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The existence of extremal solutions to nonlinear fractional integro-differential equations with advanced arguments

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Abstract

This paper deals with the existence of extremal solutions for nonlinear fractional integro-differential equations with advanced arguments. Our analysis rely on monotone iterative method based on upper and lower solutions. Also, we give an illustrative example in order to indicate the validity of our assumptions.

Keywords: Monotone iterative method, Riemann-Liouville fractional derivative, Upper and lower solutions.

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1. Introduction

Fractional calculus is a branch of mathematical analysis, that provides integrals and derivatives of any arbitrary order and due to their multiple applications in many areas of science and engineering has grown extensively. [1, 3, 4, 7, 8, 9, 10, 11, 14, 15, 16, 17]. The monotone iterative method based on upper and lower solutions is a fruitful tools that provides an efficient mechanism to prove the existence results for nonlinear differential problems. We refer the reader to the book [5] and recent papers [2, 6, 12, 13, 18, 19, 20, 21, 22].
As far as we know, few authors consider the existence of extremal solutions for nonlinear Riemann-Liouville fractional integro-differential equations with advanced arguments. So this paper is devoted to study of the following nonlinear boundary value problem:

$$\begin{cases} (D^\alpha x(t))' = f(t, x(t), D^\alpha x(t), D^\beta x(t), Tx(t), Sx(t)), \quad t \in J := [0, T], \\ D^\alpha x(0) = x^*, \ t^{1-\alpha} x(t)|_{t=0} = 0, \ 0 < \beta \le \alpha \le 1, \end{cases} \tag{1.1}$$

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where $f \in C(J \times \mathbb{R}^5, \mathbb{R})$,

$$(Tx)(t) = \int_0^t k(t,s)x(s)ds, \quad (Sx)(t) = \int_0^T h(t,s)x(s)ds,$$

$k(t,s) \in C[D, \mathbb{R}^+]$, $h(t,s) \in C[[0,T]^2, \mathbb{R}^+]$, $D = \{(t,s) \in \mathbb{R}^2 | 0 \leq s \leq t \leq T\}$ and D^α, D^β are the Riemann-Liouville fractional derivatives.

The innovation of this study is that the nonlinear term f involve unknown function $x(t)$ and it's Riemann-Liouville fractional derivatives with different orders and integral operators Tx, Sx. Therefore, from this point of view, we generalize some recent works. Moreover, with a suitable choice of upper and lower solutions and condition on function f, we obtain the existence of extremal solutions and also present iterative sequences which are convergent to them.

This paper is organized as follows: in section 2, some facts and results about fractional calculus are given, also we consider the existence of the extremal solutions for first order nonlinear differential equation, while in spire of [20] we prove the main result in section 3 and we conclude this paper by considering an example in section 4.

2. Preliminaries and some lemmas

In this section, we present some definitions and results which will be needed later.

2.1. Definition. ([4]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0,$$

provided that the right-hand side is pointwise defined.

2.2. Definition. ([4]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds \quad t > 0,$$

where $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined. In particular, for $\alpha = n$, $D^n f(t) = f^{(n)}(t)$.

1. Remark. The following properties are well known:

$D^\alpha I^\alpha f(t) = f(t), \ \alpha > 0, \ f(t) \in L^1(0, \infty)$,
$D^\beta I^\alpha f(t) = I^{\alpha-\beta} f(t), \ \alpha > \beta > 0, \ f(t) \in L^1(0, \infty)$.

2.1. Lemma. ([4]) Let $Re(\alpha) > 0$, $n = [Re(\alpha)] + 1$ and let $f_{n-\alpha}(t) = I^{n-\alpha} f(t)$ be the fractional integral of order $n - \alpha$. If $f(t) \in L^1(0,T)$ and $f_{n-\alpha} \in AC^n[0,T]$, then we have the following equality

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{i=1}^n \frac{f_{n-\alpha}^{(n-i)}(0)}{\Gamma(\alpha - i + 1)} t^{\alpha - i}.$$

