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Some Bounds and the Conditional Maximum Bound for Restricted Isometry Constants

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ABSTRACT

Compressed sensing seeks to recover an unknown sparse signal with p entries by making far fewer than p measurements. The restricted isometry Constants (RIC) has become a dominant tool used for such cases since if RIC satisfies some bound then sparse signals are guaranteed to be recovered exactly when no noise is present and sparse signals can be estimated stably in the noisy case. During the last few years, a great deal of attention has been focused on bounds of RIC, see, e. g., Candes (2008), Foucart et al (2009), Foucart (2010), Cai et al (2010), Mo et al (2011), Ji et al (2012). Finding bounds of RIC has theoretical and applied significance. In this paper, we obtain a bound of RIC. It improves the results by Cai et al (2010) and Ji et al (2012). Further, we discuss the problems related larger bound of RIC, and give the conditional maximum bound.

Keywords: Compressed sensing, L_1 minimization, restricted isometry property, sparse signal recovery.

1. INTRODUCTION

Compressed sensing aims to recover high-dimensional sparse signals based on considerably fewer linear measurements. We consider

$$y = \Phi\beta + z, \qquad (1)$$

where the matrix $\Phi \in \mathbb{R}^{n\times p}$ with $n \ll p$, $z \in \mathbb{R}^{n}$ is a vector of measurement errors, and the unknown signal $\beta \in \mathbb{R}^{p}$. Our goal is to reconstruct β based on y and Φ.

A naive approach for solving this problem is to consider

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L_0 minimization where the goal is to find the sparsest solution in the feasible set of possible solutions. However, this is NP hard and thus is computationally infeasible. It is then natural to consider the method of L_1 minimization which can be viewed as a convex relaxation of L_0 minimization. The L_1 minimization method in this context is

$$\hat{\beta} = \arg\min_{\gamma \in \mathbb{R}^p} \left\{ \|\gamma\|_1 \text{ subject to } \|y - \Phi\gamma\|_2 \leq \varepsilon \right\} \quad (2)$$

This method has been successfully used as an effective way for reconstructing a sparse signal in many settings. See, e. g., [1-8].

Recovery of high dimensional sparse signals is closely connected with Lasso and Dantzig selectors, e. g., see, [6, 9-12]. One of the most commonly used frameworks for sparse recovery via L_1 minimization is the Restricted Isometry Property (RIP) with a RIC introduced by Candes and Tao [3]. It has been shown that L_1 minimization can recover a sparse signal with a small or zero error under various conditions on δ_k and $\theta_{k,k'}$ (See Section 2). For example, the condition $\delta_k + \theta_{k,k} + \theta_{k,2k} < 1$ is used in [3], $\delta_{3k} + 3\delta_{4k} < 2$ in [4], $\delta_{2k} + \theta_{k,2k} < 1$ in [6], $\delta_{1.5k} + \theta_{k,1.5k} < 1$ in [13] and $\delta_{1.25k} + \theta_{k,1.25k} < 1$ in [8].

The RIP conditions are difficult to verify for a given matrix Φ. A widely used technique for avoiding checking the RIP directly is to generate the matrix Φ randomly and to show that the resulting random matrix satisfies the RIP with high probability using the well-known Johnson–Lindenstrauss Lemma. (See, for example, [14]). This is typically done for conditions involving only the restricted isometry constant δ. Attention has been focused on δ_{2k} as it is obviously necessary to have $\delta_{2k} < 1$ for model identifiability. In a recent paper, Davies and Gribonval [15] constructed examples which showed that if $\delta_{2k} \geq 0.7071$, exact recovery of certain k sparse signal can fail in the noiseless case. On the other hand, sufficient conditions on δ_{2k} has been given. For example, $\delta_{2k} < 0.4142$ is used in [16], $\delta_{2k} < 0.4531$ in [17], $\delta_{2k} < 0.4652$ in [18], $\delta_{2k} < 0.4721$ in [8], $\delta_{2k} < 0.4734$ in [18] and $\delta_{2k} < 0.4931$ in [19]. Some sufficient conditions on δ_k has been given. For example, $\delta_k < 0.307$ is used in [20], and $\delta_k < 0.308$ in [21] when k is even. In this paper $\delta_k < 0.308$ is given for any k, and the conditional maximum bound $\delta_k < 0.5$ is obtained.

