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q −Bernoulli Matrices and Their Some Properties

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ABSTRACT

In this study, we define q −Bernoulli matrix $\mathscr{B}(q)$ and q −Bernoulli polynomial matrix $\mathscr{B}(x,q)$ by using q −Bernoulli numbers, and polynomials respectively. We obtain some properties of $\mathscr{B}(q)$ and $\mathscr{B}(x,q)$. We obtain factorizations q −Bernoulli polynomial matrix and shifted q −Bernoulli matrix using special matrices.
Keywords: q −Bernoulli numbers, q −Bernoulli matrix, q −Vandermonde matrix.

1. INTRODUCTION

Bernoulli numbers are defined by Jacob Bernoulli ([1]). Nörlund ([2]) and Carlitz ([3]) obtained some properties of Bernoulli numbers and polynomials. Carlitz ([4, 5]) defined q −Bernoulli numbers and polynomials. Hegazi ([10]) studied q −Bernoulli numbers and polynomials.

Let n be a positive integer and $q \in (0,1)$. The quantum integer or Gauss number $[n]_q$ is defined by

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}.$$

The q −analogue of $n!$ is defined as follows

$$[n]_q! = \begin{cases} 1 & if\ n = 0, \\ [n]_q [n-1]_q \cdots [1]_q & if\ n = 1,2,\dots. \end{cases}$$

Gaussian or q −binomial coefficients are defined for integers $n \geq k \geq 1$ as

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!}$$

with $\binom{n}{0}_q = 1$ and $\binom{n}{k}_q = 0$ for $n < k$ ([6]). Some properties of q −binomial coefficients are

$$\binom{n}{k}_q = \binom{n}{n-k}_q \tag{1.1}$$

and

$$\binom{n}{k}_q \binom{k}{j}_q = \binom{n}{j}_q \binom{n-j}{k-j}_q. \tag{1.2}$$

The q −analogue of $(x-a)^n$ denoted $(x-a)_q^n$ is

$$(x-a)_q^n = \begin{cases} 1 & if\ n = 0, \\ (x-a)(x-qa)\cdots(x-q^{n-1}a) & if\ n = 1,2,\dots. \end{cases}$$

for x variable. Using definition of q −binomial coefficients it can be obtained

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$$(x+a)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} a^k x^{n-k} \tag{1.3}$$

is called Gauss's binomial formula.

2. BERNOULLI NUMBERS AND POLYNOMIALS

Firstly, we mention that Bernoulli numbers. Then using these numbers, a matrix can be delivered. This matrix is called Bernoulli matrix. Extending this matrix some matrices are obtained.

In [7], the Bernoulli numbers are defined initial condition by $B_0 = 1$ and

$$B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k \quad n = 1,2,3,\ldots. \tag{2.1}$$

The exponential generating function of Bernoulli numbers is

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{2.2}$$

Let n be a nonnegative integer, the Bernoulli polynomials $B_n(x)$ are defined by

$$B_n(x) = \sum_{n=0}^{\infty} \binom{n}{k} B_k\ x^{n-k}. \tag{2.3}$$

Zhang defined Bernoulli matrices by using Bernoulli numbers and polynomials. Also the author obtained factorization and some properties of Bernoulli matrices [8].

Definition 1. [8] Let B_n be n^{th} Bernoulli number and $B_n(x)$ be Bernoulli polynomial, $(n+1)\times(n+1)$ type Bernoulli matrix $\mathscr{B} = [b_{ij}]$ and Bernoulli polynomial matrix $\mathscr{B}(x) = [b_{ij}(x)]$ defined respectively as follows

$$b_{ij} = \begin{cases} \binom{i}{j} B_{i-j} & if\ i \geq j, \\ 0 & otherwise, \end{cases} \tag{2.4}$$

and

$$b_{ij}(x) = \begin{cases} \binom{i}{j} B_{i-j}(x) & if\ i \geq j, \\ 0 & otherwise. \end{cases} \tag{2.5}$$

It is know that the constant terms of $B_n(x)$ Bernoulli polynomials are B_n Bernoulli numbers. Therefore we obtain Bernoulli $\mathscr{B}$ matrix by using the constant term of $\mathscr{B}(x)$ Bernoulli polynomial matrix [8].

Now we give definitions of $q-$Bernoulli numbers and $q-$Bernoulli polynomials.