2.2. Lemma. The nonlinear fractional differential equation (1.1) is equivalent to the following IVP:

$$(2.1) \qquad \begin{cases} u'(t) = f(t, I^\alpha u(t), u(t), I^{\alpha-\beta} u(t), T_1 u(t), S_1 u(t)), \quad t \in J, \\ u(0) = x^*, \ 0 < \beta \leq \alpha \leq 1, \end{cases}$$

where

$$T_1u(t) = \int_0^t k_1(t,s)u(s)ds, \quad S_1u(t) = \int_0^T h_1(t,s)u(s)ds,$$

$$k_1(t,s) = \int_s^t \frac{(\tau - s)^{\alpha-1}k(t,\tau)}{\Gamma(\alpha)}d\tau, \quad h_1(t,s) = \int_s^T \frac{(\tau - s)^{\alpha-1}h(t,\tau)}{\Gamma(\alpha)}d\tau.$$

Proof. Take $D^\alpha x(t) = u(t)$ in (1.1), taking into account that $t^{1-\alpha}x(t)|_{t=0} = 0$, we get

$$x(t) = I^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)d\tau,$$

also

$$\begin{aligned}
Tx(t) = T(I^\alpha u(t)) &= \int_0^t k(t,s)(I^\alpha u(t))_{t=s}ds \\
&= \int_0^t k(t,s)\Big(\int_0^s \frac{(s-\tau)^{\alpha-1}u(\tau)}{\Gamma(\alpha)}d\tau\Big)ds \\
&= \int_0^t \Big(\int_\tau^t \frac{(s-\tau)^{\alpha-1}k(t,s)}{\Gamma(\alpha)}ds\Big)u(\tau)d\tau \\
&= \int_0^t \Big(\int_s^t \frac{(\tau-s)^{\alpha-1}k(t,\tau)}{\Gamma(\alpha)}d\tau\Big)u(s)ds \\
&= \int_0^t k_1(t,s)u(s)ds.
\end{aligned}$$

The same process can be repeated for S. So the proof is completed. □

Presently, we prove a comparison result for the first order initial value problem (2.1).

2.3. Lemma. Let $w \in C^1(J,\mathbb{R})$ satisfy the relations

$$\begin{cases} w'(t) \geq -KL_\alpha w(t) - Lw(t) - ML_{\alpha-\beta}w(t) - NT_1w(t) - PS_1w(t), \\ w(0) \geq 0, \ 0 < \beta \leq \alpha \leq 1, \end{cases} \tag{2.2}$$

where $K, L, M, N, P \geq 0$ are constants and $L_\alpha w(t) = \int_0^t \frac{(t-s)^{\alpha-1}w(s)}{\Gamma(\alpha)}ds$. If

$$\int_0^T \Big[\frac{Kt^\alpha}{\Gamma(\alpha+1)} + L + \frac{Mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + N\int_0^t k_1(t,s)ds + P\int_0^T h_1(t,s)ds\Big]dt < 1. \tag{2.3}$$

Then $w(t) \geq 0, \ \forall t \in J$.

Proof. Suppose $w(t) \geq 0$ is not true, then there exists a $t_0 \in (0,T]$ such that $w(t_0) < 0$. Let $\max\{w(t) : 0 \leq t \leq t_0\} = \lambda$, then $\lambda \geq 0$.
If $\lambda = 0$, the proof is similar to Lemma (2.1) of [20].
If $\lambda > 0$, then there exists a $t_1 \in [0,t_0]$ such that $w(t_1) = \lambda > 0$. From (2.2), we have

$$\begin{aligned}
w'(t) \geq &-\lambda\Big[\frac{Kt^\alpha}{\Gamma(\alpha+1)} + L + \frac{Mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&+ N\int_0^t k_1(t,s)ds + P\int_0^T h_1(t,s)ds\Big], \quad \forall t \in [0,t_0].
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
w(t_0) &= w(t_1) + \int_{t_1}^{t_0} w'(t)dt \\
&\geq \lambda - \lambda \int_0^T \Big[\frac{Kt^{\alpha}}{\Gamma(\alpha+1)} + L + \frac{Mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + N\int_0^t k_1(t,s)ds \\
&\qquad + P\int_0^T h_1(t,s)ds\Big]dt \\
&= \lambda\Big(1 - \int_0^T \Big[\frac{Kt^{\alpha}}{\Gamma(\alpha+1)} + L + \frac{Mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + N\int_0^t k_1(t,s)ds \\
&\qquad + P\int_0^T h_1(t,s)ds\Big]dt\Big).
\end{aligned}
$$

