

PAPER DETAILS

TITLE: Coincidence Points of Hybrid Functions on Cone Metric Spaces

AUTHORS: Kpr RAO,K PARVATHI,Md ALI

PAGES: 655-659

ORIGINAL PDF URL: <https://dergipark.org.tr/tr/download/article-file/83469>



Coincidence Points of Hybrid Functions on Cone Metric Spaces

K.P.R.RAO^{1,*}, K.Siva PARVATHI¹, Md. Mustaq ALI¹

¹*Department of Applied Mathematics, A.N.U.-Dr.M.R.Appa Row Campus,NUZVID-521 201, Krishna Dt, A.P., INDIA.*

Received: 26/03/2011 Accepted:04/07/2011

ABSTRACT

In this paper, we obtain a common coincidence point theorem for two pairs of hybrid functions on cone metric spaces.

Key words: Cone metric spaces, coincidence points, multi functions.
 2000 Mathematics Subject Classification : 47H10, 54H25.

1. INTRODUCTION

In 2007, Huang and Zhang defined cone metric spaces by substituting an ordered normed space for the real numbers ([9]). In 2008, Rezapour and Hambarani characterized types of cones ([17]). Some interesting works about fixed point and common fixed point results on cone metric spaces are [1-8, 10, 11, 13-24] etc.

In this paper, we prove a common coincidence point theorem for two pairs of hybrid functions on cone metric spaces. Our result generalizes and improves the theorems of [18, 19]. First we give some known definitions and lemmas.

Let E be a real Banach space and P a subset of E . P is called a cone whenever

- (i) P is closed, non empty and $P \neq \{0\}$
- (ii) $ax + by \in P$ for all $x, y \in P$ and non negative real numbers a and b
- (iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$, where $\text{int } P$ denotes the interior of P .

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$\|x\| \leq M \|y\|$. The least positive number satisfying the above inequality is called the normal constant of P . Rezapour and Hambarani [17] observed that there are no normal cones with $M < 1$. Hence $M \geq 1$.

Definition 1.1. [9]: Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies

- (d₁) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
- (d₂) $d(x, y) = d(y, x)$ for all $x, y \in X$,
- (d₃) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.2. [9]: Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X . Then

- (i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$, there is a natural

*Corresponding author, e-mail: kprao2004@yahoo.com

number N such that $d(x_n, x) < c$ for all $n \geq N$. we denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$,

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 < c$, there is a natural number N such

that $d(x_n, x_m) < c$ for all $n, m \geq N$,

(iii) (X, d) is a complete metric space if every Cauchy sequence in X is convergent in X .

Definition 1.3. [18]: Let (X, d) be a cone metric space and $B \subseteq X$.

(i) A point $b \in B$ is called an interior point of B whenever there is a $0 < p$ such

that $N(b, p) \subseteq B$, where $N(b, p) = \{y \in X/d(y, b) < p\}$.

(ii) A subset $A \subseteq X$ is called open if each element of A is an interior point of A .

The family $\beta = \{N(x, p) : x \in X, 0 < p\}$ is a sub basis for a topology on X .

we denote this cone topology by τ_c . Then τ_c is Hausdorff and first countable.

Recently Rezapour and Haghi [18] proved the following

Lemma 1.4. (Lemma 2.1, [18]) : Let (X, d) be a cone metric space, P a normal cone with normal constant $M = 1$ and A a compact set in (X, τ_c) . Then for every $x \in X$, there exists $a_0 \in A$ such that

$$\|d(x, a_0)\| = \inf_{a \in A} \|d(x, a)\|.$$

Lemma 1.5. [Lemma 2.2, [19]] : Let (X, d) be a cone metric space, P a normal cone with normal constant $M = 1$ and A, B two compact sets in (X, τ_c) . Then

$$\sup_{x \in B} d^1(x, A) < \infty, \text{ where } d^1(x, A) = \inf_{a \in A} \|d(x, a)\|.$$