There are several benefits for improving the bound on δ_k. First, it allows more measurement matrices to be used in compressed sensing. Secondly, for the same matrix Φ, it allows k to be larger, that is, it allows recovering a sparse signal with more nonzero elements. Furthermore, it gives better error estimation in a general problem to recover noisy compressible signals.

The rest of the paper is organized as follows. In Section 2, some basic notations and known results are introduced. Our new RIC bounds of compressed sensing matrices are presented in Section 3. In Section 4, we discuss the problems related larger bound of RIC, and give conditional maximum bound.

2. PRELIMINARIES

Let $\|u\|_0$ be the number of nonzero elements of vector $u = (u_i) \in R^p$. u is called k-sparse if $\|u\|_0 \leq k$. For an $n \times p$ matrix Φ and an integer k, $1 \leq k \leq p$, the k restricted isometry constant $\delta_k(\Phi)$ is the smallest constant such that

$$\sqrt{1-\delta_k(\Phi)}\,\|u\|_2 \le \|\Phi u\|_2 \le \sqrt{1+\delta_k(\Phi)}\,\|u\|_2$$

(3)

for every k - sparse vector u. If $k+k' \le p$, the k, k' restricted orthogonality constant $\theta_{k,k'}(\Phi)$, is the smallest number that satisfies

$$\left|\langle \Phi u, \Phi u' \rangle\right| \le \theta_{k,k'}(\Phi)\|u\|_2\|u'\|_2 \qquad (4)$$

for all u and u' such that u and u' are k -sparse and k'-sparse respectively, and have disjoint supports. For notational simplicity, we shall write δ_k for $\delta_k(\Phi)$ and $\theta_{k,k'}$ for $\theta_{k,k'}(\Phi)$ hereafter.

The following monotone properties can be easily checked

$$\delta_k \le \delta_{k'}, \quad \text{if } k \le k' \le p. \qquad (5)$$

$$\theta_{k,k'} \le \theta_{j,j'}, \quad \text{if } k \le j, \quad k' \le j' \quad \text{and} \quad j+j' \le p.$$

(6)

Candes et al [3] showed that the constants and are related by the following inequalities

$$\theta_{k,k'} \le \delta_{k+k'} \le \theta_{k,k'} + \max(\delta_k, \delta_{k'}). \qquad (7)$$

Cai et al [8] showed that for any $a \ge 1$ and positive integers k, k' such than ak' is an integer, then

$$\theta_{k,ak'} \le \sqrt{a}\,\theta_{k,k'}. \qquad (8)$$

Cai et al [20] showed that for any $x \in \mathbb{R}^n$

$$\|x\|_2 \le \frac{\|x\|_2}{\sqrt{n}} + \frac{\sqrt{n}}{4}\left(\|x\|_\infty - \|x\|_{-\infty}\right). \qquad (9)$$

where $\|x\|_\infty = \max_{1\le i\le n}|x_i|$ and $\|x\|_{-\infty} = \min_{1\le i\le n}|x_i|$.

3. NEW RIC BOUNDS OF COMPRESSED SENSING MATRICES

In this section, we consider new RIP conditions for sparse signal recovery. Suppose

$$y = \Phi\beta + z$$

with $\|z\|_2 \le \varepsilon$. Denote $\hat{\beta}$ the solution of the following L_1 minimization problem:

$$\hat{\beta} = \arg\min_{\gamma\in\mathbb{R}^p}\left\{\|\gamma\|_1 \text{ subject to } \|y-\Phi\gamma\|_2 \le \varepsilon\right\} \qquad (10)$$

The following is one of our main results of the paper.

Theorem 1. Suppose β is k sparse with $k > 1$. Then under the condition

$$\delta_k < 0.308$$

the constrained L_1 minimizer $\hat{\beta}$ given in (10) satisfies

$$\left\|\beta - \hat{\beta}\right\|_2 \le \frac{\varepsilon}{0.308-\delta_k}.$$

In particular, in the noiseless case $\hat{\beta}$ recovers β exactly.

This theorem improves $\delta_k < 0.307$ in [20] to $\delta_k < 0.308$, and k is even in [21] to any k. The proof of the theorem is very long but elementary.

