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Robust factorial ANCOVA with LTS error distributions

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Abstract

In this study, parameter estimation and hypotheses testing in the balanced factorial analysis of covariance (ANCOVA) model, when the distribution of error terms is long-tailed symmetric (LTS) are considered. The unknown model parameters are estimated using the methodology known as modified maximum likelihood (MML). New test statistics based on these estimators are also proposed for testing the main effects, interaction effect and slope parameter. Assuming LTS distributions for the error term, a Monte-Carlo simulation study is conducted to compare the efficiencies of MML estimators with corresponding least squares (LS) estimators. Power and the robustness properties of the proposed test statistics are also compared with traditional normal theory test statistics. The results of the simulation study show that MML estimators are more efficient than corresponding LS estimators. Furthermore, proposed test statistics are shown to be more powerful and robust than normal theory test statistics. In the application part, a data set, taken from the literature, is analyzed to show the implementation of the methodology presented in the study.

Keywords: Analysis of Covariance (ANCOVA), Factorial design, Long-tailed symmetric distribution, Modified likelihood, Monte Carlo simulation, Robustness.

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1. Introduction

Factorial designs, introduced by Fisher [8] and Yates [36], have a wide range of applications, i.e. they are used for the evaluation of equipment and materials, product design, performance testing, process development and so on. The reason why factorial designs draw attention of practitioners is that they provide the investigation of the effects of multiple factors simultaneously. Therefore, they are more efficient than the traditional one-factor-at-a time approach in terms of time and cost. Furthermore, they allow for the estimation and detection interactions between factors, see [20] for details.

In experimental design, analysis of covariance (ANCOVA) is another important model which is defined as a combination of regression analysis and analysis of variance (ANOVA). The ANCOVA model reduces the variability of the random error that is associated with covariates. This leads to obtain more precise estimates and more powerful tests [21, 22].

In spite of the fact that the one-way ANCOVA is mostly considered in the literature, there are limited number of papers considering factorial designs with covariates, see i.e. [33]. Therefore, in this study, we consider the following factorial ANCOVA model:

$$\begin{gathered} y_{ijk} = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + \beta(x_{ijk} - \bar{x}_{\cdots}) + \varepsilon_{ijk}, \\ i = 1, 2, \ldots, a; \quad j = 1, 2, \ldots, b; \quad k = 1, 2, \ldots, n. \end{gathered} \tag{1.1}$$

where y_{ijk}, μ, τ_i, γ_j, $(\tau\gamma)_{ij}$ have the usual interpretations. In addition, β, x_{ijk} and $\bar{x}_{\cdots}$ denote the slope parameter, the covariate term and the overall mean of the covariate terms, respectively. Without loss of generality, we assume that (i) Model (1.1) is fixed effect, i.e. $\sum_{i=1}^{a} \tau_i = 0$, $\sum_{j=1}^{b} \gamma_j = 0$ and $\sum_i (\tau\gamma)_{ij} = 0 \ \ \forall j$, $\sum_j (\tau\gamma)_{ij} = 0 \ \ \forall i$, (ii) The slopes for each treatment are homogeneous and (iii) Covariate term x is non-stochastic.

The motivation for this paper comes from the fact that the distribution of error terms is often assumed to be independently and identically distributed (i.i.d.) normal in Model (1.1). However, nonnormality is more prevalent in practice as Geary [9] indicates, "*Normality is a myth, there never was and never will be, a normal distribution*". Furthermore, least squares (LS) estimators lose efficiency and the power of the tests based on LS estimators are adversely affected in the presence of the nonnormality. Therefore, there is great interest in solving the problems that nonnormality causes [7, 10, 11, 12, 18, 19, 20, 21, 23, 32].

The originality of this paper is assumption of long-tailed (LTS) symmetric error distribution in Model (1.1). LTS distribution is used symmetric alternative of normal distribution. It also provides superiority to normal distribution for modeling outlier(s) occurred in the direction of the long tail(s) [12, 28]. Since maximum likelihood (ML) estimators cannot be obtained explicitly, we therefore derive modified maximum likelihood (MML) estimators of the model parameters [26, 27]. We also propose new test statistics based on these MML estimators for testing the main effects, the interaction effect and the significance of the slope parameter [1, 2].

It should also be noted that in the rest of the paper we consider the situation where $a = 2$ and $b = 2$ in Model (1.1) for illustration. Thus, Model (1.1) reduces to 2^2 factorial design with a covariate, i.e. we have two factors named as A and B. This reduction is made because of the fact that the results obtained for 2^2 factorial design can easily be extended to more complicated factorial designs such as 2^k [20].

The rest of the paper is organized as follows. Section 2 considers LTS distribution and its properties. In Section 3, MML estimators of the model parameters are obtained and the performances of these estimators are compared with corresponding LS estimators via a Monte-Carlo simulation study. Section 4 includes the proposed test statistics for testing the main effects, interaction effect and slope parameter. The robustness of the proposed test statistics and the normal theory test statistics are considered in Section 5.

Section 6 is reserved for an application to show the implementation of the methodology presented in the study. The paper ends with a conclusion section.

2. LTS distribution

The probability density function (pdf) of LTS distribution is given by:

$$f_{LTS}(e;p,\sigma) \propto \frac{1}{\sigma}\left(1+\frac{e^2}{q\sigma^2}\right)^{-p}, \quad q=2p-3, \quad -\infty < e < \infty \tag{2.1}$$

where p and σ are the shape and the scale parameters, respectively. Here, shape parameter p is assumed to be greater than 2. The density plots of LTS distribution are given in Figure 1 for different values of the shape parameter p.

