Gazi University Journal of Science
GU J Sci
28(1): 65-68 (2015)

Improved Bounds for the Extremal Non-Trivial Laplacian Eigenvalues

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Received:12/09/2014 Revised:04/11/2014 Accepted:05/11/2014

ABSTRACT

Let G be a simple connected graph and its Laplacian eigenvalues be $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. In this paper, we present an upper bound for the algebraic connectivity μ_{n-1} of G and a lower bound for the largest eigenvalue μ_1 of G in terms of the degree sequence $d_1, d_2, \ldots, d_n$ of G and the number $|N_i \cap N_j|$ of common vertices of i and j $(1 \leq i < j \leq n)$ and hence we improve bounds of Maden and Büyükköse [14].

Keywords: Laplacian eigenvalues, upper bounds, lower bounds, eigenvalue inequalities.
2010 MSC: 05C50, 15A18.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. We use $i \sim j$ to denote that $v_i v_j$ is an edge of G and N_i to denote that the set of neighbours of v_i. For $v_i \in V$, the degree of v_i and the average of the degrees of the vertices adjacent to v_i are denoted by d_i and m_i, respectively. We assume that $d_1 \geq d_2 \geq \cdots \geq d_n$ without lost of generality and we call $d_1, d_2, \ldots, d_n$ the degree sequence of G. Let $A(G)$ be the adjacency matrix of G and let $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. For the simplicity of notation, we write $L(G) = L$. Clearly, L is a real symmetric matrix. From this fact and Geršgorin's Theorem, it follows that its eigenvalues are nonnegative real numbers. Morever, since the sum of rows is 0, it is obvious that 0 is the smallest eigenvalue of L with the all ones vector as an eigenvector. The Laplacian eigenvalues of G are the eigenvalues of the Laplacian matrix L of G. Throughout this paper, the Laplacian eigenvalues of G are denoted by

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0.$$

In addition, by the extremal non-trivial Laplacian eigenvalues, we shall mean μ_{n-1} and μ_1. It is easy to show that $\mu_{n-1}(G) = 0$ if and only if G is not connected. Thus, μ_{n-1} is called the algebraic connectivity of the graph G [5]. In [1] it is proved that if μ is an eigenvalue of L then $\mu \leq n$ and that the multiplicity of 0 equals the number of components of G. Thus, G is a connected graph if and only if $\mu_{n-1} > 0$.

The Laplacian eigenvalues of a graph are important in the graph theory because they have a relation to numerous graph invariants, including connectivity, expanding property, isoperimetric number, maximum

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cut, independence number, genus, diameter, mean distance and bandwidth-type parameters of a graph. In many application one needs good lower bound and upper bound of extremal non-trivial Laplacian eigenvalues (see [1], [3], [4], [6], [7], [9], [10], [11], [12], [14]).

In this paper, our aim is to improve the upper bound for the algebraic connectivity μ_{n-1} of G and the lower bound for μ_1 of G given by Maden and Büyükköse [14]. We use Theorem 1 [13] and modify the technique of the proof of Lemma 3 [13], we give an upper bound for the algebraic connectivity μ_{n-1} of G and a lower bound of the largest eigenvalue μ_1 of G in terms of the degree sequence $d_1, d_2, \ldots, d_n$ of G and the number $|N_i \cap N_j|$ of common vertices of i and j $(1 \leq i < j \leq n)$.

We always assume that G is a simple connected graph of order n. The known upper and lower bounds which we used in proof of our main theorem are following:

1. Grone and Merris' bound [15]:
$$\mu_1 \geq d_1 + 1, \tag{1}$$
where d_1 is the largest degree of G.
2. Li and Pan's bound [16]:
$$\mu_2 \geq d_2, \tag{2}$$
where d_2 and μ_2 are the second largest degree and the second largest Laplacian eigenvalue of G, respectively.
3. Fidler's bound [5]: Let G be a graph different from K_n and let d_n be its minimum degree. Then
$$\mu_{n-1} \leq d_n. \tag{3}$$

2. THE MAIN RESULT

Firstly we summarize the results of Wolkowicz and Styan on the eigenvalue inequalities which are our fundamental tools in this paper.

Theorem 1. (Theorem 2.1 [13]) *Let A be an $n \times n$ complex matrix with real eigenvalues $\lambda(A)$ and let $m = \frac{trA}{n}$, $s = \sqrt{\frac{trA^2}{n} - m^2}$. Then*
$$m - s\sqrt{n-1} \leq \lambda_{min}(A) \leq m - \frac{s}{\sqrt{n-1}} \tag{4}$$
$$m + \frac{s}{\sqrt{n-1}} \leq \lambda_{max}(A) \leq m + s\sqrt{n-1}. \tag{5}$$
Equality holds on the left (right) of (4) if and only if equality holds on the left (right) of (5) if and only if the $n-1$ largest (smallest) eigenvalues are equal.

In [13] Wolkowicz and Styan proved Theorem 1 by using the following lemmas.

Lemma 2. (Lemma 2.1 [13]) *Let $C = I_n - \frac{ee^T}{n}$, $m = \frac{\lambda^T e}{n}$, $s^2 = \frac{\lambda^T C \lambda}{n}$ where W and $\lambda = (\lambda_j) \in \mathbb{R}^n$ are column vectors, and $e = (1,1,\ldots,1)^T$. Then*
$$-s\sqrt{nW^T CW} \leq W^T \lambda - mW^T e = W^T C \lambda \leq snW^T CW. \tag{6}$$
Equality holds on the left (right) of (1) if and only if $\lambda = aw + be$ for some scalars a and b, where $a < 0$ $(a > 0)$.

It should be noted that m and s^2 defined in Theorem 1 and Lemma 2 are equivalent [13].