Proof. Let s, k be positive integers, $1 \le s < k$, and

$$t=\sqrt{\frac{k}{s}}+\frac{1}{4}\sqrt{\frac{s}{k}}\,.$$

Then from Theorem 3.1 in [20], under the condition $\delta_k+t\theta_{k,s}<1$, we have

$$\left\|\beta-\hat{\beta}\right\|_2\leq\frac{2\sqrt{2}\sqrt{1+\delta_k}}{1-\delta_k-t\theta_{k,s}}\varepsilon\,.$$

By (8)

$$t\theta_{k,s}=t\theta_{\frac{k}{k-s}(k-s),s}\leq t\sqrt{\frac{k}{k-s}}\delta_k\,. \tag{11}$$

We show below that

$$\sqrt{\frac{k}{k-s}}\left(\sqrt{\frac{k}{s}}+\frac{1}{4}\sqrt{\frac{s}{k}}\right)=\frac{1}{\sqrt{x}}+\frac{5}{4}\sqrt{x}\triangleq f(x)$$

(12)

where $x=\dfrac{s}{k-s}$. The proof is of elementary trigonometric functions, but it is very clever.

Let $s=k\sin^2\alpha$, $\alpha\in(0,\frac{\pi}{2})$, then $k-s=k\cos^2\alpha$.

So

$$\sqrt{\frac{k}{k-s}}\left(\sqrt{\frac{k}{s}}+\frac{1}{4}\sqrt{\frac{s}{k}}\right)=\frac{1}{\cos\alpha}\left(\frac{1}{\sin\alpha}+\frac{\sin\alpha}{4}\right)$$

$$=\frac{1}{\tan\alpha}+\frac{5}{4}\tan\alpha=\frac{1}{\sqrt{x}}+\frac{5}{4}\sqrt{x}\,.$$

It is easy to see $f(x)$ is increasing when $x\geq\frac{4}{5}$ and decreasing when $x\leq\frac{4}{5}$. Thus $f(x)$ obtains the minimum value

$$f\left(\frac{4}{5}\right)=\sqrt{5}\,.$$

That is, if $k\equiv 0(\bmod 9)$, let $s=\frac{4}{9}k$, then under the condition $\delta_k<0.309$, we have, see [20],

$$\left\|\beta-\hat{\beta}\right\|_2\leq\frac{\varepsilon}{0.309-\delta_k}\,. \tag{13}$$

If k is even, let $s=\dfrac{k}{2}$, then

$$f(1)=2.250\,. \tag{14}$$

If $k\geq 9$ is odd, let $s=\dfrac{k-1}{2}$, then

$$f\left(\frac{4}{5}\right)\leq f\left(\frac{k-1}{k+1}\right)<f(1)\,. \tag{15}$$

since $f(x)$ is increasing when $x\geq\frac{4}{5}$.

When $k=7$, then

$$f\left(\frac{3}{4}\right)=\frac{31\sqrt{3}}{24}=2.237\,. \tag{16}$$

When $k=5$, then

$$f(x)=f\left(\frac{2}{3}\right)=\frac{11}{2\sqrt{6}}=2.245\,. \tag{17}$$

When $k=3$, we note from the remark of Theorem 3.1 in [20] that in these cases $s=1$ and $t=\sqrt{k}$, then

$$t\sqrt{\frac{k}{k-s}}=\sqrt{3}\sqrt{\frac{3}{2}}=2.121\,. \tag{18}$$

From (11) - (18) yield

$$\delta_k+t\theta_{k,\frac{k}{2}}\leq 3.25\delta_k<1$$

if k is even, and

$$\delta_k+t\theta_{k,\frac{k-1}{2}}\leq 3.25\delta_k<1$$

if k is odd. With the above relations, we can also get

$$\left\|\beta-\hat{\beta}\right\|_2\leq\frac{2\sqrt{2}\sqrt{1+\delta_k}}{1-\delta_k-t\theta_{k,s}}\varepsilon\leq\frac{\varepsilon}{0.308-\delta_k}\,.$$

Corollary 1. Suppose β is k sparse with $k\equiv 0(\bmod 9)$. Then under the condition

$$\delta_k<0.309$$

the constrained L_1 minimizer $\hat{\beta}$ given in (10) satisfies

$$\left\|\beta-\hat{\beta}\right\|_2 \le \frac{\varepsilon}{0.309-\delta_k}.$$

In particular, in the noiseless case $\hat{\beta}$ recovers β exactly.

The proof sees (11)-(13).