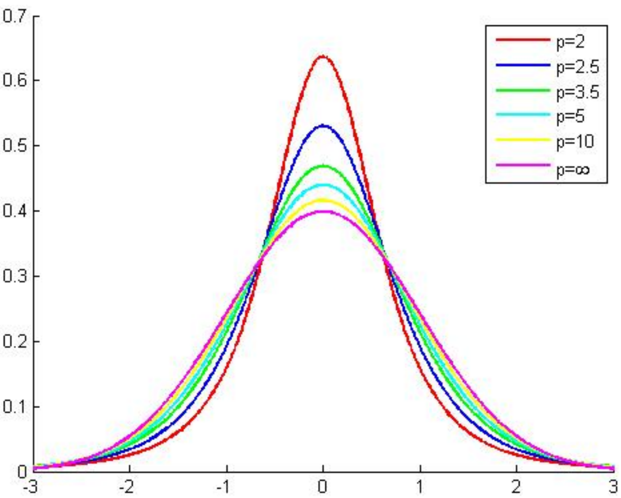

Figure 1. The density plots of LTS distribution for different values of the shape parameter p.

Since LTS is a symmetric distribution, only kurtosis values of LTS distribution are tabulated in Table 1 for some representative values of p. It is clear from Table 1 that the kurtosis of the LTS distribution is greater than 3, but it approaches 3 when p tends to ∞.

Table 1. The kurtosis values of LTS distribution for different values of the shape parameter p.

p	2.5	3.5	5	10	∞
Kurtosis	∞	9	4.2	3.4	3

It should be noted that the shape parameter p is assumed to be known in order to find the estimators of the model parameters in the rest of the paper. The reason for this assumption is that simultaneous estimation of the shape parameter, along with other parameters results in unreliable estimate of the shape parameter for small sample sizes [4, 13]. However, we estimate shape parameter using the methodology known as profile likelihood, see Section 6 for details.

3. MML estimators

The loglikelihood function ($\ln L$) for Model (1.1) with ε_{ijk} following a LTS distribution is obtained as follows:

$$\ln L \propto -2^2 n \ln \sigma - p \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n} \ln\left(1+\frac{z_{ijk}^2}{q}\right), \tag{3.1}$$

where

$$z_{ijk} = \frac{y_{ijk}-\mu-\tau_i-\gamma_j-(\tau\gamma)_{ij}-\beta(x_{ijk}-\bar{x}_{...})}{\sigma} \quad i,j=1,2; k=1,2,\ldots,n.$$

It is well-known that ML estimators of the parameters are the point where loglikelihood function attains its maximum. Therefore, in order to find ML estimators, the partial derivatives of the $\ln L$ function should be taken with respect to the parameters of interest and set them equal to zero as follows:

$$\frac{\partial \ln L}{\partial \mu}=0, \frac{\partial \ln L}{\partial \tau_i}=0, \frac{\partial \ln L}{\partial \gamma_j}=0, \frac{\partial \ln L}{\partial (\tau\gamma)_{ij}}=0, \frac{\partial \ln L}{\partial \beta}=0 \text{ and } \frac{\partial \ln L}{\partial \sigma}=0. \tag{3.2}$$

Since these equations contain a nonlinear

$$g(z_{ijk}) = \frac{z_{ijk}}{1+\frac{1}{q}z_{ijk}^2} \tag{3.3}$$

function, ML estimators of the model parameters cannot be obtained explicitly. Therefore, numerical or iterative methods should be performed. However, using numerical or iterative methods are known to have a number of drawbacks, such as; (i) wrong convergency, (ii) non-convergency, and (iii) multiple roots problems [16, 30, 35]. To avoid these difficulties, we use Tiku's [26, 27] MML methodology.

The steps of the MML method are explained as follows. First, order the z_{ijk} observations from the smallest to largest, i.e. $z_{ij(1)} \le z_{ij(2)} \le \ldots \le z_{ij(n)}$. Second, linearize the $g(\cdot)$ function using the first two terms of the Taylor series, which is expanded around the expected values of the order statistics, i.e. $t_{(k)} = E(z_{ij(k)})$, $k = 1, 2, \ldots, n$. This results in:

$$g(z_{ijk}) \cong \alpha_k + \delta_k z_{ijk} \tag{3.4}$$

where

$$\alpha_k = \frac{(2/q)t_{(k)}^3}{(1+(1/q)t_{(k)}^2)^2}, \quad \delta_k = \frac{1-(1/q)t_{(k)}^2}{(1+(1/q)t_{(k)}^2)^2}, \quad k=1,2,\ldots,n. \tag{3.5}$$

It should be noted that the exact values of $t_{(k)}$ are obtained from Tiku & Kumra [29]. Alternatively, approximate values of $t_{(k)}$ values can be obtained in the following way

$$F_{LTS}(t_{(k)}) = \int_{-\infty}^{t_{(k)}} f_{LTS}(z)dz = \frac{k}{n+1}, \quad k=1,2,\ldots,n$$

where $F_{LTS}(\cdot)$ is the cumulative distribution function (cdf) of the LTS distribution. Using approximate values of $t_{(k)}$ values instead of the exact values does not alter the efficiencies of the estimators adversely.

