

## PAPER DETAILS

TITLE: Applications of  $k$ -Fibonacci numbers for the starlike analytic functions

AUTHORS: Janusz SOKÓŁ, Ravinder Krishna RAINA, Nihal YILMAZ ÖZGÜR

PAGES: 121-127

ORIGINAL PDF URL: <https://dergipark.org.tr/tr/download/article-file/644752>

## Applications of $k$ -Fibonacci numbers for the starlike analytic functions

Janusz Sokół<sup>\*</sup>, Ravinder Krishna Raina<sup>†</sup> and Nihal Yılmaz Özgür<sup>‡</sup>

### Abstract

The  $k$ -Fibonacci numbers  $F_{k,n}$  ( $k > 0$ ), defined recursively by  $F_{k,0} = 0$ ,  $F_{k,1} = 1$  and  $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$  for  $n \geq 2$  are used to define a new class  $\mathcal{SL}^k$ . The purpose of this paper is to apply properties of  $k$ -Fibonacci numbers to consider the classical problem of estimation of the Fekete–Szegő problem for the class  $\mathcal{SL}^k$ . An application for inverse functions is also given.

*2000 AMS Classification:* Primary 30C45, secondary 30C80

**Keywords:** univalent functions; convex functions; starlike functions; subordination;  $k$ -Fibonacci numbers.

Received 27/12/2012 : Accepted 04/12/2013 Doi : 10.15672/HJMS.2015449091

### 1. Introduction

Let  $\mathbb{D} = \{z : |z| < 1\}$  denote the unit disc on the complex plane. The class of all holomorphic functions  $f$  in the open unit disc  $\mathbb{D}$  with normalization  $f(0) = 0$ ,  $f'(0) = 1$  is denoted by  $\mathcal{A}$  and the class  $\mathcal{S} \subset \mathcal{A}$  is the class which consists of univalent functions in  $\mathbb{D}$ . We say that  $f$  is subordinate to  $F$  in  $\mathbb{D}$ , written as  $f \prec F$ , if and only if  $f(z) = F(\omega(z))$  for some  $\omega \in \mathcal{A}$ ,  $|\omega(z)| < 1$ ,  $z \in \mathbb{D}$ .

Recently, N. Yılmaz Özgür and J. Sokół [5] defined and introduced the class  $\mathcal{SL}^k$  of shell-like functions as the set of functions  $f \in \mathcal{A}$  which is described in the following definition.

---

<sup>\*</sup>Department of Mathematics, Rzeszów University of Technology, Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland

Email: [jsokol@prz.edu.pl](mailto:jsokol@prz.edu.pl)

<sup>†</sup>M.P. University of Agri. and Technology, Udaipur, Rajasthan, India

Present address: 10/11 Ganpati Vihar, Opposite Sector 5, Udaipur 313002, India

Email: [rkraina\\_7@hotmail.com](mailto:rkraina_7@hotmail.com)

<sup>‡</sup>Department of Mathematics, Balıkesir University, Çağış Kampüsü, 10145 Balıkesir, Turkey

Email: [nihal@balikesir.edu.tr](mailto:nihal@balikesir.edu.tr)

**1.1. Definition.** Let  $k$  be any positive real number. The function  $f \in \mathcal{A}$  belongs to the class  $\mathcal{SL}^k$  if it satisfies the condition that

$$(1.1) \quad \frac{zf'(z)}{f(z)} \prec \tilde{p}_k(z), \quad z \in \mathbb{D},$$

