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Durrmeyer-Type Generalization of Mittag-Leffler Operators

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ABSTRACT

In this paper, we study Mittag-Leffler operators. We establish moments of these operators and estimate convergence results with the help of classical modulus of continuity. Also we give their A -statistical convergence property.

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Key words: Mittag-Leffler operators, A -statistical convergence, Gamma function, Beta function.

1. INTRODUCTION

In 1903, G.M. Mittag-Leffler [1] defined the Mittag-Leffler function by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}; (z \in \mathbb{C}, R(\alpha) > 0).$$

In 1905, A. Wiman [2] gave the definition of two-index Mittag-Leffler function by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}; (z, \beta \in \mathbb{C}, R(\alpha) > 0).$$

Note that $E_{\alpha, 1}(z) = E_{\alpha}(z)$.

M.A. Özarslan [12] investigated properties of the following Mittag-Leffler operators

$$L_n^{(\beta)}(f; x) = \frac{1}{E_{1, \beta}(\frac{nx}{b_n})} \sum_{k=0}^{\infty} f\left(\frac{k b_n}{n}\right) \frac{(nx)^k}{b_n^k \Gamma(\alpha k + \beta)}, \quad (1.1)$$

Where b_n is a sequence of positive real numbers, $\beta > 0$ is fixed, $n \in \mathbb{N}$, $C[0, \infty)$ denotes the space of continuous functions defined on $[0, \infty)$, and

$$f \in E := \left\{ f \in C[0, \infty) : \lim_{x \rightarrow \infty} \frac{|f(x)|}{1+x^2} \text{ is finite} \right\}.$$

Recall that the Banach lattice E has the norm

$$\|f\|_* = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^2}.$$

It is obvious that operators $L_n^{(\beta)}$ given by (1.1) are linear and positive. Furthermore, $L_n^{(1)}(f; x) = S_n(f; x)$, which are modified Szász-Mirakjan operators, can be easily seen.

Ozarslan [12] gave the following inequalities:

$$L_n^{(\beta)}(1; x) = 1, \quad (1.2)$$

$$\left| L_n^{(\beta)}(t; x) - x \right| \leq \frac{|1 - \beta| b_n}{n}, \quad (1.3)$$

$$\left| L_n^{(\beta)}(t^2; x) - x^2 \right| \leq \frac{(2 + |1 - \beta|) b_n}{n} x + \frac{(2|1 - \beta|^2 + |1 - \beta| + |1 - \beta||\beta - 2|) b_n^2}{n^2}, \quad (1.4)$$

$$L_n^{(\beta)}((t - x)^2; x) \leq \frac{(4|1 - \beta| + 1) b_n}{n} x + \frac{(2|1 - \beta|^2 + |1 - \beta| + |1 - \beta||\beta - 2|) b_n^2}{n^2}. \quad (1.5)$$

Now, we define the operators $D_n^{(\beta)}$ by

$$D_n^{(\beta)}(f; x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{E_{1,\beta}\left(\frac{nx}{b_n}\right) b_n^{k \equiv (k+\beta)}} \times \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} \frac{f\left(\frac{(n-1)b_n t}{n}\right)}{B(n, k+1)} dt, \quad (1.6)$$

Where $f \in C[0, \infty)$, $x \in [0, \infty)$, b_n is a sequence of positive real numbers, $\beta > 0$ is fixed, $n \in \mathbb{N}$ and $B(\cdot, \cdot)$ is the Beta function.

Lemma 1. For each $x \geq 0$ and $n \in \mathbb{N}$, we have

$$D_n^{(\beta)}(1; x) = 1, \quad (1.7)$$

$$\left| D_n^{(\beta)}(t; x) - x \right| \leq \frac{(|1 - \beta| + 1) b_n}{n} x, \quad (1.8)$$

$$\left| D_n^{(\beta)}(t^2; x) - x^2 \right| \leq \frac{x^2}{n-2} + \left(\frac{n-1}{n-2}\right) \frac{2(2 + |1 - \beta|) b_n}{n} x + \left(\frac{n-1}{n-2}\right) \frac{(2|1 - \beta|^2 + |1 - \beta| + |1 - \beta||\beta - 2|) b_n^2}{n^2}, \quad (1.9)$$