Definition 1.6. [18]: Let (X, d) be a cone metric space, P a normal cone with normal constant $M = 1$, $\mathcal{F}_c(X)$ be the set of all compact subsets of (X, τ_c) and $A \in \mathcal{F}_c(X)$. Define $h_A : \mathcal{F}_c(X) \rightarrow [0, \infty)$ and

$$d_H : \mathcal{F}_c(X) \times \mathcal{F}_c(X) \rightarrow [0, \infty) \text{ by } h_A(B) = \sup_{x \in A} d^1(x, B)$$

and $d_H(A, B) = \max\{h_A(B), h_B(A)\}$ respectively. For each $A, B \in \mathcal{F}_c(X)$ and $x, y \in A$, we have

- (i) $d^1(x, A) \leq \|d(x, y)\| + d^1(y, A)$
- (ii) $d^1(x, A) \leq d^1(x, B) + h_B(A)$
- (iii) $d^1(x, A) \leq \|d(x, y)\| + d^1(y, B) + h_B(A)$.

Definition 1.7. : Let $f : X \rightarrow X$ and $F : X \rightarrow \mathcal{F}_c(X)$. f is said to be F -weakly commuting at $x \in X$ if $f^2 x \in F f x$.

Kamran [12] defined the above in metric spaces.

Definition 1.8. : Let ϕ denote the class of all functions $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that ϕ is non decreasing,

continuous and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.

It is clear that $\phi^n(t) \rightarrow 0$ as $n \rightarrow \infty$ for all $t > 0$ and hence, we have $\phi(t) < t$, for all $t > 0$.

Now we give our main result.

2. THE MAIN RESULT

Theorem 2.1. Let (X, d) be a complete cone metric space with normal constant $M = 1$. Let $F, G : X \rightarrow \mathcal{F}_c(X)$ be two multifunctions and $f, g : X \rightarrow X$ be self maps satisfying

$$(2.1.1) d_H(Fx, Gy) \leq \phi$$

$$\left(\max \left\{ \|d(fx, gy)\|, d'(fx, Fx), d'(gy, Gy), \frac{1}{2} [d'(fx, Gy) + d'(gy, Fx)] \right\} \right)$$

for all $x, y \in X$ and $\phi \in \Phi$,

$$(2.1.2) Fx \subseteq g(X), Gx \subseteq f(X) \text{ for all } x \in X,$$

(2.1.3) one of $f(X)$ and $g(X)$ is a complete subset of X and

(2.1.4) f is F -weakly commuting and g is G -weakly commuting at their coincidence points.

Then the pairs (f, F) and (g, G) have the same coincidence point in X .

Proof. Let $x_0 \in X$. Then by Lemma 1.4, there exists

$$gx_1 \in Fx_0 \text{ such that}$$

$d^1(fx_0, Fx_0) = \|d(fx_0, gx_1)\|$. Again by Lemma 1.4, there exists $fx_2 \in Gx_1$ such

$$\text{that } d^1(gx_1, Gx_1) = \|d(gx_1, fx_2)\|.$$

Continuing in this way, we get the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$d^1(y_{2n-1}, Fx_{2n}) = \|d(y_{2n-1}, y_{2n})\| \text{ and } d^1(y_{2n}, Gx_{2n+1}) = \|d(y_{2n}, y_{2n+1})\|, \text{ where}$$

$$y_{2n} = gx_{2n+1} \in Fx_{2n} \text{ and}$$

$$y_{2n+1} = fx_{2n+2} \in Gx_{2n+1} \quad n = 0, 1, 2, \dots$$

Case(i): Suppose $y_{2n} = y_{2n+1}$ for some n .

Assume that $y_{2n+1} \neq y_{2n+2}$.

$$\begin{aligned} \|d(y_{2n+1}, y_{2n+2})\| &= d^1(y_{2n+1}, Fx_{2n+2}) \\ &\leq d_H(Fx_{2n+2}, Gx_{2n+1}) \end{aligned}$$

$$\begin{aligned}
 & \leq \phi \\
 & \left(\max \left\{ \frac{1}{2} [d(y_{2n+1}, y_{2n}), d'(y_{2n+1}, Fx_{2n+2}), d'(y_{2n}, Gx_{2n+1}), \right. \right. \\
 & \quad \left. \left. d'(y_{2n+1}, Gx_{2n+1}) + d'(y_{2n}, Fx_{2n+2}) \right] \right\} \right) \\
 & \leq \phi \\
 & \left(\max \left\{ 0, \|d(y_{2n+1}, y_{2n+2})\|, 0 \right. \right. \\
 & \quad \left. \left. \frac{1}{2} [0 + 0 + \|d(y_{2n+1}, y_{2n+2})\|] \right\} \right) \\
 & = \phi (\|d(y_{2n+1}, y_{2n+2})\|) \\
 & < \|d(y_{2n+1}, y_{2n+2})\|
 \end{aligned}$$

It is a contradiction. Hence $y_{2n+1} = y_{2n+2}$.