where

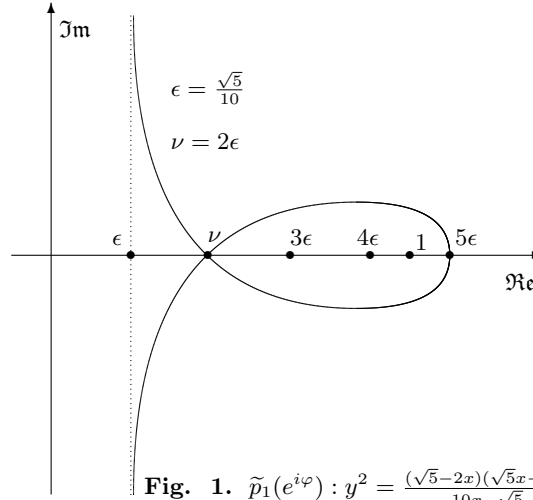
$$(1.2) \quad \tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2}, \quad \tau_k = \frac{k - \sqrt{k^2 + 4}}{2}, \quad z \in \mathbb{D}.$$

For  $k = 1$ , the class  $\mathcal{SL}^k$  becomes the class  $\mathcal{SL}$  of shell-like functions defined in [3], see also [4].

It was proved in [5] that functions in the class  $\mathcal{SL}^k$  are univalent in  $\mathbb{D}$ . Moreover, the class  $\mathcal{SL}^k$  is a subclass of the class of starlike functions  $\mathcal{S}^*$ , even more, starlike of order  $k(k^2 + 4)^{-1/2}/2$ . The name attributed to the class  $\mathcal{SL}^k$  is motivated by the shape of the curve

$$\mathcal{C} = \left\{ \tilde{p}_k(e^{it}) : t \in [0, 2\pi) \setminus \{\pi\} \right\}.$$

The curve  $\mathcal{C}$  has a shell-like shape and it is symmetric with respect to the real axis. Its graphic shape, for  $k = 1$ , is given below in Fig.1.



**Fig. 1.**  $\tilde{p}_1(e^{i\varphi}) : y^2 = \frac{(\sqrt{5}-2x)(\sqrt{5}x-1)^2}{10x-\sqrt{5}}$ .

For  $k \leq 2$ , note that we have

$$\tilde{p}_k \left( e^{\pm i \arccos(k^2/4)} \right) = k(k^2 + 4)^{-1/2},$$

and so the curve  $\mathcal{C}$  intersects itself on the real axis at the point  $w_1 = k(k^2 + 4)^{-1/2}$ . Thus  $\mathcal{C}$  has a loop intersecting the real axis also at the point  $w_2 = (k^2 + 4)/(2k)$ . For  $k > 2$ , the curve  $\mathcal{C}$  has no loops and it is like a conchoid, see for details [5]. Moreover, the coefficients of  $\tilde{p}_k$  are connected with  $k$ -Fibonacci numbers.

For any positive real number  $k$ , the  $k$ -Fibonacci number sequence  $\{F_{k,n}\}_{n=0}^{\infty}$  is defined recursively by

$$(1.3) \quad F_{k,0} = 0, \quad F_{k,1} = 1 \quad \text{and} \quad F_{k,n} = kF_{k,n-1} + F_{k,n-2} \quad \text{for } n \geq 2.$$

When  $k = 1$ , we obtain the well-known Fibonacci numbers  $F_n$ . It is known that the  $n^{th}$   $k$ -Fibonacci number is given by

$$(1.4) \quad F_{k,n} = \frac{(k - \tau_k)^n - \tau_k^n}{\sqrt{k^2 + 4}},$$

where  $\tau_k = (k - \sqrt{k^2 + 4})/2$ . If  $\tilde{p}_k(z) = 1 + \sum_{n=1}^{\infty} \tilde{p}_{k,n} z^n$ , then we have

$$(1.5) \quad \tilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\tau_k^n, \quad n = 1, 2, 3, \dots,$$

see also [5].