Definition 2. [10] Let n be a nonnegative integer and B_n be n^{th} Bernoulli numbers. The $q-$Bernoulli numbers $b_n(q)$ are defined by

$$b_n(q) = B_n \frac{[n]_q!}{n!}. \tag{2.6}$$

The $q-$Bernoulli polynomials $B_n(x,q)$ are defined by

$$B_n(x,q) = \sum_{k=0}^{n} \binom{n}{k}_q b_k(q) x^{n-k}. \tag{2.7}$$

Theorem 1. [10] For $q-$commuting variables x and y such that $xy = qxy$ we have

$$B_n(x+y,q) = \sum_{k=0}^{n} \binom{n}{k}_q y^{n-k} B_k(x,q). \tag{2.8}$$

Similar considerations apply this theorem, it can easy to check that

$$B_n(x+y,q) = \sum_{k=0}^{n} \binom{n}{k}_q x^{n-k} B_k(y,q). \tag{2.9}$$

3. q −BERNOULLI MATRICES

Zhang [8] defined generalized Bernoulli matrix by using Bernoulli numbers and polynomials. Then the author obtained factorization and some properties of the Bernoulli matrices.

Ernst [9] studied matrix form of $q-$Bernoulli polynomials and obtained recurrence formula using this matrix form. The author studied relation between $q-$Cauchy-Vandermonde matrix and the $q-$ Bernoulli matrix. Then the author obtained $q-$ analogue of the Bernoulli theorem by using the Jackson-Hahn-Cigler $q-$Bernoulli polynomials.

In this section, we define $q-$Bernoulli matrices by using $q-$Bernoulli numbers and $q-$Bernoulli polynomials, Then we obtain inverse of $q-$Bernoulli matrix and some theorems related to the generalized $q-$Bernoulli matrix.

Definition 3. Let $b_n(q)$ be n^{th} $q-$Bernoulli number. The $q-$Bernoulli matrix $\mathscr{B}(q) = [b_{ij}(q)]$ is defined by

$$b_{ij}(q) = \begin{cases} \binom{i}{j}_q b_{i-j}(q) & if\ i \geq j, \\ 0 & otherwise, \end{cases} \tag{3.1}$$

where $0 \leq i,j \leq n$.

5×5 $q-$Bernoulli matrix is

$$\mathscr{B}(q)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ -\frac{1}{2} & 1 & 0 & 0 & 0\\ \frac{[2]_q}{2\cdot 3!} & -\frac{[2]_q}{2!} & 1 & 0 & 0\\ 0 & \frac{[3]_q\cdot[2]_q}{2\cdot 3!} & -\frac{[3]_q}{2!} & 1 & 0\\ -\frac{[4]_q}{30\cdot 4!} & 0 & \frac{[4]_q\cdot[3]_q}{2\cdot 3!} & -\frac{[4]_q}{2!} & 1 \end{pmatrix}$$

Following theorem is a generalization of Theorem 2.4 in [8].

Theorem 2. Let $\mathscr{D}(q)=[d_{ij}(q)]$ be $(n+1)\times(n+1)$ matrix, is defined by

$$d_{ij}(q)=\begin{cases}\binom{i}{j}_q\dfrac{[i-j]_q!}{(i-j+1)!} & if\ i\geq j,\\ 0 & otherwise.\end{cases} \tag{3.2}$$

Then $\mathscr{D}(q)$ is the inverse of q −Bernoulli matrix.

Proof. Let $\mathscr{B}(q)$ be $q-$ Bernoulli matrix and $\mathscr{D}(q)$ defined as in (3.2).

$$\begin{aligned}\left(\mathscr{B}(q)\mathscr{D}(q)\right)_{ij} &= \sum_{k=0}^{n} b_{ik}(q)d_{kj}(q)\\ &= \sum_{k=j}^{i}\binom{i}{k}_q b_{i-k}(q)\binom{k}{j}_q\frac{[k-j]_q!}{(k-j+1)!}\\ &= \sum_{k=j}^{i}\binom{i}{k}_q\binom{k}{j}_q\frac{[k-j]_q!}{(k-j+1)!}b_{i-k}(q)\\ &= \binom{i}{j}_q\sum_{k=j}^{i}\binom{i-j}{k-j}_q\frac{[k-j]_q!}{(k-j+1)!}b_{i-k}(q)\\ &= \binom{i}{j}_q\sum_{t=0}^{i-j}\binom{i-j}{t}_q\frac{[t]_q!}{(t+1)!}b_{i-j-t}(q)\\ &= \binom{i}{j}_q\sum_{t=0}^{i-j}\binom{i-j}{t}_q\frac{[t]_q!}{(t+1)!}B_{i-j-t}\frac{[i-j-t]_q!}{(i-j-t)!}\\ &= \binom{i}{j}_q\frac{[i-j]_q!}{(i-j)!}\sum_{t=0}^{i-j}\binom{i-j}{t}_q\frac{1}{t+1}B_{i-j-t}\end{aligned}$$