Continuing in this way, we have $y_n = y_{n+k}$ for all $k = 1, 2, 3, \dots$. Hence $\{y_n\}$ is a Cauchy sequence in X .

Case (ii): Suppose that $y_n \neq y_{n+1}$ for all n . Now

$$\begin{aligned}
 \|d(y_{2n+1}, y_{2n+2})\| &= d^1(y_{2n}, Gx_{2n+1}) \\
 &\leq d_H(Fx_{2n}, Gx_{2n+1}) \\
 &\leq \phi \\
 & \left(\max \left\{ \frac{1}{2} [d(y_{2n-1}, y_{2n}), d'(y_{2n-1}, Fx_{2n}), d'(y_{2n}, Gx_{2n+1}), \right. \right. \\
 & \quad \left. \left. d'(y_{2n-1}, Gx_{2n+1}) + d'(y_{2n}, Fx_{2n}) \right] \right\} \right) \\
 &\leq \phi \\
 & \left(\max \left\{ \|d(y_{2n-1}, y_{2n})\|, \|d(y_{2n-1}, y_{2n})\|, \|d(y_{2n}, y_{2n+1})\|, \right. \right. \\
 & \quad \left. \left. \frac{1}{2} [\|d(y_{2n-1}, y_{2n+1})\| + 0] \right\} \right) \\
 &\leq \phi \\
 & \left(\max \left\{ \|d(y_{2n-1}, y_{2n})\|, \|d(y_{2n}, y_{2n+1})\|, \right. \right. \\
 & \quad \left. \left. \frac{1}{2} [\|d(y_{2n-1}, y_{2n})\| + \|d(y_{2n}, y_{2n+1})\|] \right\} \right) \\
 &= \phi (\|d(y_{2n-1}, y_{2n})\|).
 \end{aligned}$$

Similarly we can show that $\|d(y_{2n-1}, y_{2n})\| \leq \phi (\|d(y_{2n-2}, y_{2n-1})\|)$.

Thus

$$\begin{aligned}
 \|d(y_n, y_{n+1})\| &\leq \phi (\|d(y_{n-1}, y_n)\|) \leq \phi^2 \\
 (\|d(y_{n-2}, y_{n-1})\|) &\leq \dots \leq \phi^n (\|d(y_0, y_1)\|)
 \end{aligned}$$

Now for $n > m$ we have

$$\begin{aligned}
 \|d(y_n, y_m)\| &\leq \sum_{i=m+1}^n \|d(y_i, y_{i-1})\| \\
 &\leq \phi^m (\|d(y_0, y_1)\|) + \phi^{m+1} \\
 &(\|d(y_0, y_1)\|) + \dots + \phi^n (\|d(y_0, y_1)\|) \\
 &\leq \sum_{i=m}^{\infty} \phi^i (\|d(y_0, y_1)\|) \\
 &\rightarrow 0 \text{ as } m \rightarrow \infty, \text{ since } \sum_{n=1}^{\infty} \phi^n(t) < \infty
 \end{aligned}$$

for all $t > 0$.

This implies that $\lim_{m,n \rightarrow \infty} \|d(y_n, y_m)\| = 0$.

By Lemma 4, ($[2]$), $\{y_n\}$ is a Cauchy sequence in X .

Suppose $g(X)$ is complete.

Then $y_{2n} = g x_{2n+1} \rightarrow p = g v \in g(X)$ for some p and $v \in X$.