**1.2. Lemma.** [5] *If  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  belongs to the class  $\mathcal{SL}^k$ , then we have*

$$(1.6) \quad |a_n| \leq |\tau_k|^{n-1} F_{k,n},$$

where  $\tau_k = (k - \sqrt{k^2 + 4})/2$ . Equality holds in (1.6) for the function

$$(1.7) \quad \begin{aligned} f_k(z) &= \frac{z}{1 - k\tau_k z - \tau_k^2 z^2} \\ &= \sum_{n=1}^{\infty} \tau_k^{n-1} F_{k,n} z^n \\ &= z + \frac{(k - \sqrt{k^2 + 4})k}{2} z^2 + (k^2 + 1) \left( \frac{(k - \sqrt{k^2 + 4})k}{2} + 1 \right) z^3 + \dots \end{aligned}$$

## 2. The classical Fekete–Szegő functional

A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. Let  $\mathcal{S}$  be the class of univalent functions  $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$  mapping  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  into  $\mathbb{C}$  (the complex plane). The classical Fekete–Szegő functional is  $\mathcal{L}_\lambda = |a_3 - \lambda a_2^2|$ ,  $0 < \lambda \leq 1$ . Over the years, many results have been found for the classical functional  $\mathcal{L}_\lambda$ . Fekete and Szegő [1] bounded  $\mathcal{L}_\lambda$  by  $1 + 2 \exp(-2\lambda/(1 - \lambda))$ , for  $0 \leq \lambda < 1$  and  $f \in \mathcal{S}$ , where  $\mathcal{S}$  denotes the subclass of  $\mathcal{A}$  consisting of functions univalent in  $\mathbb{D}$ . This inequality is sharp for each  $\lambda$ . In particular, for  $\lambda = 1$ , one has  $|a_3 - a_2^2| \leq 1$  if  $f \in \mathcal{S}$ . Note that the quantity  $a_3 - a_2^2$  represents  $S_f(0)/6$ , where  $S_f$  denotes the Schwarzian derivative  $(f''/f')' - (f''/f')^2/2$  of locally univalent functions  $f$  in  $\mathbb{D}$ . It is interesting to consider the behavior of  $\mathcal{L}_\lambda$  for subclasses of the class  $\mathcal{S}$ . The Fekete–Szegő problem is to determine sharp upper bound for Fekete–Szegő functional  $\mathcal{L}_\lambda$  over a family  $\mathcal{F} \subset \mathcal{S}$ . In the literature, there exists a large number of results about inequalities for  $a_3 - a_2^2$  corresponding to various subclasses of  $\mathcal{S}$ . In the present paper we obtain the Fekete–Szegő inequalities for the class  $\mathcal{SL}^k$ . Before we consider how the Taylor series coefficients of functions in the class  $\mathcal{SL}^k$  might be bounded, let us first recall this problem for the Caratheodory functions. Let  $\mathcal{P}$  denote the class of analytic functions  $p$  in  $\mathbb{D}$  with  $p(0) = 1$  and  $\Re\{p(z)\} > 0$ .

**2.1. Lemma.** [2] *Let  $p \in \mathcal{P}$  with  $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ , then*

$$(2.1) \quad |c_n| \leq 2, \quad \text{for } n \geq 1.$$

*If  $|c_1| = 2$ , then  $p(z) \equiv p_1(z) = (1 + xz)/(1 - xz)$  with  $x = c_1/2$ . Conversely, if  $p(z) \equiv p_1(z)$  for some  $|x| = 1$ , then  $c_1 = 2x$ . Furthermore, we have*

$$(2.2) \quad |c_2 - c_1/2| \leq 2 - |c_1|^2/2.$$

*If  $|c_1| < 2$  and  $|c_2 - c_1/2| = 2 - |c_1|^2/2$ , then  $p(z) \equiv p_2(z)$ , where*

$$p_2(z) = \frac{1 + \bar{x}wz + z(wz + x)}{1 + \bar{x}wz - z(wz + x)}$$

and  $x = c_1/2$ ,  $w = (2c_2 - c_1^2)/(4 - |c_1|^2)$ . Conversely, if  $p(z) \equiv p_2(z)$  for some  $|x| < 1$  and  $w = 1$ , then  $c_1 = 2x$ ,  $w = (2c_2 - c_1^2)/(4 - |c_1|^2)$  and  $|c_2 - c_1/2| = 2 - |c_1|^2/2$ .