Using the orthogonality relation for Bernoulli numbers

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}B_{n-k}=\delta_{n,0} \tag{3.3}$$

(see [8]), we obtain

$$\left(\mathscr{B}(q)\mathscr{D}(q)\right)_{ij}=\binom{i}{j}_q\frac{[i-j]_q!}{(i-j)!}\delta_{i-j,0}=\delta_{i,j}.$$

Definition 4. Let $B_n(x,q)$ be n^{th} q −Bernoulli polynomial. The q −Bernoulli polynomial matrix $\mathscr{B}(x,q)=[b_{ij}(x,q)]$ is defined as follows

$$b_{ij}(x,q)=\begin{cases}\binom{i}{j}_q B_{i-j}(x,q) & if\ i\geq j,\\ 0 & otherwise.\end{cases} \tag{3.3}$$

4. q −BERNOULLI AND q −PASCAL MATRICES

Ernst [9] defined $(n+1)\times(n+1)$ generalized $q-$Pascal matrix $\mathscr{P}(x,q)=[p_{ij}(q)]$ by

$$p_{ij}(q)=\begin{cases}\binom{i}{j}_q x^{i-j} & if\ i\geq j,\\ 0 & otherwise.\end{cases} \tag{4.1}$$

The inverse of generalized $q-$ Pascal matrix $\mathscr{P}^{-1}(x,q)=[p'_{ij}(q)]$ is

$$p'_{ij}(q)=\begin{cases}\binom{i}{j}_q q^{\binom{i-j}{2}}(-x)^{i-j} & if\ i\geq j,\\ 0 & otherwise.\end{cases} \tag{4.2}$$

Now using the Zhang's methods in [8] we can generalize the factorization q −Bernoulli matrices.

Theorem 3. Let $\mathscr{B}(x,q)$ be q −Bernoulli polynomial matrix and $\mathscr{P}(x,q)$ be generalized q −Pascal matrix, then

$$\mathscr{B}(x+y,q)=\mathscr{P}(y,q)\mathscr{B}(x,q)=\mathscr{P}(x,q)\mathscr{B}(y,q) \tag{4.3}$$

and specially

$$\mathscr{B}(x,q)=\mathscr{P}(x,q)\mathscr{B}(q)\ . \tag{4.4}$$

Proof. Let $\mathscr{P}(y,q)$ be generalized $q-$Pascal matrix and $\mathscr{B}(x,q)$ be q −Bernoulli polynomial matrix. Then

$$\begin{aligned}\left(\mathscr{P}(y,q)\mathscr{B}(x,q)\right)_{ij} &= \sum_{k=0}^{n} p_{ik}(q)\,b_{kj}(x,q)\\ &= \sum_{k=j}^{i}\binom{i}{k}_q y^{i-k}\binom{k}{j}_q B_{k-j}(x,q)\\ &= \sum_{k=j}^{i}\binom{i}{j}_q\binom{i-j}{k-j}_q y^{i-k} B_{k-j}(x,q)\\ &= \binom{i}{j}_q\sum_{t=0}^{i-j}\binom{i-j}{t}_q y^{i-j-t} B_t(x,q).\end{aligned}$$

Using (2.8), we have

$$\left(\mathscr{P}(y,q)\,\mathscr{B}(x,q)\right)_{ij} = \binom{i}{j}_q B_{i-j}(x+y,q) = \left(\mathscr{B}(x+y,q)\right)_{ij}$$

and similarly it can be provide that

$$\mathscr{B}(x+y,q) = \mathscr{P}(x,q)\,\mathscr{B}(y,q).$$

Now, we show that

$$\mathscr{B}(x,q) = \mathscr{P}(x,q)\,\mathscr{B}(q).$$

$$\begin{aligned}\left(\mathscr{P}(x,q)\,\mathscr{B}(q)\right)_{ij} &= \sum_{k=0}^{n} p_{ik}(q)\, b_{kj}(q)\\ &= \sum_{k=j}^{i} \binom{i}{k}_q x^{i-k} \binom{k}{j}_q b_{k-j}(q)\\ &= \binom{i}{j}_q \sum_{k=j}^{i} \binom{i-j}{k-j}_q x^{i-k}\ b_{k-j}(q)\\ &= \binom{i}{j}_q B_{i-j}(x,q)\\ &= \left(\mathscr{B}(x,q)\right)_{ij}\end{aligned}$$