Corollary 2. Suppose β is k sparse. If $k \ge 9$ is odd, then under the condition

$$\delta_k < c_k$$

the constrained L_1 minimizer $\hat{\beta}$ given in (10) satisfies

$$\left\|\beta-\hat{\beta}\right\|_2 \le \frac{\varepsilon}{c_k-\delta_k}.$$

where

$$c_k = \frac{4\sqrt{k^2-1}}{4\sqrt{k^2-1}+9k-1}.$$

In particular, in the noiseless case $\hat{\beta}$ recovers β exactly.

The proof sees (11)-(12) and (15).

Note that $0.308 < c_k \le 0.309$ from (15).

To the best of our knowledge, this seems to be the first result for sparse recovery with conditions that only involve δ_k and k. In fact, only involving δ_k, k and only involving δ_k are equivalent.

4. THE CONDITIONAL MAXIMUM BOUND FOR RIC

Let $h=\hat{\beta}-\beta$. For any subset $Q \subset \{1,2,\cdots,p\}$, we define $h_Q = hI_Q$, where I_Q denotes the indicator function of the set Q, i.e., $I_Q(j)=1$ if $j \in Q$ and 0 if $j \notin Q$. Let T be the index set of the k largest elements (in absolute value) and let Ω be the support of β. The following fact, which is based on the minimality of $\hat{\beta}$, has been widely used, see [4].

$$\left\|h_\Omega\right\|_1 \ge \left\|h_{\Omega^c}\right\|_1. \tag{19}$$

We shall show that

$$\left\|h_T\right\|_1 \ge \left\|h_{T^c}\right\|_1, \tag{20}$$

$$\left\|h_T\right\|_2 \ge \left\|h_{T^c}\right\|_2. \tag{21}$$

In fact

$$\left\|h_T\right\|_1 + \left\|h_{T^c}\right\|_1 = \left\|h\right\|_1 = \left\|h_\Omega\right\|_1 + \left\|h_{\Omega^c}\right\|_1,$$

and T has the k largest elements (in absolute value) and Ω has at most k elements, so we have by (19)

$$\left\|h_T\right\|_1 \ge \left\|h_\Omega\right\|_1 \ge \left\|h_{\Omega^c}\right\|_1 \ge \left\|h_{T^c}\right\|_1.$$

And

$$\left\|h_{T^c}\right\|_2^2 \le \left\|h_{T^c}\right\|_1 \left\|h_{T^c}\right\|_\infty \le \left\|h_T\right\|_1 \frac{\left\|h_T\right\|_1}{k} \le \left\|h_T\right\|_2^2.$$

Definition 1. Let T_m be the index set of the m largest elements (in absolute value). The set T_m is called a sparse index set, if $\left\|h_{T_m}\right\|_1 \ge \left\|h_{T_m^c}\right\|_1$ and $m \le k$.

It is obvious that the sparse index set exists. In fact T_k is a sparse index set since $\left\|h_{T_k}\right\|_1 \ge \left\|h_{T_k^c}\right\|_1$.

Here we prove that any sparse index set T_m instead of T_k, Theorem 3.1 in [20] can be improved.

Theorem 2. Suppose β is k-sparse, and T_m is sparse index set. Let k_1, k_2 be positive integers such that

$$k_1 \ge m \text{ and } 8(k_1-m) \le k_2.$$

Let

$$t=\sqrt{\frac{k_1}{k_2}}+\frac{1}{4}\sqrt{\frac{k_2}{k_1}}-\frac{2(k_1-m)}{\sqrt{k_1k_2}}.$$

Then under the condition

$$\delta_{k_1}+t\theta_{k_1,k_2}<1$$

the L_1 minimizer defined in (10) satisfies

$$\left\|\beta-\hat{\beta}\right\|_2\le\frac{2\sqrt{2}\sqrt{1+\delta_{k_1}}}{1-\delta_{k_1}-t\theta_{k_1,k_2}}\varepsilon.$$

In particular, in the noiseless case where $y=\Phi\beta$, L_1 minimization recovers β exactly.