Then, by $w(t_0) < 0$, we get

$$
\begin{aligned}
&\int_0^T \Big[\frac{Kt^{\alpha}}{\Gamma(\alpha+1)} + L + \frac{Mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + N\int_0^t k_1(t,s)ds \\
&\quad + P\int_0^T h_1(t,s)ds\Big]dt > 1,
\end{aligned}
$$

which is contradiction. □

2.4. Lemma. If (2.3) holds. Then the linear problem

$$
\begin{cases}
u'(t) = g(t) - KI^{\alpha}u(t) - Lu(t) - ML^{\alpha-\beta}u(t) - NT_1u(t) - PS_1u(t), \\
u(0) = x^*, g \in C(J,\mathbb{R}),\ 0 < \beta \leq \alpha \leq 1,
\end{cases}
\tag{2.4}
$$

has a unique solution $u^* \in C^1(J,\mathbb{R})$.

Proof. We know that, $u(t) \in C^1(J,\mathbb{R})$ is a solution of (2.4) if and only if $u(t) \in C(J,\mathbb{R})$ is a solution of the following integral equation

$$
\begin{aligned}
u(t) &= x^* e^{-\int_0^t L ds} + \int_0^t e^{-\int_s^t L d\tau}\Big(g(s) - KI^{\alpha}u(s) - MI^{\alpha-\beta}u(s) \\
&\qquad - NT_1u(s) - PS_1u(s)\Big)ds \\
&= Au(t).
\end{aligned}
$$

For any $u, v \in C(J, \mathbb{R})$, we show that A is a contraction operator.

$$
\begin{aligned}
|Au(t) - Av(t)| &= \bigg| \int_0^t e^{L(s-t)} \Big[g(s) - KI^\alpha u(s) - MI^{\alpha-\beta} u(s) - NT_1 u(s) - PS_1 u(s) ds \Big] \\
&\quad - \int_0^t e^{L(s-t)} \Big[g(s) - KI^\alpha v(s) - MI^{\alpha-\beta} v(s) - NT_1 v(s) - PS_1 v(s) \Big] ds \bigg| \\
&= \bigg| \int_0^t e^{L(s-t)} \Big[K(I^\alpha (v-u)(s)) + M(I^{\alpha-\beta}(v-u)(s)) + N(T_1(v-u)(s)) \\
&\qquad + P(S_1(v-u)(s)) \Big] ds \bigg| \\
&\leq \int_0^T \bigg| K(I^\alpha (v-u)(s)) + M(I^{\alpha-\beta}(v-u)(s)) + N(T_1(v-u)(s)) \\
&\qquad + P(S_1(v-u)(s)) \bigg| ds \\
&\leq \int_0^T \Big[\frac{Ks^\alpha}{\Gamma(\alpha+1)} + \frac{Ms^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + N \int_0^t k_1(t,s) ds \\
&\qquad + P \int_0^T h_1(t,s) ds \Big] ds ||u - v||.
\end{aligned}
$$

Therefore, by condition (2.3), it follows

$$||Au - Av|| < ||u - v||.$$

Thus, by Banach contraction principle A has a unique fixed point u^*, which is unique solution of (2.4). □

2.1. Theorem. Let the following assumptions hold:

- (H_1) There exist $u_0, v_0 \in C^1(J, \mathbb{R})$ satisfying $u_0(t) \leq v_0(t), \ \forall t \in J$,

$$
\begin{cases} u_0'(t) \leq f(t, I^\alpha u_0(t), u_0(t), I^{\alpha-\beta} u_0(t), T_1 u_0(t), S_1 u_0(t)), & t \in J, \\ u_0(0) \leq x^*, \ 0 < \beta \leq \alpha \leq 1, \end{cases} \tag{2.5}
$$

  and v_0 satisfies inverse inequalities of (2.5).
- (H_2) There exist constants $K, L, M, N, P \geq 0$ which satisfy condition (2.3) and

$$
\begin{aligned}
f(t, x, y, z, v, w) - f(t, \bar{x}, \bar{y}, \bar{z}, \bar{v}, \bar{w}) &\geq -K(x - \bar{x}) - L(y - \bar{y}) - M(z - \bar{z}) \\
&\quad - N(u - \bar{u}) - P(w - \bar{w}).
\end{aligned}
$$

where $I^\alpha u_0(t) \leq \bar{x} \leq x \leq I^\alpha v_0(t)$, $u_0(t) \leq \bar{y} \leq y \leq v_0(t)$, $I^{\alpha-\beta} u_0(t) \leq \bar{z} \leq z \leq I^{\alpha-\beta} v_0(t)$, $T_1 u_0(t) \leq \bar{v} \leq v \leq T_1 v_0(t)$, $S_1 u_0(t) \leq \bar{w} \leq w \leq S_1 v_0(t) \ \forall t \in J$.