Finally, we incorporate (3.4) in (3.2) to obtain modified likelihood equtions, i.e.

$$\frac{\partial \ln L^*}{\partial \mu} = 0, \frac{\partial \ln L^*}{\partial \tau_i} = 0, \frac{\partial \ln L^*}{\partial \gamma_j} = 0, \frac{\partial \ln L}{\partial (\tau\gamma)_{ij}} = 0, \frac{\partial \ln L^*}{\partial \beta} = 0 \text{ and } \frac{\partial \ln L^*}{\partial \sigma} = 0.$$

The solutions of these modified likelihood equations are the following MML estimators:

$$\hat{\mu} = \hat{\mu}_{\cdot\cdot[\cdot]} - \hat{\beta}\hat{\mu}_{x\cdot\cdot[\cdot]}, \quad \hat{\tau}_i = \hat{\mu}_{i\cdot[\cdot]} - \hat{\mu}_{\cdot\cdot[\cdot]} - \hat{\beta}(\hat{\mu}_{xi\cdot[\cdot]} - \hat{\mu}_{x\cdot\cdot[\cdot]}), \tag{3.6}$$

$$\hat{\gamma}_j = \hat{\mu}_{\cdot j[\cdot]} - \hat{\mu}_{\cdot\cdot[\cdot]} - \hat{\beta}(\hat{\mu}_{x\cdot j[\cdot]} - \hat{\mu}_{x\cdot\cdot[\cdot]}), \tag{3.7}$$

$$\begin{aligned} \widehat{(\tau\gamma)}_{ij} \;=\;& \hat{\mu}_{ij[\cdot]} - \hat{\mu}_{i\cdot[\cdot]} - \hat{\mu}_{\cdot j[\cdot]} + \hat{\mu}_{\cdot\cdot[\cdot]} - \\ & \hat{\beta}(\hat{\mu}_{xij[\cdot]} - \hat{\mu}_{xi\cdot[\cdot]} - \hat{\mu}_{x\cdot j[\cdot]} + \hat{\mu}_{x\cdot\cdot[\cdot]}), \end{aligned} \tag{3.8}$$

$$\hat{\beta} = K + L\hat{\sigma} \quad \text{and} \quad \hat{\sigma} = \frac{B + \sqrt{B^2 + 4NC}}{2\sqrt{N(N - 2^2 - 1)}} \tag{3.9}$$

where

$$\hat{\mu}_{\cdot\cdot[\cdot]} = \frac{\sum_{i=1}^{2}\sum_{j=1}^{2}\hat{\mu}_{ij[\cdot]}}{2^2}, \quad \hat{\mu}_{i\cdot[\cdot]} = \frac{\sum_{j=1}^{2}\hat{\mu}_{ij[\cdot]}}{2}, \quad \hat{\mu}_{\cdot j[\cdot]} = \frac{\sum_{i=1}^{2}\hat{\mu}_{ij[\cdot]}}{2}, \quad \hat{\mu}_{ij[\cdot]} = \frac{\sum_{k=1}^{n}\delta_k y_{ij[k]}}{m},$$

$$\hat{\mu}_{x\cdot\cdot[\cdot]} = \frac{\sum_{i=1}^{2}\sum_{j=1}^{2}\hat{\mu}_{xij[\cdot]}}{2^2}, \quad \hat{\mu}_{xi\cdot[\cdot]} = \frac{\sum_{j=1}^{2}\hat{\mu}_{xij[\cdot]}}{2}, \quad \hat{\mu}_{x\cdot j[\cdot]} = \frac{\sum_{i=1}^{2}\hat{\mu}_{xij[\cdot]}}{2},$$

$$\hat{\mu}_{xij[\cdot]} = \frac{\sum_{k=1}^{n}\delta_k(x_{ij[k]} - \bar{x}_{\cdot\cdot[\cdot]})}{m}, \quad m = \sum_{k=1}^{n}\delta_k,$$

$$S^*_{xy} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}\delta_k y_{ij[k]}(x_{ij[k]} - \bar{x}_{\cdot\cdot[\cdot]}), \quad S^*_{xx} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}\delta_k(x_{ij[k]} - \bar{x}_{\cdot\cdot[\cdot]})^2,$$

$$T^*_{xy} = m\sum_{i=1}^{2}\sum_{j=1}^{2}\hat{\mu}_{ij[\cdot]}\hat{\mu}_{xij[\cdot]}, \quad T^*_{xx} = m\sum_{i=1}^{2}\sum_{j=1}^{2}\hat{\mu}^2_{xij[\cdot]}, \quad E^*_{xy} = S^*_{xy} - T^*_{xy},$$

$$E^*_{xx} = S^*_{xx} - T^*_{xx}, \quad K = \frac{E^*_{xy}}{E^*_{xx}}, \quad L = \frac{\sum_{k=1}^{n}\alpha_k(x_{ij[k]} - \bar{x}_{\cdot\cdot[\cdot]})}{E^*_{xx}}, \quad N = 2^2 n,$$

$$B = \frac{2p}{q}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}\alpha_k\left\{y_{ij[k]}-\hat{\mu}_{ij[\cdot]}+K\left[\hat{\mu}_{xij[\cdot]}-(x_{ij[k]}-\bar{x}_{\cdot\cdot[\cdot]})\right]\right\},$$

$$C = \frac{2p}{q}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}\delta_k\left\{y_{ij[k]}-\hat{\mu}_{ij[\cdot]}+K\left[\hat{\mu}_{xij[\cdot]}-(x_{ij[k]}-\bar{x}_{\cdot\cdot[\cdot]})\right]\right\}^2.$$

It is clear that MML estimators are easy to compute since they are expressed as the functions of sample observations. In MML methodology, small δ_k $(1,2,\ldots,n)$ weights are given to outlying observation(s), occurred in the direction of long tail(s). This depletes the dominant effects of outliers and makes them robust.

Remarks

(i) $2N$ is replaced by $2\sqrt{N(N-2^2-1)}$ in the denominator of $\hat{\sigma}$ for bias correction.

(ii) $(y_{ij[k]}, x_{ij[k]})$ are called concomitants corresponding to ordered $z_{ij(k)}$ observations, for further information see Islam & Tiku [12]. It should also be noted that $\bar{x}_{\cdot\cdot[\cdot]} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n} x_{ij[k]}/N$.

(iii) MML estimators are asymptotically fully efficient under some mild regularity conditions and they are as efficient as ML estimators for small sample sizes, see [3, 18, 19, 20, 31, 34, 35].