Since $\{y_n\}$ is Cauchy, we have $y_{2n+1} \rightarrow p$.

$$\begin{aligned}
 d'(p, Gv) &\leq \|d(p, y_{2n})\| + d'(y_{2n}, Gv) \\
 &\leq \|d(p, y_{2n})\| + d_H(Fx_{2n}, Gv) \\
 &\leq \|d(p, y_{2n})\| + \phi \\
 & \left(\max \left\{ \|d(y_{2n-1}, g v)\|, d'(y_{2n-1}, Fx_{2n}), d'(g v, G v), \right. \right. \\
 & \quad \left. \left. \frac{1}{2} [d'(y_{2n-1}, G v) + d'(g v, Fx_{2n})] \right\} \right) \\
 &\leq \|d(p, y_{2n})\| + \phi \\
 & \left(\max \left\{ \|d(y_{2n-1}, p)\|, \|d(y_{2n-1}, y_{2n})\|, d'(p, G v), \right. \right. \\
 & \quad \left. \left. \frac{1}{2} [\|d(y_{2n-1}, p)\| + d'(p, G v) + \|d(p, y_{2n})\|] \right\} \right)
 \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$d^1(p, Gv) \leq \phi (d^1(p, Gv)) \text{ so that } d^1(p, Gv) = 0.$$

Hence $p \in Gv$. Thus $g v = p \in G v$.

Since g is G -weakly commuting at the coincidence point v , we have $g p = g^2 v \in G g v = G p$. Thus p is a coincidence point of g and G . Since $G v \subseteq f(X)$, there exists $w \in X$ such that $p = g v = f w \in G v$.

$$\begin{aligned}
 d'(p, Fw) &\leq \|d(p, y_{2n+1})\| + d'(y_{2n+1}, Fw) \\
 &\leq \|d(p, y_{2n+1})\| + d_H(Fw, Gx_{2n+1}) \\
 &\leq \|d(p, y_{2n+1})\| + \phi
 \end{aligned}$$

$$\left(\max \left\{ \begin{aligned} &\|d(fw, y_{2n})\|, d'(fw, Fw), d'(y_{2n}, Gx_{2n+1}), \\ &\frac{1}{2}[d'(fw, Gx_{2n+1}) + d'(y_{2n}, Fw)] \end{aligned} \right\} \right) \\ \leq \|d(p, y_{2n+1})\| + \phi \\ \left(\max \left\{ \begin{aligned} &\|d(p, y_{2n})\|, d'(p, Fw), \|d(y_{2n}, y_{2n+1})\|, \\ &\frac{1}{2}[\|d(p, y_{2n+1})\| + \|d(y_{2n}, p)\| + d'(p, Fw)] \end{aligned} \right\} \right)$$

Letting $n \rightarrow \infty$, we get

$$d^1(p, Fw) \leq \phi (d^1(p, Fw)) \text{ so that } d^1(p, Fw) = 0.$$

Hence $p \in Fw$. Thus $fw = p \in Fw$.

Since f is F -weakly commuting at the coincidence point w , we have $fp = ffw \in Fw = Fp$.

Thus p is a coincidence point of f and F . Hence, the pairs (f, F) and (g, G) have the same coincidence point.

By putting $f = g = I$ (the identity map) in Theorem 2.1, we have

Corollary 2.2. Let (X, d) be a complete cone metric space with normal constant $M = 1$. Let $F, G : X \rightarrow \mathcal{F}_c(X)$ be two multi functions satisfying

$$(2.2.1) d_H(Fx, Gy) \leq \alpha$$

$$\left(\max \left\{ \begin{aligned} &\|d(x, y)\|, d'(x, Fx), d'(y, Gy), \\ &\frac{1}{2}[d'(x, Gy) + d'(y, Fx)] \end{aligned} \right\} \right)$$

for all $x, y \in X$, where $\alpha \in [0, 1)$.

Then F and G have a common fixed point in X .

Corollary 2.2 is a generalization and improvement of Theorems 1 and 2 of [19] for a pair of multi functions and of Theorems 2.6 and 2.7 of [18] for a single multi function with $G = F$.