Then there exist monotone iterative sequences $\{u_n\}, \{v_n\} \subset [u_0, v_0]$ which converge uniformly to the extremal solutions u^*, v^* of (2.1), respectively, where $\{u_n\}, \{v_n\}$ are defined by

$$\begin{aligned} u_n(t) = x^* e^{-\int_0^t L ds} + \int_0^t e^{-\int_s^t L d\tau} \Big[f\Big(s, I^\alpha u_{n-1}(s), u_{n-1}(s), I^{\alpha-\beta} u_{n-1}(s), \\ T_1 u_{n-1}(s), S_1 u_{n-1}(s)\Big) \\ - K I^\alpha (u_n - u_{n-1})(s) - L(u_n - u_{n-1})(s) - M I^{\alpha-\beta}(u_n - u_{n-1})(s) \\ - N(T_1(u_n - u_{n-1})(s)) - P(S_1(u_n - u_{n-1})(s))\Big] ds, \end{aligned}$$

and

$$\begin{aligned} v_n(t) = x^* e^{-\int_0^t L ds} + \int_0^t e^{-\int_s^t L d\tau} \Big[f\Big(s, I^\alpha v_{n-1}(s), v_{n-1}(s), I^{\alpha-\beta} v_{n-1}(s), \\ T_1 v_{n-1}(s), S_1 v_{n-1}(s)\Big) \\ - K I^\alpha (v_n - v_{n-1})(s) - L(v_n - v_{n-1})(s) - M I^{\alpha-\beta}(v_n - v_{n-1})(s) \\ - N(T_1(v_n - v_{n-1})(s)) - P(S_1(v_n - v_{n-1})(s))\Big] ds. \end{aligned}$$

Also,

$$u_0 \le u_1 \le \dots \le u_n \le \dots \le u^* \le v^* \le \dots \le v_n \le \dots \le v_1 \le v_0.$$

Proof. For $\eta \in [u_0, v_0]$, we consider

$$(2.6) \qquad \begin{cases} u'(t) = g_\eta(t) - K I^\alpha u(t) - L u(t) - M I^{\alpha-\beta} u(t) \\ \qquad -N(T_1 u(t)) - P(S_1 u(t)) \\ u(0) = x^*, \; 0 < \beta \le \alpha \le 1, \end{cases}$$

where

$$\begin{aligned} g_\eta(t) = f\Big(t, I^\alpha \eta(t), \eta(t), I^{\alpha-\beta}\eta(t), T_1\eta(t), S_1\eta(t)\Big) \\ + K I^\alpha \eta(t) + L\eta(t) + M I^{\alpha-\beta}\eta(t) \\ + N(T_1\eta(t)) + P(S_1\eta(t)). \end{aligned}$$

By Lemma (2.4), we know (2.6) has a unique solution $u \in C^1(J, \mathbb{R})$.
Denote an operator $A : [u_0, v_0] \to C(J, \mathbb{R})$ by $u = A\eta$, then

$$\begin{aligned} A\eta = x^* e^{-Lt} + \int_0^t e^{L(s-t)} \Big[f\Big(s, I^\alpha \eta(s), \eta(s), I^{\alpha-\beta}\eta(s), T_1\eta(s), S_1\eta(s)\Big) \\ + K I^\alpha \eta(s) + L\eta(s) + M I^{\alpha-\beta}\eta(s) + N(T_1\eta(s)) + P(S_1\eta(s)) \\ - K I^\alpha u(s) - L u(s) - M I^{\alpha-\beta} u(s) - N(T_1 u(s)) - P(S_1 u(s))\Big] ds. \end{aligned}$$

Now, we show that $u_0 \le Au_0$, $Av_0 \le v_0$ and A is nondecreasing.
For the first claim, let $u_1 = Au_0$, $p(t) = u_1(t) - u_0(t)$. we show that $p(t) \ge 0$. By (H_1), we get that

$$\begin{cases} p'(t) \ge -K I^\alpha p(t) - L p(t) - M I^{\alpha-\beta} p(t) \\ \qquad -N(T_1 p(t)) - P(S_1 p(t)), \\ p(0) = u_1(0) - u_0(0) = Au_0(0) - u_0(0) \ge 0. \end{cases}$$

Hence, by Lemma (2.3) $p(t) \ge 0$. Similarly, we can show $Av_0 \le v_0$.
Now, we show that A is nondecresing. Let $u_1 = Au_0$, $v_1 = Av_0$ and $p(t) = v_1(t) - u_1(t)$.