$$D_n^{(\beta)}((t - x)^2; x) \leq \frac{x^2}{n-2} + \frac{2b_n}{n} \left[\left(\frac{n-1}{n-2}\right) (2 + |1 - \beta|) + |1 - \beta| + 1 \right] x + \left(\frac{n-1}{n-2}\right) \frac{(2|1 - \beta|^2 + |1 - \beta| + |1 - \beta||\beta - 2|) b_n^2}{n^2} \quad (1.10)$$

Proof From definition of the two-index Mittag-Leffler function, we see that

$$D_n^{(\beta)}(1; x) = 1.$$

From the operators $D_n^{(\beta)}$ given by (1.6), for $n > 1$ we get

$$D_n^{(\beta)}(t; x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{E_{1,\beta}\left(\frac{nx}{b_n}\right) b_n^{k \equiv (k+\beta)}} \times \frac{(n-1)b_n}{n} \frac{B(n-1, k+2)}{B(n, k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(nx)^k}{E_{1,\beta}\left(\frac{nx}{b_n}\right) b_n^{k \equiv (k+\beta)}} \left(\frac{k+1}{n} b_n\right) = L_n^{(\beta)}(t; x) + \frac{b_n}{n} L_n^{(\beta)}(1; x).$$

Using (1.2) and (1.3), we have

$$\left| D_n^{(\beta)}(t; x) - x \right| \leq \left| L_n^{(\beta)}(t; x) - x \right| + \frac{b_n}{n} \leq \frac{(|1 - \beta| + 1) b_n}{n}.$$

From (1.6), for $n > 2$ we get

$$D_n^{(\beta)}(t^2; x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{E_{1,\beta}\left(\frac{nx}{b_n}\right) b_n^{k \equiv (k+\beta)}} \times \left[\frac{(n-1)b_n}{n} \right]^2 \frac{B(n-2, k+3)}{B(n, k+1)} = \sum_{k=0}^{\infty} \frac{(nx)^k}{E_{1,\beta}\left(\frac{nx}{b_n}\right) b_n^{k \equiv (k+\beta)}} \frac{n-1}{n-2} \left(\frac{b_n}{n}\right)^2 (k^2 + 3k + 2)$$

$$= \left(\frac{n-1}{n-2}\right) \left[L_n^{(\beta)}(t^2; x) + \frac{3b_n}{n} L_n^{(\beta)}(t; x) + \frac{2b_n^2}{n^2} L_n^{(\beta)}(1; x) \right].$$

Using (1.2)-(1.4), we have

$$\left| D_n^{(\beta)}(t^2; x) - x^2 \right| \leq \left(\frac{n-1}{n-2}\right) \left| L_n^{(\beta)}(t^2; x) - x^2 \right| + \frac{3b_n}{n} \left(\frac{n-1}{n-2}\right) \left| L_n^{(\beta)}(t; x) - x \right| + \frac{x^2}{n-2} + \frac{3b_n}{n} \left(\frac{n-1}{n-2}\right) x + \frac{2b_n^2}{n^2} \left(\frac{n-1}{n-2}\right)$$

$$\leq \left(\frac{n-1}{n-2}\right) \left[\frac{(2|1 - \beta| + 1) b_n}{n} x + \frac{(2(1 - \beta)^2 + |1 - \beta| + |1 - \beta||\beta - 2|) b_n^2}{n^2} \right] + \frac{3b_n}{n} \left(\frac{n-1}{n-2}\right) \frac{|1 - \beta| b_n}{n} + \frac{x^2}{n-2} + \frac{3b_n}{n} \left(\frac{n-1}{n-2}\right) x + \frac{2b_n^2}{n^2} \left(\frac{n-1}{n-2}\right).$$