**2.2. Theorem.** If  $p(z) = 1 + p_1z + p_2z^2 + \dots$  and

$$p(z) \prec \tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2}, \quad \tau_k = \frac{k - \sqrt{k^2 + 4}}{2}, \quad z \in \mathbb{D},$$

then we have

$$(2.3) \quad |p_1| \leq \frac{(\sqrt{k^2 + 4} - k)k}{2}$$

and

$$(2.4) \quad |p_2| \leq (k^2 + 2) \left\{ \frac{(k - \sqrt{k^2 + 4})k}{2} + 1 \right\}.$$

The above estimations are sharp.

*Proof.* If  $p \prec \tilde{p}_k$ , then there exists an analytic function  $w$  such that  $|w(z)| \leq |z|$  in  $\mathbb{D}$  and  $p(z) = \tilde{p}_k(w(z))$ . Therefore, the function

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots \quad (z \in \mathbb{D})$$

is in the class  $\mathcal{P}(0)$ . It follows that

$$(2.5) \quad w(z) = \frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right) \frac{z^2}{2} + \dots$$

and

$$\begin{aligned} \tilde{p}_k(w(z)) &= 1 + \tilde{p}_{k,1} \left\{ \frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right) \frac{z^2}{2} + \dots \right\} + \tilde{p}_{k,2} \left\{ \frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right) \frac{z^2}{2} + \dots \right\}^2 + \dots \\ &= 1 + \frac{\tilde{p}_{k,1}c_1}{2}z + \left\{ \frac{1}{2} \left(c_2 - \frac{c_1^2}{2}\right) \tilde{p}_{k,1} + \frac{1}{4}c_1^2\tilde{p}_{k,2} \right\} z^2 + \dots \end{aligned}$$

(2.6)  $\quad = p(z).$

From (1.5), we find the coefficients  $\tilde{p}_{k,n}$  of the function  $\tilde{p}_k$  given by

$$\tilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\tau^n.$$

This shows the relevant connection  $\tilde{p}_k$  with the sequence of  $k$ -Fibonacci numbers

$$\begin{aligned} \tilde{p}_k(z) &= 1 + \sum_{n=1}^{\infty} \tilde{p}_{k,n} z^n \\ &= 1 + (F_{k,0} + F_{k,2})\tau_k z + (F_{k,1} + F_{k,3})\tau_k^2 z^2 + \dots \\ (2.7) \quad &= 1 + k\tau_k z + (k^2 + 2)\tau_k^2 z^2 + (k^3 + 3k)\tau_k^3 z^3 + \dots \end{aligned}$$

If  $p(z) = 1 + p_1z + p_2z^2 + \dots$ , then by (2.6) and (2.7), we have

$$(2.8) \quad p_1 = \frac{k\tau_k c_1}{2}$$

and

$$(2.9) \quad p_2 = \frac{k\tau_k}{2} \left(c_2 - \frac{c_1^2}{2}\right) + \frac{(k^2 + 2)}{4} c_1^2 \tau_k^2.$$

From (2.8) and (2.1) we directly obtain (2.3). From (2.9) and (2.2), we obtain

$$\begin{aligned}
|p_2| &= \left| \frac{k\tau_k}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{(k^2+2)}{4} c_1^2 \tau_k^2 \right| \\
&\leq \left| \frac{k\tau_k}{2} \left( c_2 - \frac{c_1^2}{2} \right) \right| + \left| \frac{(k^2+2)}{4} c_1^2 \tau_k^2 \right| \\
&\leq \frac{k|\tau_k|}{2} \left( 2 - \frac{1}{2} |c_1|^2 \right) + \frac{(k^2+2)}{4} |c_1|^2 \tau_k^2 \\
(2.10) \quad &= k|\tau_k| + \frac{|c_1|^2}{4} ((k^2+2)\tau_k^2 - k|\tau_k|).
\end{aligned}$$