By (H_2), we have

$$\begin{cases} p'(t) \geq -KI^{\alpha}p(t) - L(t) - MI^{\alpha-\beta}p(t) \\ \qquad\quad -N(T_1p(t)) - P(S_1p(t)), \\ p(0) = v_1(0) - u_1(0) > 0. \end{cases}$$

So A is nondecreasing.

Next, let $u_n = Au_{n-1}, v_n = Av_{n-1}, n = 1, 2, \dots$. By the properties of the operator A, we obtain that

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq u^* \leq v^* \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0.$$

Clearly, u_n, v_n satisfy

$$\begin{cases} u_n'(t) = f(t, I^{\alpha}u_{n-1}, u_{n-1}, I^{\alpha-\beta}u_{n-1}, T_1u_{n-1}, S_1u_{n-1}) \\ \qquad -K(I^{\alpha}(u_n - u_{n-1})) - L(u_n - u_{n-1}) - M(I^{\alpha-\beta}(u_n - u_{n-1})) \\ \qquad -N(T_1(u_n - u_{n-1})) - P(S_1(u_n - u_{n-1})), \\ u_n(0) = x^*, \end{cases}$$

$$\begin{cases} v_n'(t) = f(t, I^{\alpha}v_{n-1}, v_{n-1}, I^{\alpha-\beta}v_{n-1}, T_1v_{n-1}, S_1v_{n-1}) \\ \qquad -K(I^{\alpha}(v_n - v_{n-1})) - L(v_n - v_{n-1}) - M(I^{\alpha-\beta}(v_n - v_{n-1})) \\ \qquad -N(T_1(v_n - v_{n-1})) - P(S_1(v_n - v_{n-1})), \\ v_n(0) = x^*. \end{cases}$$

The sequences u_n, v_n are uniformly bounded and equicontinuous, so by Arzela-Ascoli Theorem, we find that $\lim_{n\to\infty} u_n(t) = u^*(t)$ and $\lim_{n\to\infty} v_n(t) = v^*(t)$ uniformly on J, and $u^*(t), v^*(t)$ are solutions of (2.1).

Finally, we prove that u^*, v^* are the extremal solutions of (2.1) in $[u_0, v_0]$. Let $w \in [u_0, v_0]$ be any solution of (2.1), then $Aw = w$. By $u_0 \leq w \leq v_0$ and the properties of A, we have

$$u_n \leq w \leq v_n, \quad n = 1, 2, \dots .$$

Thus, taking limit as $n \to \infty$, we have $u^* \leq w \leq v^*$. That is, u^*, v^* are the extremal solutions of (2.1) in $[u_0, v_0]$.

This completes the proof. □

3. Main result

In this section we prove the existence of extremal solutions of (1.1).

Let $C_{1-\alpha}(J, \mathbb{R}) = \{u \in C(0, T];\ t^{1-\alpha}u \in C(J, \mathbb{R})\}$ and

$DC_{1-\alpha}(J, \mathbb{R}) = \{u \in C_{1-\alpha}(J, \mathbb{R});\ D^{\alpha}u \in C^1(J, \mathbb{R})\}$.

3.1. Theorem. Assume that:

(H_1') There exist $y_0, z_0 \in DC_{1-\alpha}(J, \mathbb{R})$ such that $y_0(t) \leq z_0(t)$ and $D^{\alpha}y_0(t) \leq D^{\alpha}z_0(t)$, are lower and upper solution of (1.1),

$$\begin{cases} (D^{\alpha}y_0(t))' \leq f(t, y_0(t), D^{\alpha}y_0(t), D^{\beta}y_0(t), Ty_0, Sy_0(t)) \\ D^{\alpha}y_0(0) \leq x^*, \quad t^{1-\alpha}y_0(t)|_{t=0} = 0. \end{cases} \tag{3.1}$$

and z_0 satisfies inverse inequalities of (3.1).