If we simplify the above inequality, then we obtain

$$\begin{aligned} \left| D_n^{(\beta)}(t^2; x) - x^2 \right| &\leq \frac{x^2}{n-2} \\ &+ \left(\frac{n-1}{n-2} \right) \frac{(2|1-\beta|+4)b_n}{n} x \\ &+ \frac{b_n^2}{n^2} \left(\frac{n-1}{n-2} \right) (2(1-\beta)^2 \\ &+ 4|1-\beta| + |1-\beta||\beta-2| + 2). \end{aligned}$$

On the other hand, we can give the second moment as

$$\begin{aligned} D_n^{(\beta)}((t-x)^2; x) &\leq \left| D_n^{(\beta)}(t^2; x) - x^2 \right| \\ &+ 2x \left| D_n^{(\beta)}(t; x) - x \right| \\ &\leq \frac{2b_n}{n} \left[\left(\frac{n-1}{n-2} \right) (|1-\beta| + 2) + |1-\beta| + 1 \right] x \\ &+ \frac{x^2}{n-2} + \frac{b_n^2}{n^2} \left(\frac{n-1}{n-2} \right) (2(1-\beta)^2 + 4|1-\beta| \\ &+ |1-\beta||\beta-2| + 2). \end{aligned}$$

2. RATE OF CONVERGENCE

We start with the following lemma, which proves that $D_n^{(\beta)}$ maps E into itself.

Lemma 2. Let $\left(\frac{b_n}{n}\right)$ be a bounded sequence of positive numbers and $\beta > 0$ be fixed. Then there exists a constant $M(\beta)$ such that, for $\omega(x) = \frac{1}{1+x^2}$, we have

$$\omega(x) D_n^{(\beta)}\left(\frac{1}{\omega}; x\right) \leq M(\beta)$$

holds for all $x \in [0, \infty)$ and $n \in \mathbb{N}$. Furthermore, for all $f \in E$, we have

$$\left\| D_n^{(\beta)}(f) \right\|_* \leq M(\beta) \|f\|_*.$$

Proof. From (1.7) and (1.9), we have

$$\begin{aligned} \omega(x) D_n^{(\beta)}\left(\frac{1}{\omega}; x\right) &= \frac{1}{1+x^2} \left[D_n^{(\beta)}(1; x) + D_n^{(\beta)}(t^2; x) \right] \\ &\leq \frac{1}{1+x^2} \left\{ 1 \right. \\ &+ \frac{n-1}{n-2} \left[\frac{(2|1-\beta|+1)b_n}{n} x + \frac{3b_n^2}{n^2} |1-\beta| + x^2 \right. \\ &+ \frac{3b_n}{n} x + \frac{2b_n^2}{n^2} \\ &\left. \left. + \frac{(2|1-\beta|^2 + |1-\beta| + |1-\beta||\beta-2|)b_n^2}{n^2} \right] \right\} \end{aligned}$$

$$\begin{aligned} &\leq 1 + \frac{n-1}{n-2} \left[\frac{(2|1-\beta|+1)b_n}{2n} + \frac{3b_n^2}{n^2} |1-\beta| + 1 \right. \\ &+ \frac{3b_n}{2n} + \frac{2b_n^2}{n^2} \\ &\left. + \frac{(2|1-\beta|^2 + |1-\beta| + |1-\beta||\beta-2|)b_n^2}{n^2} \right] \end{aligned}$$

$$\begin{aligned} &= \frac{2n-3}{n-2} + \frac{n-1}{n-2} \frac{b_n}{n} (|1-\beta| + 2) \\ &+ \frac{n-1}{n-2} \frac{b_n^2}{n^2} (2|1-\beta|^2 + 4|1-\beta| \\ &+ |1-\beta||\beta-2| + 2) = M(\beta). \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} \omega(x) \left| D_n^{(\beta)}(f; x) \right| &\leq \omega(x) \left| D_n^{(\beta)}\left(\omega \frac{f}{\omega}; x\right) \right| \\ &\leq \|f\|_* \omega(x) D_n^{(\beta)}\left(\frac{1}{\omega}; x\right) \\ &\leq M(\beta) \|f\|_*. \end{aligned}$$

Taking supremum on both sides of above inequality, we easily prove the results.