(iv) When the shape parameter p is small and the sample size is large, some of the δ_k $(k=1,2,\ldots,n)$ values may be negative. This may cause nonreal or negative estimates of $\hat{\sigma}$. To avoid this problem, Islam & Tiku [12] suggest taking the following versions of the α_k and δ_k:

$$\alpha_k^* = \frac{(1/q)t_{(k)}^3}{(1+(1/q)t_{(k)}^2)^2}, \quad \delta_k^* = \frac{1}{(1+(1/q)t_{(k)}^2)^2}, \quad k=1,2,\ldots,n,$$

respectively. Islam & Tiku [12] indicate that this modification does not alter the asymptotic properties of the MML estimators.

It can be shown that the difference $g(z_{ij(k)}) - (\alpha_k + \beta_k z_{ij(k)})$ converges to zero as n tends to ∞ for all $p \geq 2$. As a consequence,

$$\lim_{n\to\infty}\frac{1}{n}\left|\frac{\partial \ln L}{\partial\theta}-\frac{\partial \ln L^*}{\partial\theta}\right| = 0 \tag{3.10}$$

for all the parameters in Model (1.1) and σ. This has been proven for simple linear regression model by Tiku et al. [32] and the one-way ANOVA model by Senoglu [17] for different types of error distributions. Since the proof for factorial ANCOVA can be made by following the same lines in the mentioned papers, we do not reproduce for the brevity.

Equation (3.10) shows that MML estimators are asymptotically equivalent to corresponding ML estimators. They are also asymptotically minimum variance bound (MVB) estimators, see Bhattacharyya [3] and Vaughan and Tiku [34] for further information.

The asymptotic properties of the MML estimators of parameters in Model (1.1) are given in the following theorems.

3.1. Theorem. *$\hat{\mu}_i$, $\hat{\mu}_j$ and $\hat{\mu}_{ij}$ $(i, j = 1, 2)$ are asymptotically normally distributed, i.e.*

$$\hat{\mu}_i \overset{asym}{\sim} N\left(\mu_i, \frac{q}{2^2mp}\sigma^2\right), \tag{3.11}$$

$$\hat{\mu}_j \overset{asym}{\sim} N\left(\mu_j, \frac{q}{2^2mp}\sigma^2\right) \tag{3.12}$$

and

$$\hat{\mu}_{ij} \overset{asym}{\sim} N\left(\mu_{ij}, \frac{q}{2mp}\sigma^2\right) \tag{3.13}$$

where $\hat{\mu}_i$, $\hat{\mu}_j$ and $\hat{\mu}_{ij}$ have the usual interpretations.

3.2. Theorem. *$\hat{\beta}$ estimator is asymptotically normally distributed with mean β and variance $\sigma^2 \Big/ \left(\frac{2p}{q}E^*_{xx}\right)$.*

3.3. Theorem. *The asymptotic distribution of $(N - 2^2 - 1)\hat{\sigma}^2/\sigma^2$ is a Chi-square with degrees of freedom $\nu = N - 2^2 - 1$, $N = 2^2n$.*

Reader is referred to Kendall and Stuart [14], Senoglu & Tiku [18] and Senoglu [20] for the proof of the these theorems.

Simulations

We conduct a Monte-Carlo simulation in this part of the study to compare the performances of the traditional LS estimators with the corresponding MML estimators, In the simulation study, we take $\mu = \tau_i = \gamma_j = (\tau\gamma)_{ij} = 0$ $(i, j = 1, 2)$, $\beta = 1$ and $\sigma = 1$ in Model (1.1) without loss of generality. The error terms are generated from LTS distribution for different values of the shape parameter, i.e. $p = 2,\ 2.5,\ 3.5$ and 5. The covariate terms x are generated from standard normal distribution. All simulations are replicated 10,000 times in which the sample size is taken to be $n = 10$ and 20. MATLAB software is used for all computations.

The efficiencies of the LS and the MML estimators of the model parameters are compared in terms of their means, variances and mean squared errors (MSE), see Table 2. *RE* values (REs) are calculated using the following formula:

$$RE = \frac{MSE_{MML}}{MSE_{LS}} \times 100. \tag{3.14}$$

MML estimators are said to be more efficient than LS estimators when the REs are smaller than 100.

It should also be noted that we use $\tilde{\mu}$, $\tilde{\tau}_i$, $\widetilde{(\tau\gamma)}_{ij}$, $\tilde{\beta}$ and $\tilde{\sigma}$ notations for the corresponding LS estimators of the model parameters in (1.1). The formulas of the LS estimators are not given here for the sake of brevity. However, they can be found in Senoglu and Acitas [24].

It is clear from Table 2 that the MML estimators are more efficient than the corresponding LS estimators in general. As shape parameter p increases from 2 to 5, the MSE values of the MML and the LS estimators are very close to each other

Table 2. Simulated means, variances ($n \times Var$), MSEs ($n \times MSE$) and RE values of the LS and the MML estimators of the parameters μ, τ_1, γ_1, $(\tau\gamma)_{11}$, β and σ.