(H_2') There exist constants $K, L, M, N, P \geq 0$ which satisfy condition (2.3) such that

$$\begin{aligned} f(t, x, y, z, u, w) - f(t, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{w}) \geq & -K(x - \bar{x}) - L(y - \bar{y}) - M(z - \bar{z}) \\ & - N(u - \bar{u}) - P(w - \bar{w}), \end{aligned}$$

where $y_0(t) \leq \bar{x} \leq x \leq z_0(t)$, $D^{\alpha}y_0(t) \leq \bar{y} \leq y \leq D^{\alpha}z_0(t)$, $D^{\beta}z_0(t) \leq \bar{z} \leq z \leq D^{\beta}z_0(t)$, $Ty_0(t) \leq \bar{u} \leq u \leq Tz_0(t)$, $Sy_0(t) \leq \bar{w} \leq w \leq Sz_0(t)$.

Then there exist iterative sequences $\{y_n\}$, $\{z_n\}$ which converge uniformly to the extremal solutions y^*, z^* of (1.1), respectively.

Proof. Let $D^\alpha x(t) = u(t)$ in (1.1), then Equation (1.1) is transformed into first order integro-differential equation (2.1). Now, we prove that all the conditions of Theorem (2.1) hold. Let $u_0(t) = D^\alpha y_0(t), v_0(t) = D^\alpha z_0(t)$, we have $u_0(t) \leq v_0(t)$. Also $y_0(t) = I^\alpha u_0(t),\ z_0(t) = I^\alpha v_0(t)$, so by (H_1') u_0, v_0 satisfy (H_1). By (H_2'), it is easy to see that the condition (H_2) holds. Therefore, by Theorem (2.1), we obtain that (2.1) has extremal solutions $u^*, v^* \in C^1(J,\mathbb{R})$ in $[u_0, v_0]$. Let $y^* = I^\alpha u^*,\ z^* = I^\alpha v^*$ so it follows that

$$\begin{cases} D^\alpha y^*(t) = u^*(t) \\ t^{1-\alpha} y^*(t)|_{t=0} = 0 \end{cases} \tag{3.2}$$

Since u^* satisfies (2.1) and y^* satisfies (3.2), then y^* is a solution of (1.1). Similarly, we can show that z^* is a solution of (1.1). It is easy to show that y^*, z^* are extremal solutions of (1.1). This completes the proof. ☐

4. Example

Consider the following problem:

$$\begin{cases} (D^{\frac{1}{2}} x(t))' = \frac{-1}{10} x(t) - \frac{1+t}{15} D^{\frac{1}{2}} x(t) - \frac{1+t^2}{20} D^{\frac{1}{4}} x(t) \\ \qquad\qquad - \frac{1+t^3}{30} \int_0^t tsx(s)ds - \frac{1+t^4}{40} \int_0^1 sx(s)ds, \quad t \in [0,1], \\ D^{\frac{1}{2}} x(0) = 0, \quad t^{\frac{1}{2}} x(t)|_{t=0} = 0, \end{cases} \tag{4.1}$$

where $\alpha = \frac{1}{2}, \beta = \frac{1}{4},\ k(t,s) = ts,\ h(t,s) = s$. Here,

$$f(t,x,y,z,u,w) = \frac{-1}{10} x - \frac{1+t}{15} y - \frac{1+t^2}{20} z - \frac{1+t^3}{30} u - \frac{1+t^4}{40} w.$$

By easy computation, we have
$K = \frac{1}{10},\ L = \frac{2}{15},\ M = \frac{1}{10},\ N = \frac{1}{15},\ P = \frac{1}{20}$.
Also,

$$\int_0^T \Big[\frac{Kt^\alpha}{\Gamma(\alpha+1)} + L + \frac{Mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + N\int_0^t k_1(t,s)ds + P\int_0^T h_1(t,s)ds\Big] dt = 0.324 < 1.$$

Now, take $u_0(t) = 0,\ v_0(t) = t^2$. It is easy to see that $u_0,\ v_0$ are lower and upper solution of (4.1). So all the conditions of Theorem (3.1) hold.
Thus there exist iterative sequences $\{u_n\}, \{v_n\}$ which converge uniformly to the extremal solutions u^*, v^* of (4.1), respectively.

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