Recall that the usual modulus of continuity of f on the closed interval $[0, B]$ is defined by

$$\omega_B(f, \delta) = \sup\{|f(t) - f(x)| : x, t \in [0, B], |t - x| \leq \delta\}.$$

It is well known that, for a function $f \in C[0, B]$, we have $\lim_{\delta \rightarrow 0} \omega_B(f, \delta) = 0$.

Now, we acquire the rate of convergence of the operators $D_n^{(\beta)} f$ to f , for all $f \in C[0, B]$.

Theorem 1. Let $\beta > 0$ be fixed, $\left(\frac{b_n}{n}\right)$ be a bounded sequence of positive numbers with $\frac{b_n}{n} \rightarrow 0$ as $n \rightarrow \infty$, $f \in C[0, B]$, and $\omega_{B+1}(f, \delta)$ ($B > 0$) be modulus of continuity of f on the finite interval $[0, B+1] \subset [0, \infty)$. Then

$$\begin{aligned} \left\| D_n^{(\beta)}(f; x) - f(x) \right\|_{C[0, B]} &\leq M_f(\beta, B) \delta_n^2(\beta, B) \\ &+ 2\omega_{B+1}(f, \delta_n(\beta, B)) \end{aligned}$$

where

$$\begin{aligned} \delta_n(\beta, B) &= \left\{ \frac{B^2}{n-2} + \frac{2b_n}{n} \left[\left(\frac{n-1}{n-2} \right) (|1-\beta| \right. \right. \\ &+ 2) + |1-\beta| + 1 \right] B \\ &+ \frac{b_n^2}{n^2} \left(\frac{n-1}{n-2} \right) [2|1-\beta|^2 + 4|1-\beta| \\ &+ |1-\beta||\beta-2| + 2] \}^{1/2} \end{aligned}$$

and $M_f(\beta, B)$ is an absolute constant depending on f , β and B .

Proof. Let $\beta > 0$ be fixed. For $x \in [0, B]$ and $t \leq B + 1$, we have

$$|f(t) - f(x)| \leq \omega_{B+1}(f, |t - x|) \leq \left(1 + \frac{|t - x|}{\delta}\right) \omega_{B+1}(f, \delta)$$

where $\delta > 0$. On the other hand, for $x \in [0, B]$ and $t > B + 1$, we get $t - x > 1$, we can write

$$|f(t) - f(x)| \leq A_f(1 + x^2 + t^2) \leq A_f(2 + 3x^2 + 2(t - x)^2) \leq 6A_f(1 + B^2)(t - x)^2.$$

By the above two inequalities, for $x \in [0, B]$ and $t \geq 0$, we get

$$|f(t) - f(x)| \leq 6A_f(1 + B^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right) \omega_{B+1}(f, \delta).$$

Using Cauchy-Schwarz inequality and (1.10), we have

$$\begin{aligned} |D_n^{(\beta)}(f; x) - f(x)| &\leq 6A_f(1 + B^2)D_n^{(\beta)}((t - x)^2; x) \\ &\quad + \omega_{B+1}(f, \delta) \left(1 + \frac{1}{\delta} \sqrt{D_n^{(\beta)}((t - x)^2; x)}\right) \\ &\leq M_f(\beta, B)\delta_n^2(\beta, B) + 2\omega_{B+1}(f, \delta_n(\beta, B)), \end{aligned}$$

where

$$\begin{aligned} \delta_n(\beta, B) &= \left\{ \frac{B^2}{n-2} + \frac{2b_n}{n} \left[\left(\frac{n-1}{n-2} \right) (|1-\beta| + 2) + |1-\beta| + 1 \right] B \right. \\ &\quad \left. + \frac{b_n^2}{n^2} \left(\frac{n-1}{n-2} \right) [2|1-\beta|^2 + 4|1-\beta| + |1-\beta||\beta-2| + 2] + 2 \right\}^{1/2} \end{aligned}$$

and $M_f(\beta, B) = 6(1 + B^2)A_f$. So, we have the desired results.

3. A-STATISTICAL CONVERGENCE

Recently, some authors deal with A -statistically convergence of linear positive operators [8, 10, 11].