Proof. For any sparse index set T_m, let $S_0\supset T_m$ be the index set of the k_1 largest elements (in absolute value). Rearrange the indices of S_0^c if necessary according to the descending order of $|h_i|$, $i\in S_0^c$. Partition S_0^c into $S_0^c=\sum_{i\ge1}S_i$, where $|S_i|=k_2$, the last S_i satisfies $|S_i|\le k_2$. If $h_{S_0}=0$, then the theorem is trivially true. So here, we assume that $h_{S_0}\neq 0$. Then it follows from (9) that

$$\sum_{i\ge1}\left\|h_{S_i}\right\|_2\le\frac{1}{\sqrt{k_2}}\sum_{i\ge1}\left\|h_{S_i}\right\|_1+\frac{\sqrt{k_2}}{4}\sum_{i\ge1}\left(\left\|h_{S_i}\right\|_\infty-\left\|h_{S_i}\right\|_{-\infty}\right)$$

$$\le\frac{1}{\sqrt{k_2}}\sum_{i\ge1}\left\|h_{S_i}\right\|_1+\frac{\sqrt{k_2}}{4}\left\|h_{S_1}\right\|_\infty$$

$$=\frac{1}{\sqrt{k_2}}\left\|h_{S_0^c}\right\|_1+\frac{\sqrt{k_2}}{4}\left\|h_{S_1}\right\|_\infty$$

$$=\frac{1}{\sqrt{k_2}}\left(\left\|h_{T_m^c}\right\|_1-\left\|h_{S_0\cap T_m^c}\right\|_1\right)+\frac{\sqrt{k_2}}{4}\left\|h_{S_1}\right\|_\infty$$

$$\le\frac{1}{\sqrt{k_2}}\left(\left\|h_{T_m}\right\|_1-\left\|h_{S_0\cap T_m^c}\right\|_1\right)+\frac{\sqrt{k_2}}{4}\left\|h_{S_1}\right\|_\infty$$

$$=\frac{1}{\sqrt{k_2}}\left(\left\|h_{S_0}\right\|_1-2\left\|h_{S_0\cap T_m^c}\right\|_1\right)+\frac{\sqrt{k_2}}{4}\left\|h_{S_1}\right\|_\infty$$

$$\le\frac{1}{\sqrt{k_2}}\left(\left\|h_{S_0}\right\|_1-2(k_1-m)\left\|h_{S_1}\right\|_\infty\right)+\frac{\sqrt{k_2}}{4}\left\|h_{S_1}\right\|_\infty$$

$$=\frac{1}{\sqrt{k_2}}\left\|h_{S_0}\right\|_1+\left(\frac{\sqrt{k_2}}{4}-\frac{2(k_1-m)}{\sqrt{k_2}}\right)\left\|h_{S_1}\right\|_\infty$$

$$\le\left(\frac{\sqrt{k_1}}{\sqrt{k_2}}+\frac{\sqrt{k_2}}{4\sqrt{k_1}}-\frac{2(k_1-m)}{\sqrt{k_1k_2}}\right)\left\|h_{S_0}\right\|_2=t\left\|h_{S_0}\right\|_2$$

Now

$$\left|\left\langle\Phi h,\Phi h_{S_0}\right\rangle\right|=\left|\left\langle\Phi h_{S_0},\Phi h_{S_0}\right\rangle+\sum_{i\ge1}\left\langle\Phi h_{S_i},\Phi h_{S_0}\right\rangle\right|$$

$$\ge\left(1-\delta_{k_1}\right)\left\|h_{S_0}\right\|_2^2-\theta_{k_1,k_2}\left\|h_{S_0}\right\|_2\sum_{i\ge1}\left\|h_{S_i}\right\|_2$$

$$\ge\left(1-\delta_{k_1}-t\theta_{k_1,k_2}\right)\left\|h_{S_0}\right\|_2^2.$$

Note that

$$\left\|\Phi h\right\|_2\le\left\|\Phi\beta-y\right\|_2+\left\|\Phi\hat{\beta}-y\right\|_2\le2\varepsilon.$$

$$\left|\left\langle\Phi h,\Phi h_{S_0}\right\rangle\right|\le\left\|\Phi h\right\|_2\left\|\Phi h_{S_0}\right\|_2\le2\varepsilon\sqrt{1+\delta_{k_1}}\left\|h_{S_0}\right\|_2.$$

Also the next relation

$$\left\|h_{S_0^c}\right\|_2^2\le\left\|h_{T_m^c}\right\|_2^2\le\left\|h_{T_m^c}\right\|_1\left\|h_{T_m^c}\right\|_\infty$$

$$\le\left\|h_{T_m}\right\|_1\frac{\left\|h_{T_m}\right\|_1}{m}\le\left\|h_{T_m}\right\|_2^2\le\left\|h_{S_0}\right\|_2^2$$

implies

$$\left\|h\right\|_2^2=\left\|h_{S_0}\right\|_2^2+\left\|h_{S_0^c}\right\|_2^2\le2\left\|h_{S_0}\right\|_2^2.$$

Putting them together we get

$$\left\|h\right\|_2\le\sqrt{2}\left\|h_{S_0}\right\|_2\le\frac{2\sqrt{2}\sqrt{1+\delta_{k_1}}}{1-\delta_{k_1}-t\theta_{k_1,k_2}}\varepsilon.$$

If let $m=k$, then Theorem 2 is Theorem 3.1 in [20].