We give two examples of this theorem for 3×3 and 4×4 q −Bernoulli polynomial matrix and q −Pascal matrix.

$$\begin{aligned}&\left(\mathscr{P}(y,q)\,\mathscr{B}(x,q)\right)_{ij}\\ &= \begin{pmatrix} 1 & 0 & 0\\ y & 1 & 0\\ y^2 & [2]_q y & 1\end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0\\ x-\frac{1}{2} & 1 & 0\\ x^2-\frac{[2]_q}{2}x+\frac{[2]_q}{12} & [2]_q x-\frac{[2]_q}{2} & 1\end{pmatrix}\\ &= \begin{pmatrix} 1 & 0 & 0\\ (x+y)-\frac{1}{2} & 1 & 0\\ x^2+[2]_q xy+y^2-\frac{[2]_q}{2}(x+y)+\frac{[2]_q}{12} & [2]_q(x+y)-\frac{[2]_q}{2} & 1\end{pmatrix}\\ &= \mathscr{B}(x+y,q)\end{aligned}$$

$$\begin{aligned}&\left(\mathscr{P}(x,q)\,\mathscr{B}(q)\right)_{ij}\\ &= \begin{pmatrix} 1 & 0 & 0 & 0\\ x & 1 & 0 & 0\\ x^2 & [2]_q x & 1 & 0\\ x^3 & [3]_q x^2 & [3]_q x & 1\end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0\\ -\frac{1}{2} & 1 & 0 & 0\\ \frac{[2]_q}{2\cdot 3!} & -\frac{[2]_q}{2!} & 1 & 0\\ 0 & \frac{[2]_q[3]_q}{2\cdot 3!} & -\frac{[3]_q}{2!} & 1\end{pmatrix}\\ &= \begin{pmatrix} 1 & 0 & 0 & 0\\ x-\frac{1}{2} & 1 & 0 & 0\\ x^2-\frac{[2]_q}{2}x+\frac{[2]_q}{12} & [2]_q x-\frac{[2]_q}{2} & 1 & 0\\ x^3-\frac{[3]_q}{2}x^2+\frac{[2]_q[3]_q}{12}x & [3]_q x^2-\frac{[2]_q[3]_q}{2}x+\frac{[2]_q[3]_q}{12} & [3]_q x-\frac{[3]_q}{2} & 1\end{pmatrix}\\ &= \mathscr{B}(x,q)\end{aligned}$$

Corollary 1. Let $\mathscr{B}(x,q)$ be q −Bernoulli polynomial matrix then $\mathscr{B}^{-1}(x,q) = \left[c_{ij}(q)\right]$ is

$$c_{ij}(q) = \begin{cases} \dfrac{[i]_q!}{[j]_q!} \displaystyle\sum_{t=0}^{i-j} \frac{q^{\binom{t}{2}}(-x)^t}{[t]_q!\,(i-j-t+1)!} & if\ i \ge j,\\ 0 & otherwise. \end{cases} \tag{4.5}$$

Proof. Let $\mathscr{B}(q)$ be q −Bernoulli matrix and $\mathscr{P}(x,q)$ be generalized q −Pascal matrix. Using factorization of $\mathscr{B}(x,q)$ in (4.4)

$$\mathscr{B}^{-1}(x,q) = \mathscr{B}^{-1}(q)\,\mathscr{P}^{-1}(x,q) = \mathscr{D}(q)\,\mathscr{P}^{-1}(x,q)$$