Lemma 3. (Lemma 2.2 [13]) *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, m and s be defined as in Lemma 2 and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then*
$$\lambda_n \leq m - \frac{s}{\sqrt{n-1}} \leq m + \frac{s}{\sqrt{n-1}} \leq \lambda_1. \tag{7}$$

Using Theorem 1 Maden and Büyükköse [14] gave upper and lower bounds for μ_{n-1} and μ_1.

Theorem 4. (Theorem 3 and Corollary 5 [14]) *Let G be a simple graph. Then*
$$\sqrt{m - s\sqrt{\frac{n-2}{2}}} \leq \mu_{n-1} \leq \sqrt{m - s\sqrt{\frac{1}{n-1}}} \tag{8}$$
and
$$\sqrt{m + \frac{s}{\sqrt{n-1}}} \leq \mu_1 \leq \sqrt{m + s\sqrt{n-1}}, \tag{9}$$
where $m = \frac{1}{n}\sum_{i=1}^{n} d_i\,(d_i + 1)$ and
$$s^2 = \frac{1}{n}\left(\sum_{i=1}^{n}(d_i^2 + d_i)^2 + 2\sum_{\substack{i<j \\ i\sim j}}(d_i + d_j)(d_i + d_j - 2|N_i \cap N_j|) + 2_{i<j}|N_i \cap N_j|^2\right) - m^2.$$

□

Now, we reprove Lemma 3 for the Laplacian matrix L of G and hence we improve the upper bound for λ_{n-1} in (8) and the lower bound for λ_1 in (9).

Theorem 5. *Let G be a simple graph and let m and s be defined as in Theorem 4. Then*

$$\mu_{n-1} \leq \left(m - \left(\frac{ns^2 + 2((d_1+1)^2 - d_{n-1}^2)(d_2^2 - d_{n-1}^2)}{n^2 - n} \right)^{1/2} \right)^{1/2} \quad (10)$$

and

$$\mu_1 \geq \left(m + \left(\frac{ns^2 + 2(d_1+1)^2((d_1+1)^2 - d_n^2)}{n^2 - n} \right)^{1/2} \right)^{1/2}. \quad (11)$$

Proof. Let G be a simple graph and let m and s be defined as in Theorem 4. Then we have that

$$n^2(m - \mu_{n-1}^2)^2 = n^2 \left(\frac{1}{n} \sum_{i=1}^{n} (\mu_i^2 - \mu_{n-1}^2) \right)^2$$
$$= \sum_{i=1}^{n} (\mu_i^2 - \mu_{n-1}^2)^2 + \sum_{j \neq k} (\mu_j^2 - \mu_{n-1}^2)(\mu_k^2 - \mu_{n-1}^2)$$

By using (1)-(3) we have that

$$\sum_{j \neq k} (\mu_j^2 - \mu_{n-1}^2)(\mu_k^2 - \mu_{n-1}^2) \geq 2((d_1+1)^2 - d_{n-1}^2)(d_2^2 - d_{n-1}^2).$$

On the other hand,

$$\sum_{i=1}^{n} (\mu_i^2 - \mu_{n-1}^2)^2 = \sum_{i=1}^{n} (\mu_i^2 - m + m - \mu_{n-1}^2)^2 = \sum_{i=1}^{n} [(\mu_i^2 - m)(\mu_i^2 + m - 2\mu_{n-1}^2)] + n(m - \mu_{n-1}^2)^2$$
$$= ns^2 + n(m - \mu_{n-1}^2)^2.$$

Finally, we have that

$$n^2(m - \mu_{n-1}^2)^2 \geq ns^2 + n(m - \mu_{n-1}^2)^2 + 2((d_1+1)^2 - d_{n-1}^2)(d_2^2 - d_{n-1}^2).$$

Solving this inequality for μ_{n-1}^2 we obtain the inequality in (10).

Now we similarly expand $n^2(\mu_1^2 - m)$. Then we have

$$n^2(\mu_1^2 - m)^2 = \left(n\mu_1^2 - \sum_{i=1}^{n} \mu_i^2 \right)^2 = \sum_{i=1}^{n} (\mu_1^2 - \mu_i^2)^2 + \sum_{j \neq k} (\mu_1^2 - \mu_j^2)(\mu_1^2 - \mu_k^2).$$

By using (1)-(3), we have that

$$\sum_{j \neq k} (\mu_1^2 - \mu_j^2)(\mu_1^2 - \mu_k^2) \geq 2(d_1+1)^2((d_1+1)^2 - d_n^2).$$

We have that

$$n^2(\mu_1^2 - m)^2 \geq n(\mu_1^2 - m)^2 + ns^2 + 2(d_1+1)^2((d_1+1)^2 - d_n^2).$$

Solving this inequality for μ_1^2 we obtain the inequality in (11). □

In the proof of Lemma 3 in [13, Lemma 2.2], the second sum is omitted but we consider it to improve the upper bound for μ_{n-1} in (8) and the lower bound for μ_1 in (9). Now we compare our bounds with the bounds of Maden and Büyükköse [14].

Exercise 6. Let $G = (V, E)$ with $V = \{1,2,3,4,5,6,7,8\}$ and

$$E = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{2,5\},\{2,6\},\{2,8\},\{4,5\},\{4,6\},\{4,8\},\{6,7\},\{7,8\}\}.$$

For this graph $\mu_7 = 1.13$ and $\mu_1 = 7.1$. We present aforesaid upper bounds for μ_7 and lower bounds for μ_1 of the graph G as follows:

	μ_7	(8)	(10)		μ_1	(9)	(11)
G	1.13	4.72	2.76		7,10	3.02	5.17

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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