real value of the first missing observation is (10), the real value of the second missing observation is 10.2. Estimations of the missing observations have been made with LSM and Model 1 (M1), Model 2 (M2) and the estimation values found with LSM method are (9.8) and (10.1) and the ones found with suggested models are (9.8) and (10.2).

It is observed that parameter estimation values obtained from respective methods are either equal or very close to each other. The goal programming model suggested here is alternative for the LSM method.

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APPENDIX

A-) Problems used to estimate completed random block design parameters.

Problem 1 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	0.05	0.03	0.06	0.01	0.15
2	0	0.04	0.03	0.02	0.09
3	0.01	0.05	0	0.04	0.1
4	0.06	0.01	0.04	0	0.11
$Y_{.j}$	0.12	0.13	0.13	0.07	0.45

Problem 2 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	0.06	0.065	0.065	0.07	0.26
2	0.07	0.075	0.07	0.065	0.28
3	0.067	0.072	0.069	0.060	0.268
$Y_{.j}$	0.197	0.212	0.204	0.195	0.808

Problem 3 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	18	21	19	20	78
2	16	17	19	19	71
3	19	23	20	24	86
$Y_{.j}$	53	61	58	63	235

Problem 4 (Montgomery 1997)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	9.3	9.4	9.6	10	38.3
2	9.4	9.3	9.8	9.9	38.4
3	9.2	9.4	9.5	9.7	37.8
4	9.7	9.6	10	10.2	39.5
$Y_{.j}$	37.6	37.7	38.9	39.8	154

Problem 5 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	9.3	9.4	9	9.2	36.9
2	9.2	9.5	9.3	9.4	37.4
3	9.1	9	9.2	9.5	36.8
4	9.2	9.3	9.5	9	37
$Y_{.j}$	36.8	37.2	37	37.1	148.1

B-) Problems used to estimate balanced-incomplete random block design estimated parameters.

Problem 1 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	3.55	3.8	3.3	-	10.65
2	4	3.5	-	4.6	12.1
3	3	-	3.5	3.25	9.75
4	-	3.25	4.5	3.55	11.3
$Y_{.j}$	10.55	10.55	11.3	11.4	43.8

Problem 2 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	4	3	0.6	-	7.6
2	3	3.6	-	2.3	8.9
3	3.5	-	2	1	6.5
4	-	3.4	1.8	1.5	6.7
$Y_{.j}$	10.5	10	4.4	4.8	29.7

Problem 3 (Montgomery 1997)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	73	74	-	71	218
2	-	75	67	72	214
3	73	75	68	-	216
4	75	-	72	75	222
$Y_{.j}$	221	224	207	218	870

Problem 4 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	4.37	-	8.32	8.03	20.72
2	-	8.3	7.82	8.05	24.17
3	8.32	8.73	8.91	-	25.96
4	8.03	8.31	-	9.28	25.62
$Y_{.j}$	20.72	25.34	25.05	25.36	96.47

Problem 5 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	73	74	-	71	218
2	-	75	72	72	219
3	73	75	73	-	221
4	74	-	72	75	221
$Y_{.j}$	220	224	217	218	879

C-) Problems used to estimate random block design parameters in the case of a single missing observation.

Problem 1 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	0.05	0.03	0.06	0.01	0.15
2	0	0.04	0.03	0.02	0.09
3	0.01	0.05	0	0.04	0.1
4	y	0.01	0.04	0	0.05+y
$Y_{.j}$	0.06+y	0.13	0.13	0.07	0.39+y

Problem 2 (Montgomery 1997)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	9.3	9.4	9.6	10	38.3
2	9.4	9.3	9.8	9.9	38.4
3	9.2	9.4	9.5	9.7	37.8
4	9.7	9.6	10	y	29.3+y
$Y_{.j}$	37.6	37.7	38.9	29.6+y	143.8+y

Problem 3 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	9.3	9.4	9	9.2	36.9
2	9.2	9.5	9.3	9.4	37.4
3	9.1	9	9.2	y	27.3+y
4	9.2	9.3	9.5	9	138.6+y
$Y_{.j}$	36.8	37.2	37	27.6+y	148.1

Problem 4 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	4	1	0	0	5
2	1	1	0	-5	-3
3	-1	-1	y	-4	y-6
4	0	-2	-2	-4	-8
$Y_{.j}$	4	-1	y-2	-13	y-12

Problem 5 (Montgomery 1997)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	-2	-1	1	5	3
2	-1	-2	y	4	1+y
3	-3	-1	0	2	-2
4	2	1	5	7	15
$Y_{.j}$	-4	-3	6+y	18	17+y

D-) Problems used to estimate random block design parameters in the case of two missing observations.

Problem 1 (Montgomery 1997)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	9.3	9.4	9.6	y_1	$28.3 + y_1$
2	9.4	9.3	9.8	9.9	38.4
3	9.2	9.4	9.5	9.7	37.8
4	9.7	9.6	10	y_2	$29.3 + y_2$
$Y_{.j}$	37.6	37.7	38.9	$19.6+y_1+y_2$	$133.8 + y_1 + y_2$

Problem 2 (Artificial Data)

Treatment	Block 1	2	3	4	$Y_{i.}$
1	0.05	0.03	0.06	0.01	0.15
2	0	0.04	0.03	0.02	0.09
3	0.01	y_1	0	0.04	$0.05 + y_1$
4	y_2	0.01	0.04	0	$0.05 + y_2$
$Y_{.j}$	$0.06+y_2$	$0.08 + y_1$	0.13	0.07	$0.34 + y_1 + y_2$