We recall concepts of A -statistical convergence. Let $A = (a_{jk})$ be a non-negative regular summability matrix. The A -density of a subset K of $\mathbb{N}$ is given by

$$\delta_A(K) = \lim_j \sum_{k \in K} a_{j,k},$$

provided that limit exists (see [5]). A sequence $x = (x_n)$ is said to be A -statistically convergent to l and denoted by $st_A - \lim x = l$ if for every $\varepsilon > 0$,

$$\lim_{k \rightarrow \infty} \sum_{n: |x_k - l| \geq \varepsilon} a_{k,n} = 0$$

or $\delta_A\{n \in \mathbb{N}: |x_k - l| \geq \varepsilon\} = 0$ (see [4, 9]).

In the special case of $A = C_1$, Cesàro matrix of order one, the A -statistical convergence reduces to statistical convergence [3, 7]. If we choose $A = I$, the identity matrix, then A -statistical convergence reduces to ordinary convergence. Kolk [6] proved that in the case of $\lim_j \max_n |a_{j,n}| = 0$, A -statistical convergence is stronger than ordinary convergence.

Assuming that $(b_n)_{n \in \mathbb{N}}$ is a sequence satisfying

$$st_A - \lim_n \frac{b_n}{n} = 0.$$

Then we see

$$st_A - \lim_n \frac{b_n^2}{n^2} = 0.$$

For an example, take $A = C_1$ and define

$$b_n := \begin{cases} n: n = m^2 \ (m \in \mathbb{N}) \\ n^{\frac{1}{3}}: \text{otherwise} \end{cases}. \quad (3.1)$$

Then we easily see that $st_A - \lim_n \frac{b_n}{n} = st_A - \lim_n \frac{b_n^2}{n^2} = 0$ (see [12]).

Theorem 2. Let $A = (a_{jk})$ be a non-negative regular summability matrix and $\beta > 0$ be fixed. If $st_A - \lim_n \frac{b_n}{n} = 0$, then

$$st_A - \left\| D_n^{(\beta)}(f; x) - f(x) \right\|_{C[0, B]} = 0$$

holds for every $f \in E$.

Proof. Given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. For fixed $\beta > 0$, define the following sets:

$$U = \{n: \delta_n(\beta, B) \geq r\},$$

$$U_1 = \left\{ n: \frac{\beta^2}{n-2} \geq \frac{r-\varepsilon}{3} \right\},$$

$$U_2 := \left\{ n: 2B \frac{b_n}{n} \left[(|1 - \beta| + 2) \frac{n-1}{n-2} + |1 - \beta| + 1 \right] \geq \frac{r - \varepsilon}{3} \right\},$$

$$U_3 := \left\{ n: \left(\frac{n-1}{n-2} \right) [2(1 - \beta)^2 + 4|1 - \beta| + |1 - \beta||\beta - 2| + 2] \frac{b_n^2}{n^2} \geq \frac{r - \varepsilon}{3} \right\}.$$

Then $U \subset U_1 \cup U_2 \cup U_3$ can be seen. So, we can get

$$\sum_{k \in U} a_{jk} \leq \sum_{k \in U_1} a_{jk} + \sum_{k \in U_2} a_{jk} + \sum_{k \in U_3} a_{jk}.$$

For $j \rightarrow \infty$ in the above inequality and $st_A - \lim_n \frac{b_n}{n} = 0$, we have $\lim_j \sum_{k \in U} a_{jk} = 0$.

So this shows that $st_A - \lim_n \delta_n(\beta, B) = 0$ which implies

$$st_A - \lim_n \omega_{B+1}(f, \delta_n(\beta, B)) = 0$$

due to the right continuity of $\omega_{B+1}(f, \cdot)$ at zero. Using the previous theorem, we get the desired result.

Remark 1. Note that choosing the sequence $(b_n)_{n \in \mathbb{N}}$ as in (3.1), the statistical approximation results in the last theorem works, but its classical case does not work since $\left(\frac{b_n}{n} \right)_{n \in \mathbb{N}}$ is not convergent in the ordinary sense.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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