Let $m_0\le m$ be smallest positive integer so that

$$\left\|h_{T_m}\right\|_1 \geq \left\|h_{T_m^c}\right\|_1 .$$

Then we have

Theorem 3. Suppose β is k -sparse. Let be k_1, k_2 positive integers such that $k_1 \geq k \geq m_0$ and $8(k_1 - m_0) \leq k_2$. Let

$$t = \sqrt{\frac{k_1}{k_2}} + \frac{1}{4}\sqrt{\frac{k_2}{k_1}} - \frac{2(k_1 - m_0)}{\sqrt{k_1 k_2}} .$$

Then under the condition

$$\delta_{k_1} + t\theta_{k_1,k_2} < 1$$

the L_1 minimizer defined in (10) satisfies

$$\left\|\beta - \hat{\beta}\right\|_2 \leq \frac{2\sqrt{2}\sqrt{1+\delta_{k_1}}}{1 - \delta_{k_1} - t\theta_{k_1,k_2}} \varepsilon .$$

In particular, in the noiseless case where $y = \Phi\beta$, L_1 minimization recovers β exactly.
The proof is similar to of Theorem 2.

Note that k is independent of h, but m and m_0 are dependent of h, i.e. $m = m(h)$ and $m_0 = m_0(h)$.

The following is one of our main results of the paper. It is the consequence of Theorem 2.

Theorem 4. Suppose β is k sparse with $k > 1$. If $k \equiv 0 (\mathrm{mod}\, 5)$ and $T_{\frac{k}{5}}$ is sparse index set, then under the condition $\delta_k < 0.5$ the constrained L_1 minimizer $\hat{\beta}$ given in (10) satisfies

$$\left\|\beta - \hat{\beta}\right\|_2 \leq \frac{\sqrt{3}}{0.5 - \delta_k} \varepsilon .$$

In particular, in the noiseless case $\hat{\beta}$ recovers β exactly.

Proof. If $k \equiv 0 (\mathrm{mod}\, 5)$ and $T_{\frac{k}{5}}$ is sparse index set, then in Theorem 2, set $k_1 = \frac{k}{5}$, $k_2 = \frac{4k}{5}$. Thus

$$t = \sqrt{\frac{k_1}{k_2}} + \frac{1}{4}\sqrt{\frac{k_2}{k_1}} - \frac{2(k_1 - \frac{k}{5})}{\sqrt{k_1 k_2}} = 1 .$$

Then under the condition

$$\delta_{\frac{k}{5}} + \theta_{\frac{k}{5},\frac{4k}{5}} < 1$$

we have

$$\left\|\beta - \hat{\beta}\right\|_2 \leq \frac{2\sqrt{2}\sqrt{1+\delta_{\frac{k}{5}}}}{1 - \delta_{\frac{k}{5}} - \theta_{\frac{k}{5},\frac{4k}{5}}} \varepsilon .$$

By (5) and (7) we get

$$\delta_{\frac{k}{5}} + \theta_{\frac{k}{5},\frac{4k}{5}} \leq 2\delta_k < 1 .$$

In this case

$$\left\|\beta - \hat{\beta}\right\|_2 \leq \frac{2\sqrt{2}\sqrt{1+\delta_{\frac{k}{5}}}}{1 - \delta_{\frac{k}{5}} - \theta_{\frac{k}{5},\frac{4k}{5}}} \varepsilon \leq \frac{2\sqrt{2}\sqrt{1+\delta_k}}{1 - 2\delta_k} \varepsilon \leq \frac{\sqrt{3}}{0.5 - \delta_k} \varepsilon$$

.An explicitly example in [20] is constructed in which $\delta_k < 0.5$, but it is impossible to recover certain k sparse signals. Therefore, the bound for δ_k cannot go beyond 0.5 in general in order to guarantee stable recovery of k sparse signals.

CONFLICT OF INTEREST
No conflict of interest was declared by the authors.

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