Since  $\tau_k = (k - \sqrt{k^2+4})/2$ , so it is easily verified that

$$(2.11) \quad (k^2+2)\tau_k^2 - k|\tau_k| = \frac{(k(k - \sqrt{k^2+4}))(k^2+3)}{2} + k^2 + 2.$$

We want to show that (2.11) is positive for  $k > 0$ . Notice that

$$(2.12) \quad \frac{(k - \sqrt{k^2+4})(k^3+3k)}{2} + k^2 + 2 = \frac{(k^2+2)\sqrt{k^2+4} - k^3 - 4k}{k + \sqrt{k^2+4}}.$$

Thus, (2.11) is positive when

$$(2.13) \quad (k^2+2)\sqrt{k^2+4} > k^3 + 4k, \quad k > 0,$$

or equivalently, when

$$(2.14) \quad \left\{ (k^2+2)\sqrt{k^2+4} \right\}^2 > \{k^3+4k\}^2, \quad k > 0.$$

The inequality (2.14) yields the inequality

$$(2.15) \quad 4k^2 + 16 > 0, \quad k > 0,$$

which is evidently true, and hence (2.11) is positive. Therefore,  $(k^2+2)\tau_k^2 - |\tau_k| > 0$  and from (2.10), we obtain

$$\begin{aligned}
|p_2| &\leq k|\tau_k| + \frac{|c_1|^2}{4} ((k^2+2)\tau_k^2 - k|\tau_k|) \\
&\leq k|\tau_k| + (k^2+2)\tau_k^2 - k|\tau_k| \\
&= (k^2+2)\tau_k^2 \\
&= (k^2+2) \left\{ \frac{(k - \sqrt{k^2+4})k}{2} + 1 \right\}.
\end{aligned}$$

Thus, the equality in estimations (2.3), (2.4) are attained by the coefficients of the function given by (2.7).  $\square$

**2.3. Theorem.** Let  $\lambda$  be real. If  $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$  belongs to  $\mathcal{SL}^k$ , then

$$(2.16) \quad |a_3 - \lambda a_2^2| \leq (k(k - \sqrt{k^2+4})/2 + 1)(k^2 + 1 + k^2|\lambda|).$$

The above estimation is sharp. If  $\lambda \leq 0$ , then the equality in (2.16) is attained by the function  $f_k$  given in (1.6), and by the function  $-f_k(-z)$  when  $\lambda \geq 0$ .

*Proof.* For given  $f \in \mathcal{SL}^k$ , define  $p(z) = 1 + p_1 z + p_2 z^2 + \dots$  by

$$\frac{zf'(z)}{f(z)} = p(z) \quad (z \in \mathbb{D}),$$

where  $p \prec \tilde{p}_k$  in  $\mathbb{D}$ . Hence

$$z + 2a_2 z^2 + 3a_3 z^3 + \dots = \{z + a_2 z^2 + a_3 z^3 + \dots\} \{1 + p_1 z + p_2 z^2 + \dots\}$$

and

$$a_2 = p_1, \quad 2a_3 = p_1 a_2 + p_2.$$

Therefore,  $|a_3 - \lambda a_2| = |(p_1 a_2 + p_2)/2 + \lambda p_1^2|$ . Using this and the bounds (2.3), (2.4) and (1.6), we obtain

$$\begin{aligned} |a_3 - \lambda a_2^2| &= |(p_1 a_2 + p_2)/2 - \lambda p_1^2| \\ &\leq \frac{|p_1||a_2| + |p_2|}{2} + |\lambda||p_1^2| \\ &\leq \frac{k(k - \sqrt{k^2 + 4})/2 \cdot k(k - \sqrt{k^2 + 4})/2 + (k^2 + 2)(k(k - \sqrt{k^2 + 4})/2 + 1)}{2} \\ &\quad + |\lambda| \left\{ \frac{(\sqrt{k^2 + 4} - k)k}{2} \right\}^2 \\ &= \frac{k^2(k(k - \sqrt{k^2 + 4})/2 + 1) + (k^2 + 2)(k(k - \sqrt{k^2 + 4})/2 + 1)}{2} \\ &\quad + |\lambda| \left\{ \frac{(\sqrt{k^2 + 4} - k)k}{2} \right\}^2 \\ &= (k^2 + 1)(k(k - \sqrt{k^2 + 4})/2 + 1) + |\lambda| \left\{ \frac{(\sqrt{k^2 + 4} - k)k}{2} \right\}^2 \\ &= (k(k - \sqrt{k^2 + 4})/2 + 1)(k^2 + 1 + k^2|\lambda|). \end{aligned}$$