Parameter	$Mean$		$n \times Var$		$n \times MSE$		RE
	LS	MML	LS	MML	LS	MML	
			$n = 10,\ p = 2$				
μ	0.0018	0.0018	0.2493	0.1471	0.2493	0.1471	59
τ_1	0.0000	-0.0006	0.2617	0.1572	0.2617	0.1572	60
γ_1	-0.0010	0.0009	0.2571	0.1535	0.2571	0.1535	60
$(\tau\gamma)_{11}$	0.0040	0.0028	0.2570	0.1526	0.2572	0.1526	59
β	1.0002	0.9996	0.2894	0.2066	0.2894	0.2066	71
σ	0.9355	1.1608	12.454	0.9464	12.870	12.048	94
			$n = 10,\ p = 2.5$				
μ	-0.0006	-0.0001	0.2504	0.1899	0.2504	0.1899	76
τ_1	-0.0023	-0.0020	0.2572	0.1943	0.2573	0.1943	76
γ_1	-0.0023	-0.0014	0.2593	0.1951	0.2593	0.1952	75
$(\tau\gamma)_{11}$	0.0005	-0.0009	0.2580	0.1968	0.2580	0.1968	76
β	1.0014	1.0013	0.3000	0.2319	0.3000	0.2319	77
σ	0.9697	1.0739	0.5182	0.3788	0.5273	0.4334	82
			$n = 10,\ p = 3.5$				
μ	-0.0014	-0.0012	0.2575	0.2301	0.2576	0.2301	89
τ_1	0.0015	0.0013	0.2521	0.2259	0.2522	0.2259	90
γ_1	-0.0019	-0.0023	0.2590	0.2343	0.2591	0.2344	90
$(\tau\gamma)_{11}$	-0.0021	-0.0015	0.2577	0.2331	0.2578	0.2331	90
β	0.9996	0.9996	0.2871	0.2632	0.2871	0.2632	92
σ	0.9808	1.0535	0.2655	0.2467	0.2692	0.2753	102
			$n = 10,\ p = 5$				
μ	0.0007	0.0007	0.2452	0.2346	0.2452	0.2346	96
τ_1	-0.0022	-0.0023	0.2645	0.2519	0.2646	0.2519	95
γ_1	0.0014	0.0019	0.2548	0.2452	0.2549	0.2453	96
$(\tau\gamma)_{11}$	-0.0002	-0.0002	0.2598	0.2485	0.2598	0.2485	96
β	0.9998	1.0000	0.2978	0.2860	0.2978	0.2860	96
σ	0.9883	1.0370	0.2094	0.2147	0.2107	0.2284	108
			$n = 20,\ p = 2$				
μ	-0.0004	0.0003	0.2558	0.1383	0.2558	0.1383	54
τ_1	-0.0016	-0.0013	0.2540	0.1392	0.2541	0.1392	55
γ_1	0.0014	0.0007	0.2609	0.1424	0.2609	0.1424	55
$(\tau\gamma)_{11}$	-0.0010	-0.0012	0.2607	0.1424	0.2607	0.1424	55
β	1.0008	1.0005	0.2584	0.1520	0.2584	0.1520	59
σ	0.9556	1.0927	21.482	0.7126	21.876	0.8844	40
			$n = 20,\ p = 2.5$				
μ	0.0002	-0.0006	0.2499	0.1799	0.2499	0.1799	72
τ_1	-0.0013	-0.0005	0.2544	0.1870	0.2544	0.1870	74
γ_1	-0.0010	-0.0001	0.2522	0.1859	0.2522	0.1859	74
$(\tau\gamma)_{11}$	-0.0005	-0.0004	0.2545	0.1845	0.2545	0.1845	72
β	0.9992	0.9993	0.2728	0.1997	0.2728	0.1997	73
σ	0.9858	1.0415	0.6542	0.3528	0.6583	0.3873	59
			$n = 20,\ p = 3.5$				
μ	-0.0009	-0.0011	0.2462	0.2152	0.2462	0.2152	87
τ_1	0.0011	0.0006	0.2583	0.2262	0.2583	0.2262	88
γ_1	-0.0001	0.0004	0.2439	0.2144	0.2439	0.2144	88
$(\tau\gamma)_{11}$	0.0002	-0.0008	0.2497	0.2174	0.2497	0.2174	87
β	0.9992	0.9995	0.2733	0.2392	0.2733	0.2392	88
σ	0.9929	1.0319	0.2842	0.2211	0.2852	0.2415	85
			$n = 20,\ p = 5$				
μ	-0.0007	-0.0005	0.2552	0.2399	0.2552	0.2399	94
τ_1	-0.0010	-0.0011	0.2617	0.2475	0.2618	0.2476	95
γ_1	0.0012	0.0011	0.2550	0.2395	0.2550	0.2395	94
$(\tau\gamma)_{11}$	-0.0015	-0.0013	0.2559	0.2411	0.2560	0.2411	94
β	0.9998	1.0001	0.2671	0.2543	0.2671	0.2543	95
σ	0.9962	1.0247	0.2030	0.1908	0.2033	0.2029	100

as expected. For certain cases, the MSE of $\tilde{\sigma}$ is better than $\hat{\sigma}$, see i.e. $n = 10$, $p = 3.5$ and $p = 5$. However, when sample size increases, $\hat{\sigma}$ gains efficiency. It should also be noted that we just reproduce the results for τ_1, γ_1 and $(\tau\gamma)_{11}$ for the brevity.

4. Hypotheses testing

The null hypotheses are given for testing the main effects, interaction effect and the slope parameter as follows:

$$
\begin{array}{llll}
H_{01} & : & \forall \tau_i = 0, \quad i = 1,2 & \text{(for testing main effect of factor A)} \\
H_{02} & : & \forall \gamma_j = 0, \quad j = 1,2 & \text{(for testing main effect of factor B)} \\
H_{03} & : & \forall (\tau\gamma)_{ij} = 0, \quad i = 1,2; j = 1,2 & \text{(for testing interaction effect AB)} \\
H_{04} & : & \beta = 0 & \text{(for testing slope parameter } \beta\text{).}
\end{array}
$$

We propose following test statistics

$$
(4.1) \quad F_A^* = \frac{\left(\frac{2^2 mp}{q}\right) \sum_{i=1}^{2} (\hat{\mu}_i - \bar{\mu}_i)^2}{\hat{\sigma}^2}, \quad F_B^* = \frac{\left(\frac{2^2 mp}{q}\right) \sum_{j=1}^{2} (\hat{\mu}_j - \bar{\mu}_j)^2}{\hat{\sigma}^2},
$$

$$
(4.2) \qquad F_{AB}^* = \frac{\left(\frac{2mp}{q}\right) \sum_{i=1}^{2} \sum_{j=1}^{2} (\hat{\mu}_{ij} - \bar{\mu}_{ij})^2}{\hat{\sigma}^2} \quad \text{and} \quad F_\beta^* = \frac{2p}{q} E_{xx}^* \frac{\hat{\beta}^2}{\hat{\sigma}^2},
$$

for testing the null hypotheses H_{01}, H_{02}, H_{03}, and H_{04}, respectively.