and inverse of generalized q −Pascal matrix (4.2), we obtain

$$\begin{aligned}\left(\mathscr{D}(q)\,\mathscr{P}^{-1}(x,q)\right)_{ij} &= \sum_{k=0}^{n} d_{ik}(q) p'_{kj}(q)\\ &= \sum_{k=j}^{i} \binom{i}{k}_q \frac{[i-k]_q!}{(i-k+1)!}(-x)^{k-j}\, q^{\binom{k-j}{2}} \binom{k}{j}_q\\ &= \sum_{k=j}^{i} \binom{i}{j}_q \binom{i-j}{k-j}_q \frac{[i-k]_q!\, q^{\binom{k-j}{2}}}{(i-k+1)!}(-x)^{k-j}\\ &= \binom{i}{j}_q \sum_{t=0}^{i-j} \binom{i-j}{t}_q \frac{[i-j-t]_q!\, q^{\binom{t}{2}}}{(i-j-t+1)!}(-x)^t\\ &= \frac{[i]_q!}{[j]_q!} \sum_{t=0}^{i-j} \frac{q^{\binom{t}{2}}(-x)^t}{[t]_q!\,(i-j-t+1)!}\\ &= c_{ij}(q)\end{aligned}$$

5. SHIFTED q −BERNOULLI AND q −VANDERMONDE MATRICES

In [8] Zhang defined shifted Bernoulli matrix, and obtained some relations between shifted Bernoulli matrix and Vandermonde matrix.

In this section we define $q-$ Vandermonde matrix. and $q-$ shifted Bernoulli matrix by using $q-$ Bernoulli polynomials and give its relation with q −Vandermonde matrix.

Definition 5. Let $B_n(x,q)$ be q −Bernoulli polynomial. The shifted q −Bernoulli matrix $\widetilde{\mathscr{B}}(y,q) = \left[\tilde{b}_{ij}(y,q)\right]$ is defined by

$$\tilde{b}_{ij}(y,q) = B_i(y+j,q) \tag{5.1}$$

where $0 \le i,j \le n$.

Definition 6. ([11]) The $(n+1)\times(n+1)$ type q −Vandermonde matrix $\mathrm{V}(y,q) = \left[v_{ij}(y,q)\right]$ is defined by

$$v_{ij}(y,q) = (y+j)_q^i. \tag{5.2}$$

4×4 q −Vandermonde matrix is

$$\mathrm{V}(y,q) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ y & y+1 & y+2 & y+3 \\ y^2 & (y+1)_q^2 & (y+2)_q^2 & (y+3)_q^2 \\ y^3 & (y+1)_q^3 & (y+2)_q^3 & (y+3)_q^3 \end{pmatrix}.$$

In the following theorem, we obtain factorization shifted $q-$ Bernoulli matrix by using $q-$ Bernoulli matrix and q −Vandermonde matrix.

Theorem 4. Let $\mathrm{V}(y,q)$ be $q-$Vandermonde matrix and $\mathscr{B}(q)$ be q −Bernoulli matrix. Then

$$\widetilde{\mathscr{B}}(y,q) = \mathscr{B}(q)\mathrm{V}(y,q).$$

Proof.

$$\left(\mathscr{B}(q)\mathrm{V}(y,q)\right)_{ij} = \sum_{k=0}^{n} b_{ik}(q)\, v_{kj}(y,q)$$
$$= \sum_{k=0}^{i} \binom{i}{k}_q b_{i-k}(q)\,(y+j)_q^k$$

If we use definition of q −Bernoulli polynomial, then

$$\left(\mathscr{B}(q)\mathrm{V}(y,q)\right)_{ij} = B_i(y+j,q)$$

we obtain

$$\mathscr{B}(q)\mathrm{V}(y,q) = \widetilde{\mathscr{B}}(y,q).$$

For $q \to 1^-$, we can obtain Theorem 5.2 in [8].

The factorization of 3×3 shifted q −Bernoulli matrix is as follows.

$$\mathscr{B}(q)\mathrm{V}(y,q)$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{[2]_q}{2\cdot 3!} & -\frac{[2]_q}{2!} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ y & y+1 & y+2 \\ y^2 & (y+1)_q^2 & (y+2)_q^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 \\ y-\frac{1}{2} & y+\frac{1}{2} & y+\frac{3}{2} \\ y^2-\frac{[2]_q}{2}y+\frac{[2]_q}{2\cdot 3!} & y^2+\frac{[2]_q}{2}y+\frac{7\cdot[2]_q}{2\cdot 3!}-1 & y^2+\frac{3\cdot[2]_q}{2}y+\frac{37\cdot[2]_q}{2\cdot 3!}-4 \end{pmatrix}$$
$$= \widetilde{\mathscr{B}}(y,q).$$

CONFLICT OF INTEREST

The authors declare that there is no conflict of interests regarding the publication of this paper.

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