□

**2.4. Corollary.** *If  $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ ,  $|z| < r_0(g)$ ,  $r_0(g) \geq 1/4$ , is an inverse to  $f \in \mathcal{SL}^k$ , then we have*

$$(2.17) \quad |b_2| \leq \frac{(k - \sqrt{k^2 + 4})k}{2},$$

$$(2.18) \quad |b_3| \leq (k(k - \sqrt{k^2 + 4})/2 + 1)(3k^2 + 1).$$

*The above estimation is sharp. The equalities are attained by the function  $-if_k^{-1}(iz)$ , where  $f_k$  is given in (1.6).*

*Proof.* For each  $f \in \mathcal{S}$ , the Koebe one-quarter theorem ensures that the image of  $\mathbb{D}$  under  $f$  contains the disc of radius  $1/4$ . If  $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$  is univalent in  $\mathbb{D}$  then,  $f$  has the inverse  $f^{-1}$  with the expansion

$$(2.19) \quad f^{-1}(z) = z - a_2 z^2 + (2a_2^2 - a_3) z^3 + \dots, \quad |z| < r_0(f), \quad r_0(f) \geq 1/4.$$

It was proved in [5] that functions in the class  $\mathcal{SL}^k$  are univalent in  $\mathbb{D}$ . From Lemma 1.2 and (2.19), we obtain the inequality (2.17). Also, from Theorem 2.3 (with  $\lambda = 2$ ) and (2.19), we obtain the inequality (2.18). If  $f \in \mathcal{SL}^k$ , then the function  $-if_k(iz)$  satisfies

(1.1), so it belongs to the class  $\mathcal{SL}^k$  too. Moreover, from (1.6), we have

$$\begin{aligned}
& -if_k^{-1}(iz) \\
& = z + i \frac{(k - \sqrt{k^2 + 4})k}{2} z^2 \\
& \quad - \left\{ 2 \left( \frac{(k - \sqrt{k^2 + 4})k}{2} \right)^2 + (k^2 + 1) \left( \frac{(k - \sqrt{k^2 + 4})k}{2} + 1 \right) \right\} z^3 + \dots \\
& = z + i \frac{(k - \sqrt{k^2 + 4})k}{2} z^2 - (k(k - \sqrt{k^2 + 4})/2 + 1)(3k^2 + 1)z^3 + \dots
\end{aligned}$$

This shows that the equalities in (2.17) and (2.18) are attained by the second and third coefficients of the function  $-if_k^{-1}(iz)$ .  $\square$

## References

- [1] M. Fekete, G. Szegő, Eine Bemerkung über ungerade schlichte Functionen, J. Lond. Math. Soc. **8**(1933) 85–89.
- [2] C. Pommerenke, Univalent Functions, in: Studia Mathematica Mathematicische Lehrbücher, Vandenhoeck and Ruprecht, 1975.
- [3] J. Sokół, On starlike functions connected with Fibonacci numbers, Folia Scient. Univ. Tech. Resoviensis 175(23)(1999), 111–116.
- [4] J. Sokół, Remarks on shell-like functions, Folia Scient. Univ. Tech. Resoviensis 181(24)(2000), 111–115.
- [5] N. Yilmaz Özgür, J. Sokół, On starlike functions connected with  $k$ -Fibonacci numbers, Bull. Malaysian Math. Sci. Soc. 38(1)(2015), 249–258.