As a result of Theorem 3.1, Theorem 3.2 and Theorem 3.3 for large n, the null distribution of all the test statistics F_A^*, F_B^*, F_{AB}^* and F_β^* statistics are central F with degrees of freedom $\nu_1 = 1$ and $\nu_2 = 2^2 n - 2^2 - 1$.

Simulations

To evaluate the accuracy of the central F distribution, we simulate the Type I error probabilities of F_A^*, F_B^*, F_{AB}^* and F_β^* test statistics. Type I error probabilities are found by computing the following probabilities:

$$
P_1 = Prob(F_A^* \geq F_{\nu_1,\nu_2} | H_{01}), \quad P_2 = \quad Prob(F_B^* \geq F_{\nu_1,\nu_2} | H_{02}),
$$

$$
P_3 = Prob(F_{AB}^* \geq F_{\nu_1,\nu_2} | H_{03}), \quad \text{and} \quad P_4 = Prob(F_\beta^* \geq F_{\nu_1,\nu_2} | H_{04})
$$

where F_{ν_1,ν_2} is the table value for the F distribution for $\alpha = 0.05$, $\nu_1 = 1$ and $\nu_2 = 2^2 n - 2^2 - 1$. In Table 3, the values of type I error for the test statistics F_A^*, F_B^*, F_{AB}^* and F_β^* are given.

Table 3. Simulated values of the Type I error probabilities: $\alpha = 0.05$.

	$n = 10$				$n = 20$			
	$p = 2$	$p = 2.5$	$p = 3.5$	$p = 5$	$p = 2$	$p = 2.5$	$p = 3.5$	$p = 5$
F_A^*	0.050	0.041	0.046	0.047	0.054	0.046	0.045	0.047
F_B^*	0.048	0.043	0.046	0.051	0.055	0.048	0.047	0.046
F_{AB}^*	0.051	0.047	0.045	0.048	0.051	0.047	0.045	0.048
F_β^*	0.043	0.039	0.042	0.045	0.050	0.050	0.044	0.046

It is clear from Table 3 that F_A^*, F_B^*, F_{AB}^* and F_β^* tests exhibit good approximation to the pre-assumed $\alpha = 0.05$ value for small n. This indicates that they have F distribution even for small n.

It should be noted that here and in the rest of the paper, we present results only for the testing of the main effect of factor A and the slope parameter β, since the results for the main effect of factor B and the interaction effect AB are similar to those given for factor A.

The power of normal theory test statistics and the proposed test statistics are also compared via the Monte-Carlo simulation study. The simulation setup is taken as given in the previous section.

Corresponding normal theory test statistics for testing the null hypotheses H_{01}, H_{02}, H_{03}, and H_{04} are given below:

$$
\begin{aligned}
F_A &= \frac{A_{yyadj}}{\tilde{\sigma}^2}, \quad F_B = \frac{B_{yyadj}}{\tilde{\sigma}^2}, \\
F_{AB} &= \frac{AB_{yyadj}}{\tilde{\sigma}^2} \quad \text{and} \quad F_\beta = \frac{E_{yy} - E_{yyadj}}{\tilde{\sigma}^2},
\end{aligned}
\tag{4.3}
$$

respectively, for further information see Senoglu and Acitas [24].

Power values of the test statistics for testing the main effect of factor A and the slope parameter β are given in Table 4. It should be noted that the power values of F_A and F_A^* tests are obtained by adding and subtracting a constant d to the observations in the low level and the high level of factor A, respectively. Similarly, the power values of F_β and F_β^* tests are obtained adding a constant d to the true vale of β.

It is obvious from Table 4 that proposed tests are more powerful than classical normal theory tests. It should also be noted that the lines corresponding to $d = 0$ give the type I error.

Table 4. Power values of the F_A and F_A^*; F_β and F_β^* tests: $\alpha = 0.05$.

	$p = 2$		$p = 2.5$		$p = 3.5$		$p = 5$	
	F_A	F_A^*	F_A	F_A^*	F_A	F_A^*	F_A	F_A^*
d				$n = 10$				
0.00	0.045	0.050	0.044	0.041	0.048	0.046	0.048	0.047
0.15	0.17	0.22	0.16	0.17	0.15	0.16	0.15	0.15
0.30	0.49	0.66	0.45	0.54	0.42	0.47	0.41	0.46
0.45	0.77	0.93	0.74	0.86	0.72	0.81	0.71	0.79
0.60	0.92	0.99	0.91	0.98	0.9	0.96	0.90	0.95
				$n = 20$				
0.00	0.043	0.054	0.046	0.046	0.049	0.045	0.050	0.047
0.10	0.16	0.24	0.14	0.17	0.14	0.14	0.14	0.14
0.20	0.47	0.68	0.43	0.54	0.41	0.46	0.40	0.42
0.30	0.76	0.94	0.74	0.87	0.73	0.80	0.72	0.77
0.40	0.92	0.99	0.91	0.98	0.91	0.96	0.92	0.95
	$p = 2$		$p = 2.5$		$p = 3.5$		$p = 5$	
	F_β	F_β^*	F_β	F_β^*	F_β	F_β^*	F_β	F_β^*
d				$n = 10$				
0.00	0.050	0.043	0.049	0.039	0.046	0.042	0.048	0.045
0.15	0.18	0.2	0.16	0.15	0.15	0.13	0.14	0.14
0.30	0.51	0.59	0.45	0.47	0.43	0.43	0.43	0.42
0.45	0.79	0.88	0.76	0.8	0.74	0.75	0.74	0.74
0.60	0.92	0.97	0.91	0.94	0.91	0.92	0.91	0.92
				$n = 20$				
0.00	0.053	0.050	0.053	0.050	0.047	0.044	0.050	0.046
0.10	0.16	0.22	0.16	0.22	0.14	0.14	0.14	0.14
0.20	0.47	0.64	0.47	0.64	0.41	0.43	0.41	0.41
0.30	0.78	0.92	0.78	0.92	0.73	0.76	0.73	0.74
0.40	0.92	0.99	0.92	0.99	0.92	0.94	0.92	0.93