REFERENCES

- [1] .Abbas M, and Jungck, G., "Common fixed point result for non commuting mappings without continuity in cone metric spaces", *J.Math.Anal.Appl.* **341**:416-420(2008).
- [2] Azam, A., Arshad M., and Beg., I., "Common fixed points of two maps in cone metric spaces", *Rend.Ciric.Mat.Palermo*, **57**:433-441 (2008).
- [3] A.Azam, M.Arshad, "Common fixed points of generalized contractive maps in cone uniform spaces", *Bull.Iranian Math.Soc.*, **35**(2) 255-264. (2009).
- [4] Arshad, M., Azam, A., Vetro, P., "Some common fixed point results in cone uniform spaces", *Fixed Point Theory and Appl.* 2009, Article ID 493965, 11 Pages, doi: 10.1155/2009/493965.
- [5] Bari C.Di, and Vetro. P., " ϕ -pairs and common fixed points in cone metric spaces", *Rend.Ciric.Mat. Palermo*, **57**:279-285 (2008).
- [6] Bari C.Di, and Vetro P., "Weakly ϕ -pairs and common fixed points in cone metric spaces" *Ciric.Mat.Palermo*, **58** 125-132 (2009).
- [7] Haghi, R.H., Rezapour, Sh., "Fixed points of multifunctions on regular cone metric spaces", *Expo.Math.*, **28**:71-77 (2010).
- [8] Harjani, J., Sadarangani, K., "Generalized contractions in partially ordered metric spaces and Applications to ordinary differential equations", *Nonlinear Analysis*, **72**(2010), 1188-1197.
- [9] L.G.Huang and X.Zhang., "Cone metric spaces and fixed point theorems of contractive mappings", *J.Math.Anal.Appl.* **332**(2007), 1468-1476.
- [10] D.Ilic and V.Rakocevic., "Common fixed point result for maps on cone metric spaces", *J.Math. Anal.Appl.* **341**(2008), 867-882.
- [11] S.Jankovic, Z.Kadelburg, S.Radonevic, B.E.Rhoades, "Assad-Kirk type fixed point theorems for a pair of nonself mappings on cone metric spaces", *Fixed point Theory and Appl.* Vol.2009, Article ID 761086, 16 Pages, doi:10.1155/2009/761086.
- [12] T.Kamran., "Coincidence and fixed points for hybrid strict contractions", *J.Math.Anal.Appl.* 299464-468(2007).
- [13] D.Kim, D.Wardowski, "Dynamics processes and fixed points of set-valued nonlinear contractions in cone metric spaces", *Nonlinear Analysis* **71**, 5170-5175(2009).
- [14] Z.Kadelburg, S.Radonevic, B.Rosic, "Strict contractive conditions and common fixed point theorems in cone metric spaces", *Fixed point Theory and Appl.* Vol.2009, Article ID 173838, 14 Pages, doi:10.1155/2009/173838.
- [15] Z.Kadelburg, S.Radonevic, V.Rakocevic, "Remarks on quasi-contraction on a cone metric space", *Appl.Math.Lett.*, **22**:1674-1679(2009).
- [16] Radonevic, S., "Common fixed points under contractive conditions in cone metric spaces", *Computer and Math.with Appl.*, **58** 1273-1278(2009).
- [17] .Rezapour Sh., and Hamalbarani, R., "Some notes on pair cone metric spaces and fixed point theorems of contractive mappings", *J.Math.Anal.Appl.* **345**(2008), 719-724.
- [18] Rezapour Sh, and Haghi., R.H, "Fixed points of multi functions on cone metric spaces", *Numerical Functional Analysis and optimization*, **30**:7(2009), 825-832. DOI:10.1080-01630560903123346.

- [19] Rezapour S., Khandani H., and Vaezpour., S., M., “Efficacy of cones on topological vector spaces and application to common fixed points of multi functions”, **Rend.Circ.Mat. Palermo**,59 185-197(2010).
- [20] Rezapour, Sh., Haghi, R.H., Shahzad, N., “Some notes on fixed points of quasi-contraction maps”, **Appl.Math.Lett.**,23:498-502(2010).
- [21] Wei-Shih Du, “A note on cone metric fixed point theory and its equivalence”,**Nonlinear Analysis**, 72:2259-2261(2010).
- [22] Vetro., P., “Common fixed points in cone metric spaces” , **Rend. Circ. Mat. Palermo**, 56:464-468 (2007).
- [23] Włodarczyk, K., Plebaniak, R., Obczynski, C., “Convergence theorems,best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces”, **Nonlinear Analysis** 72:794-805(2010).
- [24] Zhao, Z., Chen, X., “Fixed points of decreasing operators in ordered Banach spaces and applications to nonlinear second order elliptic equations”,**Computer and Math.with Appl.** 58:1223-1229 (2009).