5. Robustness

In practice, the true distribution cannot be determined exactly or uniquely. The shape parameter may be misspecified or the data may contain outliers, or it may be contaminated. In this case, a question arises: How do the deviations from an assumed model affect the type I error and power of the proposed and the normal theory tests? In other words, how robust they are to departures from an assumed distribution, see i.e. [20, 25]. Therefore, this section is devoted to exploring the robustness of the proposed and the normal theory tests.

We assume that the underlying distribution for the error terms is $LTS(p = 3.5, \sigma = 1)$. We consider the following plausible alternatives:

Model I: $LTS(p = 2, \sigma)$
Model II: $LTS(p = 2.5, \sigma)$
Model III: (Dixon's outlier model)

$$(n - r)LTS(p = 3.5, \sigma) + rLTS(p = 3.5, 4\sigma), \quad r = 1, 2.$$

Table 5. Power values of the F_A and F_A^*; F_β and F_β^* tests for alternatives to $LTS(p = 3.5, \sigma)$: $\alpha = 0.05$.

	True Model		Model I		Model II		Model III		Model IV		Model V	
	F_A	F_A^*	F_A	F_A^*	F_A	F_A^*	F_A	F_A^*	F_A	F_A^*	F_A	F_A^*
d						$n = 10$						
0.00	0.048	0.046	0.046	0.034	0.051	0.045	0.044	0.039	0.043	0.037	0.049	0.044
0.15	0.15	0.16	0.18	0.18	0.15	0.15	0.12	0.12	0.13	0.13	0.17	0.17
0.30	0.42	0.47	0.49	0.58	0.43	0.50	0.35	0.40	0.36	0.40	0.46	0.52
0.45	0.72	0.81	0.77	0.89	0.74	0.85	0.62	0.72	0.63	0.74	0.75	0.85
0.60	0.90	0.96	0.91	0.98	0.91	0.97	0.83	0.92	0.83	0.92	0.92	0.97
						$n = 20$						
0.00	0.049	0.045	0.050	0.035	0.046	0.038	0.050	0.041	0.049	0.042	0.047	0.040
0.10	0.14	0.14	0.16	0.17	0.14	0.15	0.12	0.12	0.12	0.13	0.16	0.16
0.20	0.41	0.46	0.47	0.59	0.43	0.50	0.34	0.39	0.34	0.39	0.45	0.51
0.30	0.73	0.80	0.77	0.91	0.74	0.84	0.62	0.71	0.62	0.72	0.76	0.84
0.40	0.91	0.96	0.92	0.99	0.91	0.98	0.84	0.92	0.84	0.92	0.93	0.98

	True Model		Model I		Model II		Model III		Model IV		Model V	
	F_β	F_β^*	F_β	F_β^*	F_β	F_β^*	F_β	F_β^*	F_β	F_β^*	F_β	F_β^*
d						$n = 10$						
0.00	0.046	0.042	0.051	0.032	0.045	0.037	0.049	0.036	0.053	0.040	0.052	0.044
0.15	0.15	0.13	0.17	0.16	0.16	0.15	0.13	0.12	0.13	0.11	0.16	0.15
0.30	0.43	0.43	0.50	0.53	0.45	0.46	0.36	0.36	0.36	0.36	0.46	0.47
0.45	0.74	0.75	0.79	0.84	0.76	0.78	0.65	0.66	0.65	0.67	0.78	0.80
0.60	0.91	0.92	0.92	0.96	0.92	0.94	0.84	0.87	0.85	0.87	0.94	0.95
						$n = 20$						
0.00	0.047	0.044	0.050	0.032	0.053	0.040	0.051	0.041	0.047	0.038	0.048	0.039
0.10	0.14	0.14	0.16	0.16	0.14	0.14	0.12	0.11	0.12	0.12	0.15	0.15
0.20	0.41	0.43	0.47	0.56	0.42	0.46	0.34	0.37	0.34	0.37	0.44	0.48
0.30	0.73	0.76	0.78	0.88	0.74	0.81	0.63	0.68	0.64	0.69	0.77	0.82
0.40	0.92	0.94	0.92	0.98	0.92	0.96	0.84	0.89	0.85	0.90	0.94	0.96

Here n is the number of replications for each combination of levels of factors A and B.

Model IV: (Mixture model)

$$0.90LTS(p = 3.5, \sigma) + 0.10LTS(p = 3.5, 4\sigma).$$

Model V: (Contaminated model)

$$0.90LTS(p = 3.5, \sigma) + 0.10Uniform(-0.5, 0.5).$$

In this study, following definitions of robustness definitions are used, see Box [5] and Box & Tiao [6]:

Criterion Robustness: A hypothesis testing procedure is said to have criterion robustness if its type I error is never substantially higher than a pre-assigned value for plausible alternatives to an assumed model.

Inference Robustness: A hypothesis testing procedure is said to have inference robustness if its power is high, at any rate for plausible alternatives.

Given in Table 5 are the power values of the F_A, F_A^* and F_β, F_β^* tests under the true and alternative models. We see that all tests have criterion robustness since the simulated value of the Type I error is about pre-assigned value $\alpha = 0.05$. However, F_A^* and F_β^* tests are more preferable than the traditional F_A and F_β tests in terms of inference robustness, since they have higher power under the plausible alternatives.

The results of the Monte-Carlo simulation studies show that the MML estimators and the test statistics based on them are more preferable than the corresponding normal theory estimators and test statistics when the distribution of error terms is LTS. If we increase p further and end up with normal distribution, the performances of the proposed estimators and test statistics are exactly the same as their LS counterparts.

6. Application

Montgomery [15] considers an example in the context of factorial experiments with covariates. The experiment contains A and B factors having two levels (-1 and 1) and a covariate term x. The data set is given in Table 6.

Table 6. Data set for the application.

A	B	x	y
-1	-1	4.05	-30.73
-1	-1	3.58	-26.46
-1	-1	5.38	-26.39
-1	-1	2.48	-8.94
-1	1	5.03	39.72
-1	1	15.53	103.01
-1	1	-0.67	15.89
-1	1	4.10	44.54
1	-1	1.06	10.94
1	-1	0.36	9.07
1	-1	8.63	54.58
1	-1	13.64	73.72
1	1	11.44	66.2
1	1	5.13	38.57
1	1	1.96	16.3
1	1	2.92	20.44

Montgomery [15] assumes that error terms are normally distributed and analyzes the data using traditional LS estimators and normal theory test statistics, see page 621 in [15]. However, the Shapiro-Wilk test rejects the normality assumption of the error terms calculated from LS estimators at a significance level $\alpha = 0.05$. We also provide a Q-Q plot of the error terms, see Figure 2. It is clear from this figure that there is an outlying observation. Both the result of the Shapiro-Wilk test and the Q-Q plot lead us to assume a non-normal error distribution.

Therefore, we here use LTS as an alternative error distribution. However, before starting to analyze the data we should identify the plausible value of the shape

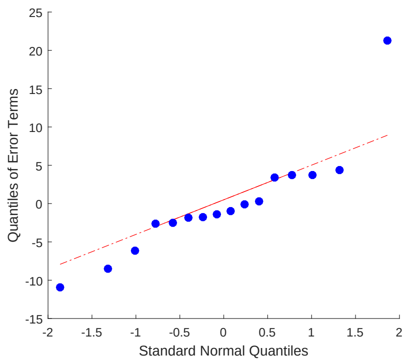

Figure 2. Normal Q-Q plot of the error terms.

parameter p. In this study, we use a method called as the profile likelihood to identify the plausible value of the shape parameter p. The steps of the method are given as follows:

(i) Calculate the MML estimators of the model parameters $\hat{\mu}$, $\hat{\tau}_i$, $\hat{\gamma}_j$, $\widehat{(\tau\gamma)}_{ij}$, $\hat{\beta}$ and $\hat{\sigma}$ $(i,j=1,2)$ for given p.

(ii) Calculate the log-likelihood value using the following equation:

$$\ln L(\hat{\mu},\hat{\tau}_i,\hat{\gamma}_j,\widehat{(\tau\gamma)}_{ij},\hat{\beta},\hat{\sigma}) \approx -2^2 n \ln\hat{\sigma} - p\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}\ln\left(1+\frac{\hat{z}_{ijk}^2}{q}\right) \tag{6.1}$$

where

$$\hat{z}_{ijk} = \big(y_{ijk}-\hat{\mu}-\hat{\tau}_i-\hat{\gamma}_j-\widehat{(\tau\gamma)}_{ij}-\hat{\beta}(x_{ijk}-\bar{x}_{...})\big)/\hat{\sigma} \quad (i,j=1,2;k=1,2,...,n).$$

(iii) Repeat steps (i) and (ii) for a serious values of p.

(iv) p value maximizing the log-likelihood function among the others is chosen as a plausible value of the shape parameter.

See [12] for more detailed information. After following these steps, we see that the plausible value of the shape parameter p is 2. MML estimates of the model parameters and the tests based on them are computed for $p=2$, see Table 7 and Table 8.

Table 7. The LS and the MML estimates of the model parameters.

	μ	τ_1	γ_1	$(\tau\gamma)_{11}$	β	σ
LS	25.03	-9.40	-16.06	-15.49	5.09	8.33
MML	26.93	-11.19	-16.30	-15.48	8.03	9.29

Table 8 shows that the test statistics F_A, F_B, F_{AB}, F_β and F_A^*, F_B^*, F_{AB}^*, F_β^* tests are in agreement rejecting all the null hypotheses given in Section 4. In other words, main effects, interaction effect and the slope parameter are significant at the $\alpha = 0.05$ level according to the results of the proposed and normal theory tests. However, the calculated values of the proposed test statistics are much more greater than the corresponding values of normal theory test statistics. This is another indication of the superiority of proposed test statistics. It should be noted that this conclusion is in accordance with the results given in Table 4.

Table 8. Calculated values of the proposed and normal theory tests statistics.

	A	B	AB	β
F	20.24	59.05	54.10	119.43
F^*	43.83	93.09	83.92	159.45

7. Conclusion

In this study, we consider a factorial ANCOVA model in which error terms are i.i.d. LTS. We derive MML estimators of the model parameters using Tiku's [26, 27] methodology. We also propose new test statistics based on these estimators for testing the main effects, the interaction effect and slope parameter. In the simulation study, MML estimators are shown to be more efficient than LS estimators. The results of the simulation study also demonstrate that proposed test statistics are more powerful and robust than corresponding normal theory test statistics even for small sample sizes.

It should be noted that we assume a balanced design in this study. However, the proposed method cannot easily be transferred to an unbalanced design, see for example [15] in the context of factorial design.

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